Arba Minch University

Applied Mathematics One(Math 1051) Lecture Note

"Mathematics is the most beautiful and most powerful creation of the human spirit."

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Contents

Chapter 1

Matrices and Determinants

Objective:

- By the end of this chapter, students are expected to:
  - Define and identify different types of matrices
  - Understand the arithmetic operations on matrices
  - Reduce the given matrix to row reduced echelon form using elementary row operations
  - Find the inverse of some matrices using elementary row operations
  - Define system of linear equations in terms of matrices
  - Apply Gaussian elimination method, Gaussian Jordan method, and matrix inversion method to solve the given system of linear equations
  - Define and compute eigenvalue and eigenvectors

1.1 Definition of matrix and basic operations

Definition 1.1

Matrix is a rectangular array or arrangement of numbers of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

is called a matrix of size m by n (written as $m \times n$).

Each number $\mathbf{a_{ij}}$ is called an element or entry of the matrix and it is an element appearing in the i^{th} row and j^{th} column of a matrix. Elements in the horizontal line are said to form rows, and elements in the vertical lines are said to form columns. Here we say A has m rows and n columns. The i^{th} row of matrix A is $R_i = \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \end{bmatrix} 1 \leq i \leq m$

The j^{th} column of the matrix A is

$$c_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{bmatrix} 1 \leq j \leq n.$$

The order of a matrix denotes the number of rows and columns in the matrix. Thus, a matrix of order $m \times n$ has **m** rows and **n** columns. We often write the matrix A as $A = [a_{ij}]_{m \times n},\ 1 \leq i \leq m,\ 1 \leq j \leq n$, i-denotes the row and j- denotes the column.
For example, in the matrix

$$A = \begin{bmatrix} 5 & 9 & 6 & 8 \\ 3 & 2 & 3 & 1 \\ 1 & 0 & 4 & 7 \end{bmatrix}$$

, there are 3 rows and 4 columns. Therefore, matrix A can be called a matrix of order or size 3×4.
The rows are

$$\begin{bmatrix} 5 & 9 & 6 & 8 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 3 & 1 \end{bmatrix} \& \begin{bmatrix} 1 & 0 & 4 & 7 \end{bmatrix}$$

, and the columns

$$\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} \& \begin{bmatrix} 8 \\ 1 \\ 7 \end{bmatrix}.$$

Here $a_{11} = 5,\ a_{12} = 9, a_{13} = 6,\ a_{14} = 8,\ a_{21} = 3,\ a_{22} = 2,\ a_{23} = 3,\ a_{24} = 1$, etc. and A has 12 elements.

Definition 1.2

Two matrices A and B are said to be equal, written $A = B$, if they are of the same order and if all corresponding entries are equal i.e. $a_{ij} = b_{ij}$.

For example,

$$\begin{bmatrix} 5 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2+3 & 1 & 0 \\ 2 & 3 & 2 \times 2 \end{bmatrix}$$

but

$$\begin{bmatrix} 2 \\ 9 \end{bmatrix} \neq \begin{bmatrix} 2 & 9 \end{bmatrix}$$

. Why?

Example 1.1

Given the matrix equation

$$\begin{bmatrix} x+y & 8 \\ x-y & 6 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 1 & 6 \end{bmatrix}.$$

Find x and y.
Solution: *By the definition of equality of matrices,*
$x + y = 3$ and $x - y = 1$ solving this system of equations gives $x = 2$ and $y = 1$.

Exercise 1.1

1. *Find the values of x, y, z and w which satisfy the matrix equation*

(a) $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$ (b) $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4w+6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2w \end{bmatrix}$

1.1.1 Types of Matrices

1. A matrix having exactly one row is called a row matrix.

Example 1.2

$$\begin{pmatrix} 1 & 0 & 4 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 5 \end{pmatrix}$$

, are row matrices. A row matrix is often referred to as a row vector.

2. A matrix having exactly one column is called a column matrix

Example 1.3

$$\begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}$$

is a column matrix.

3. A zero matrix or null matrix is a matrix in which all of its elements are zero.

Example 1.4

The matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \& \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are the zero matrices

4. A matrix, in which the numbers of rows and the number of columns are equal, that is, an $n \times n$ matrix, is called a square matrix of order n.

Example 1.5

$$\begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}, \& \begin{bmatrix} 1 & 2 & 8 \\ 4 & 6 & 0 \\ 1 & 3 & 5 \end{bmatrix}$$

are square matrices.

If $A = (a_{ij})_{n \times n}$ is a square matrix the elements a_{ii}'s are called the diagonal elements. The main diagonal or simply diagonal of $\mathbf{A}$ consists of the elements $a_{11},\ a_{22},\ a_{33},\ \cdots,\ a_{nn}$.

5. A square matrix in which all the non-diagonal elements are zero is called a diagonal matrix.

Example 1.6

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \& \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

are diagonal matrices.

Note that the diagonal elements in a diagonal matrix may also be zero.

Example 1.7

$$\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \& \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are also diagonal matrix.

6. A diagonal matrix whose diagonal elements are equal is called a scalar matrix

Example 1.8

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \& \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \& \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are scalar matrices

7. Diagonal matrix of order n in which every diagonal element is unity (one) is called the identity matrix or unit matrix of order n . The identity matrix of order n is denoted by I_n.

Example 1.9

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is an identity matrix of order two.

8. A square matrix having only zeros below its diagonal is called upper triangular matrix. A square matrix having only zeros above its diagonal is called lower triangular matrix. A matrix that is either upper triangular or lower triangular is called **triangular matrix**.

Example 1.10

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 4 & 3 \end{bmatrix}, \ \& \begin{bmatrix} 3 & 0 \\ 6 & 4 \end{bmatrix}$$

are lower triangular matrices and

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \& \begin{bmatrix} 3 & 7 \\ 0 & 4 \end{bmatrix}$$

are upper triangular matrices.

Definition 1.3

A matrix obtained by deleting one or more rows and/or columns of A is called sub matrix of A.

Example 1.11

If $A = \begin{bmatrix} 4 & 6 & 1 \\ 3 & 8 & 2 \\ 2 & 0 & 3 \end{bmatrix}$, *then* $\begin{bmatrix} 8 & 2 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$ *and* $\begin{bmatrix} 4 & 6 \\ 3 & 8 \\ 2 & 0 \end{bmatrix}$ *are some of the sub matrices of A.*

1.1.2 Operations on Matrices

Addition and Subtraction of matrices

Definition 1.4

If A and B are matrices of the same order, then sum of A and B , denoted by $A + B$, is the new matrix of the same order obtained by adding the corresponding elements of A and B. Similarly, the difference of A and B , denoted by $A - B$, is the matrix obtained by subtracting the corresponding elements of$A$ and B.

Example 1.12

1. *If*$A = \begin{bmatrix} 1 & 2 & 8 \\ 4 & 6 & 0 \\ 1 & 3 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 0 & 3 \\ 3 & 8 & 2 \\ 4 & 6 & 1 \end{bmatrix}$, *then we have that*

   (a) $$A + B = \begin{bmatrix} 1 & 2 & 8 \\ 4 & 6 & 0 \\ 1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 3 \\ 3 & 8 & 2 \\ 4 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+0 & 8+3 \\ 4+3 & 6+8 & 0+2 \\ 1+4 & 3+6 & 5+1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 11 \\ 7 & 14 & 2 \\ 5 & 9 & 6 \end{bmatrix}$$

   (b) $$A - B = \begin{bmatrix} 1 & 2 & 8 \\ 4 & 6 & 0 \\ 1 & 3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 3 \\ 3 & 8 & 2 \\ 4 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1-2 & 2-0 & 8-3 \\ 4-3 & 6-8 & 0-2 \\ 1-4 & 3-6 & 5-1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 5 \\ 1 & -2 & -2 \\ -3 & -3 & 4 \end{bmatrix}.$$

1.1.3 Multiplication of Matrices by Scalar

Let A be any matrix and α be a scalar (real number), then αA is the matrix obtained from A multiplying each element of A by α. This operation is called scalar multiplication.
In particular, $-A$ is the matrix obtained from A by multiplying each element of A by -1 and is called the additive inverse of A.

Example 1.13

If $A = \begin{bmatrix} 1 & 2 & 6 \\ 5 & 0 & 4 \\ 3 & 1 & 2 \end{bmatrix}$ *, then*

$$3A = 3 \begin{bmatrix} 1 & 2 & 6 \\ 5 & 0 & 4 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(6) \\ 3(5) & 3(0) & 3(4) \\ 3(3) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 18 \\ 15 & 0 & 12 \\ 9 & 3 & 6 \end{bmatrix}$$

Properties on Matrix addition and Scalar Multiplication

Let A, B and C be $m \times n$ matrices and $\mathbf{0}$ be a zero matrix of size $m \times n$ and α, β be scalars. Then

1. $A + B = B + A$ (Commutative law for addition)
2. $(A + B) + C = A + (B + C)$ (Associative law for addition)
3. $A + \mathbf{0} = \mathbf{0} + A = A$ ($\mathbf{0}$ is called Additive identity)
4. For each matrix A , there exists a unique $m \times n$ matrix $-A$ such that $A + (-A) = \mathbf{0} = -A + A$
5. $\alpha(A + B) = \alpha A + \alpha B$
6. $(\alpha\beta)A = \alpha(\beta A) = \beta(\alpha A)$
7. $(\alpha + \beta)A = \alpha A + \beta A$

From the above properties, the set of all matrices having the same order forms a vector space with the operations addition and scalar multiplication.

1.2 Product and Transpose of a matrix

Definition 1.5: (Matrix Product)

Let A be an $m \times r$ matrix and B be an $r \times n$. The $(ij)^{th}$ entry of $C = AB$ is the dot product of the i^{th} row vector of A and the j^{th} column vector of B:

$$\begin{aligned} c_{ij} &= \begin{bmatrix} a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ir} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ \vdots \\ b_{rj} \end{bmatrix} \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj} \\ &= \Sigma_{k=1}^{r} a_{ik}b_{kj} \end{aligned}$$

The product C has order $m \times n$.

Example 1.14

1. *If* $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 5 \\ 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$ *then*

$$\begin{aligned} AB &= \begin{bmatrix} 1(2)+1(2)+3(4) & 1(5)+1(3)+3(1) \\ 1(2)+0(2)+5(4) & 1(5)+0(3)+5(1) \\ 3(2)+2(2)+1(4) & 3(5)+2(3)+1(1) \end{bmatrix} \\ &= \begin{bmatrix} 2+2+12 & 5+3+3 \\ 2+0+20 & 5+0+5 \\ 6+4+4 & 15+6+1 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 11 \\ 22 & 10 \\ 14 & 22 \end{bmatrix} \end{aligned}$$

2. $A = \begin{bmatrix} 2 & 3 \\ 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$ *, then* $AB = \begin{bmatrix} 6 & 9 \\ 0 & 21 \end{bmatrix}$ *and* $BA = \begin{bmatrix} 21 & 0 \\ 11 & 6 \end{bmatrix}$. *So,* $AB \neq BA$

Note: In general matrix multiplication is not commutative.

Properties of Matrix Multiplication

Let A, B and C be three matrices of the appropriate sizes. Let α be a scalar. Then

1. $A(BC) = (AB)C$.
2. $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$.
3. $\alpha(AB) = (\alpha A)B = A(\alpha B)$

1.2.1 The Transpose of a Matrix

Definition 1.6

A matrix obtained from a given matrix $\boldsymbol{A}$ *by interchanging the rows and columns is called the transpose of* A *and it is denoted by* A^t *.That is, If* $A = (a_{ij})_{m\times n}$ *then* $A^t = (a_{ji})_{n\times m}$

Example 1.15

1. If $A = \begin{bmatrix} 2 & 5 \\ -2 & 3 \\ 4 & 1 \end{bmatrix}$, then $A^t = \begin{bmatrix} 2 & -2 & 4 \\ 5 & 3 & 1 \end{bmatrix}$

2. If $B = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}$, then $B^t = \begin{bmatrix} 4 & 3 & -6 \end{bmatrix}$

Let $A = (a_{ij})_{n\times n}$ be a square matrix. Then A is said to be

i) Symmetric matrix if $A^t = A$

ii) Skew symmetric if $A^t = -A$

Example 1.16

1. $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 0 & -2 \\ 4 & -2 & 1 \end{bmatrix}, A^t = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 0 & -2 \\ 4 & -2 & 1 \end{bmatrix},$
   $\implies A^t = A$, therefore, A is symmetric.

2. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A^t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, therefore, A is skew-symmetric.

Properties of Transpose of Matrix

Let A and B be matrices such that addition and multiplication is defined. Then

1. $(A^t)^t = A$
2. $(A + B)^t = A^t + B^t$ And $(AB)^t = B^t A^t$
3. $(\alpha A)^t = \alpha A^t, \alpha -$ is a scalar
4. $A = \underbrace{\frac{1}{2}(A + A^t)}_{symmetric} + \underbrace{\frac{1}{2}(A - A^t)}_{skew}$

1.2.2 Trace of a Matrix

Definition 1.7

let $A = (a_{ij})_{n\times n}$*, be a square matrix of order* n *.Then trace of* A *is defined to be the sum of the diagonal elements of* A*. That is* $trace(A) = \sum_{i=1}^{n} a_{ii}$.

Notation: The trace of a matrix A is also commonly denoted as $trace(A)$ or $tr(A)$.

Properties of trace of a matrix

If A and B are square matrices, then

- $trace(A + B) = trace(A) + trace(B)$
- $trace(A) + trace(A^t)$
- $trace(cA) = c(trace(A))$
- $trace(AB) = trace(BA)$

Example 1.17

Find the trace of $A = \begin{bmatrix} 15 & 6 & 7 \\ 2 & -4 & 2 \\ 3 & 2 & 6 \end{bmatrix}$

Solution: $tr(A) = \sum_{i=1}^{3} a_{ii} = 15 + -4 + 6 = 17$

1.3 Elementary Row Operations and its properties

Definition 1.8: (Elementary row operation)

Given any matrix Aof order $m \times n$. Any one of the following operations on the matrix is called elementary row operation.

1. Interchanging any two rows of A $R_i \Leftrightarrow R_j$ (Interchange the i^{th} and j^{th} row)
2. Multiplying a row of **A** by a nonzero constant **k** $R_i \implies kR_i$ (Multiply the i^{th} row by scalar **k**)
3. Adding a multiple of one row of A to another row of A . $R_j \implies R_j + kR_i$ (add **k** times i^{th} row to j^{th} row).

Example 1.18

1. *Give a matrix* $A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix}$

   (a) *Interchange rows* 1 *and* 3 *of* A

   $$\Longrightarrow \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

   (b) *Multiply the third row of* A *by* $1/3$

   $$\Longrightarrow \quad \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}$$

   (c) *Multiply the second row of* A *by* -2*, then add to the third row of* A

   $$\Longrightarrow \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}$$

Definition 1.9

Two matrices A *and* B *are called row equivalent or simply (equivalent matrices) if one matrix can be obtained from the other matrix by applying finite number of elementary operations. In this case we write* $A \sim B$*.*

Example 1.19

As we observe from the above example,

$$\begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2/3 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}$$

Definition 1.10: (Matrix in reduced row echelon form):

A matrix in reduced row echelon form has the following properties:

1. *All rows consisting entirely of* 0 *are at the bottom of the matrix.*
2. *For each nonzero row, the first entry is* 1*. The first entry is called a leading* 1*.*
3. *For two successive non zero rows, the leading* 1 *in the higher row appears farther to the left than the leading* 1 *in the lower row.*
4. *If a column contains a leading* 1*, then all other entries in that column are* 0*.*

Note: A matrix is in row echelon form as the matrix has the first 3 properties.

Example 1.20

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are the matrices in reduced row echelon form. Where as the matrix.

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is not in reduced row echelon form but it is row echelon form since the matrix has the first 3 properties and all the other entries above the leading 1 in the third column are not **0**. The matrix

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is not in row echelon form (also not in reduced row echelon form) since the leading 1 in the second row is not in the left of the leading 1 in the third row and all the other entries above the leading 1 in the third column are not **0**.

Definition 1.11: (Rank of a matrix)

The rank of a matrix A , denoted by $rank(A)$, is the number of nonzero rows remaining after it has been changed into row echelon or reduced row echelon form.

Remark: If A is zero matrix then **rank(A)** is **0**.

Example 1.21

Determine the rank of the following matrices.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 3 & 1 & 2 & 6 \\ -1 & 2 & 5 & -4 \\ 2 & 3 & 7 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 6 & 9 & -3 \\ 2 & 4 & 6 & -2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 3 & 1 & 2 & 6 \\ -1 & 2 & 5 & -4 \\ 2 & 3 & 7 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & -3 \\ 0 & 3 & 7 & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 3R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 1 & -9 \end{bmatrix}$$

$$R_4 \to R_4 - R_3 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to R_2 - 2R_3 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, $\textbf{rank(A)} = \textbf{3}$ *by using elementary row operations.*

$$B = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 6 & 9 & -3 \\ 2 & 4 & 6 & -2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $\textbf{rank(B)} = 1$

Exercise 1.2

1. *Transform the following matrix in to reduced row echelon form and determine the rank of the following matrices.*

   (a) $A = \begin{bmatrix} 10 & -8 & 0 \\ 1 & 3 & -5 \\ 7 & 0 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix}$

   (b) $C = \begin{bmatrix} 2 & 6 & 7 & 9 \\ 3 & 4 & 5 & -1 \\ 1 & 2 & 3 & 1 \\ 2 & 5 & 8 & 4 \\ -1 & 2 & 2 & 10 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 8 & 0 & 3 \end{bmatrix}$

1.4 Inverse of a matrix and its properties

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is invertible or non-singular and B is the inverse of A. In this situation, we write $B = A^{-1}$. Notice that if B is the inverse of A, then we can just as easily say A is the inverse of B, or A and B are inverses of each other.

Example 1.22

Show that $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ *is an inverse for the matrix* $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$*:*

Solution:-*By the definition there are two multiplications to confirm. (We will show later that this isn't necessary, but right now we are working strictly from the definition.) We have*

$$\begin{aligned} AB &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2(2) + (-1)1 & 2(1) + (-1)2 \\ (-1)1 + 1(1) & -1(1) + 1(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I_2 \end{aligned}$$

and similarly

$$\begin{aligned} BA &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(2) + 1(-1) & 1(-1) + 1(1) \\ 1(2) + 2(1) & 1(-1) + 2(2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I_2 \end{aligned}$$

Therefore the definition for inverse is satisfied, so that A *and* B *work as inverses to each other.*

Example 1.23

Matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *cannot have an inverse.*

Theorem 1.1

Suppose that $\boldsymbol{A}$ *is invertible and that both* $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *are inverses of* $\boldsymbol{A}$*. Then* $\boldsymbol{B} = \boldsymbol{C}$ *and we will denote the inverse as* A^{-1} *.*

Computing the Inverse of a Non Singular Matrix

Suppose A is a non singular square matrix of size **n**. Create the $n \times n$ matrix **M** by placing the $n \times n$ identity matrix in to the right of the matrix A. Let **N** be a matrix that is row-equivalent to **M** and in reduced row-echelon form then the first $\boldsymbol{n}$ columns of **N** is I_n and the last $\boldsymbol{n}$ columns of **N** is A^{-1}.

Example 1.24

Computing a Matrix Inverse of $B = \begin{bmatrix} -7 & 6 & 12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$

Solution:*The augmented matrix is*

$$[B \mid I] = \begin{bmatrix} -7 & -6 & -12 & 1 & 0 & 0 \\ 5 & 5 & 7 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{bmatrix}$$

by applying elementary row operation the equivalent reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & -10 & -12 & 9 \\ 0 & 1 & 0 & 13/2 & 8 & 11/2 \\ 0 & 0 & 1 & 5/2 & 3 & 5/2 \end{bmatrix}$$

So

$$B^{-1} = \begin{bmatrix} -10 & -12 & 9 \\ 13/2 & 8 & 11/2 \\ 5/2 & 3 & 5/2 \end{bmatrix}$$

Properties of inverse matrix

Let **A, B, C** be matrices of the appropriate sizes so that the following multiplications make sense, **I** a suitably sized identity matrix, and α a nonzero scalar. Then

1. (Uniqueness) The matrix A has at most one inverse, henceforth denoted as A^{-1}, provided A is invertible.

2. (Double Inverse) If A is invertible, then $(A^{-1})^{-1} = A$:

3. (2=3 Rule) If any two of the three matrices **A, B** and **AB** are invertible, then so is the third, and moreover $(AB)^{-1} = B^{-1}A^{-1}$:

4. A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$.

5. If A is invertible, then $(\alpha A)^{-1} = (\frac{1}{\alpha})A^{-1}$:

6. (Inverse/Transpose) If A is invertible, then $(A^t)^{-1} = (A^{-1})^t$.

7. (Cancellation) Suppose A is invertible. If $AB = AC$ or $BA = CA$, then $B = C$:

1.5 Determinant of a matrix and its properties

Definition 1.12: (Determinant of a matrix)

Let A be an $n \times n$ matrix. Then the determinant of A denoted as $det(A)$ or $\mid A \mid$ is defined recursively by:
If $A = [a]$ is a 1×1 matrix, then $det(A) = a$. If A is a matrix of size **n** *within ,$n \geqslant 2$ then*

$$\det(A) = A_{11}det(A_{11}) - A_{12}\det(A_{12}) + A_{13}\det(A_{13}) - \ \cdots \ + (-1)^{n+1}A_{1n}\det(A_{1n})$$

where A_{1j} a sub matrix of A obtaining by deleting the fist row and the j^{th} column.

So to compute the determinant of a 5×5 matrix we must build 5 sub matrices, each of size 4. To compute the determinants of each the 4×4 matrices we need to create 4 sub matrices each, these now of size 3 and so on. To compute the determinant of a 10×10 matrix would require computing the determinant of $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 3,628,800$ 1×1 matrices. Fortunately there are better ways. However this does suggest an excellent computer programming exercise to write a recursive procedure to compute a determinant.
Lets compute the determinant of a reasonable sized matrix by hand.

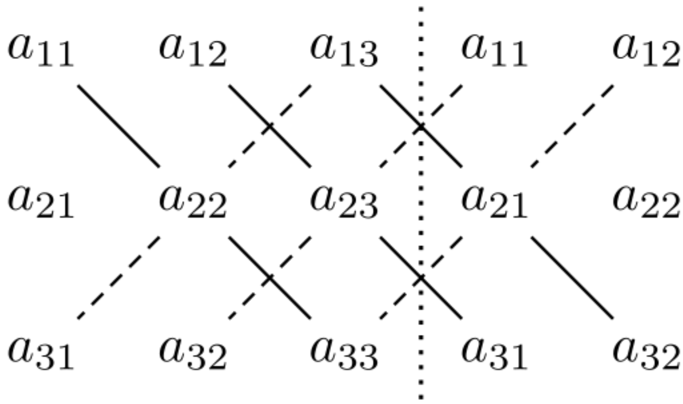
Suppose that we have the 3×3 matrix $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{bmatrix}$ then

$$\begin{aligned} det(A) &= \mid A \mid \\ &= \begin{vmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{vmatrix} \\ &= 3\begin{vmatrix} 1 & 6 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ -3 & 2 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 \\ -3 & -1 \end{vmatrix} \\ &= 3(1\left|2\right| - 6\left|-1\right|) - 2(4\left|2\right| - 6\left|-3\right|) - (4\left|-1\right| - \left|-3\right|) \\ &= 3(1(2) - 6(-1)) - 2(4(2) - 6(-3)) - (4(-1) - (-3)) \\ &= 24 - 52 + 1 \\ &= -27 \end{aligned}$$

Theorem 1.2: (Exchanging Columns Changes the Sign of a Determinant).

If the matrix A' is obtained from A by interchanging any two columns, and their determinants exist, then $|A'| = -|A|$.

The rule of **Sarrus** is a mnemonic for the 3×3 matrix determinant: the sum of the products of three diagonal north-west to south-east lines of matrix elements, minus the sum of the products of three diagonal south-west to north-east lines of elements, when the copies of the first two columns of the matrix are written beside it as in the illustration. This scheme for calculating the determinant of a 3×3 matrix does not carry over into higher dimensions.



Properties of determinants of matrix

1. If A is a triangular matrix, then the determinant of A is the product of all the diagonal elements of A.
2. If B is obtained from A by multiplying one row of A by the scalar α, then $\det(B) = \alpha(\det(A))$.
3. If B is obtained from A by adding a multiple of one row of A to another row of A, then $\det(B) = \det(A)$.
4. The matrix A is invertible if and only if $\det(A) \neq 0$ and $det(A^{-1}) = \dfrac{1}{\det(A)}$.
5. The determinant of a product of two matrices is the product of their determinants. That is,
$$\det(AB) = \det(A)\det(B) \implies \det(A^n) = (\det(A))^n$$
6. If B is the transpose of a matrix A, then $\det(B) = \det(A)$

Minor In a Matrix

Suppose A is an $n \times n$ matrix and A_{ij} is the $(n-1) \times (n-1)$ sub matrix formed by removing row i and column j. Then the **minor** for A at location $i\ j$ is the determinant of the sub matrix, $M_{ij}(A) = det(A_{ij})$.

Co factor In a Matrix

Suppose A is an $n \times n$ matrix and A_{ij} is the $(n-1) \times (n-1)$ sub matrix formed by removing row i and column j. Then the **Co factor** for A at location ij is the determinant of the sub matrix, $C_{ij}(A) = (-1)^{i+j} det(A_{ij})$.

Definition 1.13: (Adjoint)

*If $A = (a_{ij})$ is an $n \times n$ matrix, the adjoint of A, denoted by **adj A**, is the transpose of the matrix of cofactors.*

Hence
$$adj\,A = \begin{bmatrix} c_{11} & c_{21} & c_{31} & \cdots & c_{n1} \\ c_{12} & c_{22} & c_{32} & \cdots & c_{n2} \\ c_{13} & c_{23} & c_{33} & \cdots & c_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & c_{3n} & \cdots & c_{nn} \end{bmatrix}$$

Theorem 1.3

Let A be an $n \times n$ matrix. Then

$$A(adj\ A) = (\det\ A)I_n = (adj\ A)A.$$

Note If the $\det(A) \neq 0$, then $A^{-1} = \frac{1}{\det(A)} adj(A)$

Example 1.25

Let $\begin{bmatrix} 2 & 3 & -1 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$.

1. *Determine*

   (a) *The minors of all elements A.*

   (b) *The co factors of all elements of A.*

   (c) *The $adj(A)$.*

   (d) *The inverse of A.*

Solution: Exercise

1.6 Solving system of linear equations

Definition 1.14

A linear equation in the variables $x_1,\ x_2,\ \cdots,\ x_n$ is an equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ where the coefficients $a_1,\ a_2,\ \cdots,\ a_n$ and right hand side constant term b are given constants.

Definition 1.15

A general system of m linear equations with n unknowns can be written as

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & +\cdots+ & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & +\cdots+ & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & +\cdots+ & a_{mn}x_n & = & b_m. \end{array} \tag{1.1}$$

Here $x_1, x_2, \ldots, x_n$, are the unknowns, $a_{11}, a_{12}, \ldots, a_{mn}$ are the coefficients of the system, and $b_1, b_2, \ldots, b_m$ are the constant terms.

A solution of a linear equation is any n-tuple of values $(s_1, s_2, ..., s_n)$ which satisfies the linear equation. For example, $(-1, -1)$ is a solution of the linear equation $x + 3y = -4$ since$-1 + (3 \times -1) = -1 + (-3) = -4$, but $(1, 5)$ is not.

Similarly, a solution to a linear system is any n-tuple of values $(s_1, s_2, ..., s_n)$ which simultaneously satisfies all the linear equations given in the system.

For example,

$$\begin{aligned} 3x + 2y - z &= 1 \\ 2x - 2y + 4z &= -2 \\ -x + \tfrac{1}{2}y - z &= 0 \end{aligned}$$

has as its solution $(1, -2, -2)$. This can also be written as:

$$\begin{aligned} x &= 1 \\ y &= -2 \\ z &= -2 \end{aligned}$$

We also refer to the collection of all possible solutions as the solution set.

In general, for any linear system of equations there are three possibilities regarding solutions:

1. **A unique solution** In this case only one specific solution set exists. Geometrically this implies the n-planes specified by each equation of the linear system all intersect at a unique point in the space that is specified by the variables of the system.
2. **No solution:** The equations are termed inconsistent and specify n-planes in space which do not intersect or overlap. It is not possible to specify a solution set that satisfies all equations of the system.
3. **An infinite range of solutions:** The equations specify n-planes whose intersection is an m-plane where $m \leq n$. This being the case, it is possible to show that an infinite set of solutions within a specific range exists that satisfy the set of linear equations.

Example 1.26

Given the system of linear equations,

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 7 \\ x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

we have $n = 4$ variables and $m = 3$ equations. Also,

$$\begin{aligned} a_{11} &= 1 a_{12} = 2 & a_{13} &= 0 a_{14} = 1 & b_1 &= 7 \\ a_{21} &= 1 a_{22} = 1 & a_{23} &= 1 a_{24} = -1 & b_2 &= 3 \\ a_{31} &= 3 a_{32} = 1 & a_{33} &= 5 a_{34} = -7 & b_3 &= 1 \end{aligned}$$

Additionally, convince yourself that $x_1 = -2, \quad x_2 = 4, \quad x_3 = 2, \quad x_4 = 1$ is one solution (but it is not the only one!).

Note that the above system can be written concisely as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

we may write the above simultaneous equations as

$$AX = b$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A matrix **A** is called the coefficient matrix of the system, while the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

obtained by adjoining **b** to **A** is called the augmented matrix of the system.
Remark: If $b_i = 0, \quad \forall i = 1,\ 2,\ \cdots,\ m$ then the linear system is called Homogenous otherwise, non Homogenous.

Theorem 1.4

If $[A|b]$ and $[C|d]$ are row equivalent, then the systems $AX = b$ and $CX = d$ have exactly the same solutions.

Remark: If **A** is an **m** by **n** matrix then the linear system $AX = \mathbf{0}$ has trivial solution $X = \mathbf{0}$.

Theorem 1.5

*If **A** is an **m** by **n** matrix then the equation $AX = \mathbf{0}$ has non trivial solution only, when $Rank(A) < n$ otherwise if $Rank(A) = n$ then the trivial solution is unique.*

The system of equation $AX = b$ can be directly solved in the following cases.

1. If $A = D$, the equation (1.1) become

$$\begin{aligned} a_{11}x_1 \quad\quad\quad\quad\quad\quad\quad\quad &= b_1 \\ a_{22}x_2 \quad\quad\quad\quad &= b_2 \\ \ddots \quad\quad &\ \ \vdots \\ a_{nn}x_n &= b_n \end{aligned}$$

   The solution is given by $x_i = \dfrac{b_i}{a_{ii}},\ a_{ii} \neq 0$

2. If $A = L$, the equation (1.1) become

$$\begin{aligned} a_{11}x_1 \quad\quad\quad\quad\quad\quad\quad\quad\quad &= b_1 \\ a_{21}x_1 \quad +a_{22}x_2 \quad\quad\quad\quad\quad &= b_2 \\ \ddots \quad &\ \ \vdots \\ a_{n1}x_1 \quad +a_{n2} \quad +\cdots+ \quad a_{nn}x_n &= b_n \end{aligned}$$

Solving the first equation and then successively solving the 2nd, 3rd and so on.
We obtain $x_1 = \frac{b_1}{a_{11}}, \quad x_2 = \frac{b_2 - a_{21}x_1}{a_{22}}, \; \cdots, \quad x_n = \frac{b_2 - (a_{n1}x_1 + a_{n2} + \cdots + a_{n(n-1)}x_n)}{a_{nn}},$
where $a_{ii} \neq 0, \quad i = 1,\ 2,\ \cdots, n$
This method of solving equation is called forward substitution method.

3. If $A = U$, the equation (1.1) become

$$\begin{array}{ccccccc} a_{11}x_1 & +a_{12}x_2 & +a_{13}x_3 & +\cdots & +a_{1n}x_n & = b_1 \\ & a_{22}x_2 & +a_{23}x_3 & +\cdots & +a_{2n}x_n & = b_2 \\ & & \ddots & & & \vdots \\ & & & & a_{nn}x_n & = b_n \end{array}$$

Solving for the unknowns in the in the order $x_n, \quad x_{n-1}, \cdots, x_1$, we get

$$x_n = \frac{b_n}{a_{nn}}, \quad x_{n-1} = \frac{b_{n-1} - a_{(n-1)nx_n}}{a_{(n-1)(n-1)}}, \quad \cdots, \quad x_1 = \frac{b_1 - (a_{12}x_2 + a_{13}x_3 + \cdots \ + a_{1n}x_n)}{a_{11}}$$

This method of solving equation is called backward substitution method. Therefore, matrix **A** is solvable if it can be transformed in to any one of the forms **D**, **U**, **L**.

Theorem 1.6

Consider an ***m*** *equations with* ***n*** *variables $AX = b$ then*

a) *If $rank(A|b) = rank(A) = n$ then the system has unique solution*

b) *If $rank(A|b) = rank(A) < n$ then the system has infinitely many solutions.*

c) *If $rank(A|b) > n$ then the system has no solution.*

To solve a linear system , we have the following Methods;

1.6.1 Cramer's rule

If $AX = b$ is a linear system of **n** equations in **n** unknowns, and if $detA \neq 0$, then the system has unique solution which can be determined by. $x_i = \frac{|\ A_i\ |}{|\ A\ |}, \quad i = 1,\ 2,\ \ \cdots,\ n$
Where A_i is the matrix obtained from **A** when i^{th} column of A is replaced by **b**.
Consider the linear system

$$\begin{cases} a_1x + b_1y & = c_1 \\ a_2x + b_2y & = c_2 \end{cases}$$

which in matrix format is

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Assume $a_1b_2 - b_1a_2$ nonzero. Then, with help of determinants x and y can be found with Cramer's rule as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{c_1b_2 - b_1c_2}{a_1b_2 - b_1a_2}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{a_1c_2 - c_1a_2}{a_1b_2 - b_1a_2} \quad .$$

The rules for 3×3 matrices are similar. Given

$$\begin{cases} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{cases}$$

which in matrix format is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

Then the values of x, y and z can be found as follows:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \text{ and } z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Example 1.27

Solve the following system by cramer's rule.

$$\begin{gathered} 2x_1 + 3x_2 + 4x_3 = 19 \\ x_1 + 2x_2 + x_3 = 4 \\ 3x_1 - x_2 + x_3 = 9 \end{gathered}$$

Solution: *The coefficient matrix is*

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \\ 3 & -1 & 1 \end{bmatrix}$$

and column matrix

$$b = \begin{bmatrix} 19 \\ 4 \\ 9 \end{bmatrix}$$

, then

$$det(A) = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 4 + 9 - 4 - 24 - 3 + 2 = -16 \neq 0$$

then the system has unique solution.

Example: Cont.

$$A_1 = \begin{bmatrix} 19 & 3 & 4 \\ 4 & 2 & 1 \\ 9 & -1 & 1 \end{bmatrix} \,\&\, \det(A_1) = \begin{vmatrix} 19 & 3 & 4 \\ 4 & 2 & 1 \\ 9 & -1 & 1 \end{vmatrix} = 38 + 27 - 16 - 72 - 12 + 19 = -16$$

$$A_2 = \begin{bmatrix} 2 & 19 & 4 \\ 1 & 4 & 1 \\ 3 & 9 & 1 \end{bmatrix} \,\&\, \det(A_2 = \begin{vmatrix} 2 & 19 & 4 \\ 1 & 4 & 1 \\ 3 & 9 & 1 \end{vmatrix} = 8 + 57 + 36 - 48 - 19 - 18 = 16$$

$$A_3 = \begin{bmatrix} 2 & 3 & 19 \\ 1 & 2 & 4 \\ 3 & -1 & 9 \end{bmatrix} \,\&\, \det(A_3) = \begin{vmatrix} 2 & 3 & 19 \\ 1 & 2 & 4 \\ 3 & -1 & 9 \end{vmatrix} = 36 + 36 - 19 - 114 - 27 + 8 = -80$$

$$\therefore\ x_1 = \frac{\det(A_1)}{det(A)} = \frac{-16}{-16} = 1$$
$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{16}{-16} = -1$$
$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-80}{-16} = 5.$$

This is the solution of the system.

1. **Problem 1:** Use Cramer's Rule to solve each for each of the variables.

$$\begin{array}{rcrcr} x & - & y & = & 4 \\ -x & + & 2y & = & -7 \end{array}$$

$$\begin{array}{rcrcr} -2x & + & y & = & -2 \\ x & - & 2y & = & -2 \end{array}$$

2. **Problem 2:** Use Cramer's Rule to solve this system for z.

$$\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 1 \\ 3x & & & + & z & = & 4 \\ x & - & y & - & z & = & 2 \end{array}$$

1.6.2 Gaussian elimination method

Gauss elimination method is used to solve system of linear equations. In this method the linear system of equation is reduced to an upper triangular system by using successive elementary row operations. Finally we solve the value variables by using back ward substitution method. This method will be fail if any of the pivot element $a_{ii},\ \ i = 1,\ 2,\ \cdots,\ n$ becomes zero. In such case we re-write equation in such manner so that pivots are non zero. This procedure is called pivoting.
Consider system $AX = b$

Step 1: Form the augmented matrix $[A|b]$

Step 2: Transform $[A|b]$ to row echelon form $[U|d]$ using row operations.

Step 3: Solve the system $UX = d$ by back substitution.

Example 1.28

Solve the following system using Gauss elimination method.

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 5 \\ 4x_1 + 14x_2 + 12x_3 &= 10 \\ 6x_1 + x_2 + 5x_3 &= 9 \end{aligned}$$

***Solution**: The augmented matrix of the system is*

$$\begin{bmatrix} 2 & -3 & 1 & 5 \\ 4 & 14 & 12 & 10 \\ 6 & 1 & 5 & 9 \end{bmatrix}$$

Applying, elementary row operations on this matrix to change into its echelon form.

$$\begin{bmatrix} 2 & -3 & 1 & 5 \\ 4 & 14 & 12 & 10 \\ 6 & 1 & 5 & 9 \end{bmatrix} \begin{matrix} R_2 \longrightarrow R_2 - 2R_1 \\ R_3 \longrightarrow R_3 - 3R_1 \end{matrix} \begin{bmatrix} 2 & -3 & 1 & 5 \\ 0 & 20 & 10 & 0 \\ 0 & 10 & 2 & -6 \end{bmatrix}$$

$$R_3 \longrightarrow R_3 - 1/2R_2 \begin{bmatrix} 2 & -3 & 1 & 5 \\ 0 & 20 & 10 & 0 \\ 0 & 0 & -3 & -6 \end{bmatrix}$$

Since $rank(\underset{-}{A}) = rank(A) = 3 = n$ the solution exists and is unique.

$$\begin{aligned} 2x_1 \quad -3x_2 \quad +x_3 &= 5 \\ 20x_2 \quad +10x_3 &= 0 \\ -3x_3 &= -6 \end{aligned}$$

From this we get $x_3 = 2$. And using back substitution we have $x_2 = -1$ and $x_1 = 0$
Hence $(0, -1, 2)$ is the solution of the system.

1.6.3 Inverse matrix method

Let $AX = b$ is a system of **n** linear equations with **n** unknowns and A is invertible, then the system has unique solution given by inversion method $X = A^{-1}b$.
Note:- When A is not square or is singular, the system may not have a solution or may have more than one solution.

Example 1.29

Use the inverse of the coefficient matrix to solve the following system

$$3x_1 + x_2 = 6$$
$$-x_1 + 2x_2 + 2x_3 = -7$$
$$5x_1 - x_3 = 10$$

Solution: *Okay, let's first write down the matrix form of this system.*

$$\begin{bmatrix} 3 & 9 & 0 \\ -1 & 2 & 2 \\ 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 10 \end{bmatrix}$$

Now, we found the inverse of the coefficient matrix by using methods of finding Inverses and is the following;

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 2 \\ 5 & 0 & -1 \end{bmatrix}$$
$$\implies C_A = \begin{bmatrix} -2 & 9 & -10 \\ 1 & 3 & 5 \\ 2 & -6 & 7 \end{bmatrix}$$
$$\implies adj(A) = \begin{bmatrix} 2 & -1 & 2 \\ 9 & -3 & -6 \\ -10 & 5 & 7 \end{bmatrix}$$

and $det(A) = 3(-2) + 1(9) + 0(-10) = -6 + 9 = 3$, then

$$A^{-1} = 1/3 \begin{bmatrix} 2 & -1 & 2 \\ 9 & -3 & -6 \\ -10 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ 3 & -1 & -2 \\ -10/3 & 5/3 & 7/3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ 3 & -1 & -2 \\ -10/3 & 5/3 & 7/3 \end{bmatrix} \begin{bmatrix} 6 \\ -7 \\ 10 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 5 \\ -25/3 \end{bmatrix}$$

Now each of the entries of X are $x_1 = 1/3,\ x_2 = 5$ and $x_3 = -25/3$

Exercise 1.3

1. *Solve the following linear system of equation by using Cramer's rule, Gaussian elimination method, and inverse method.*

(a) $\begin{aligned} 2x_1 + 5x_2 + 3x_3 &= 9 \\ 3x_1 + x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 - x_3 &= 6 \end{aligned}$ (b) $\begin{aligned} x + z &= 1 \\ 2x + y + z &= 0 \\ x + y + 2z &= 1 \end{aligned}$ (c) $\begin{aligned} x + 2y + z &= 3 \\ 2x + 5y - z &= -4 \\ 3x - 2y - z &= 5 \end{aligned}$

2. *Use rank of matrix to determine the values of a, b and c so that the following system has:*

(a) *no solution* (b) *more than one solution* (c) *a unique solution and solve it.*

i) $\begin{aligned} 1x + y - bz &= 1 \\ 2x + 3y + az &= 3 \\ x + ay + 3z &= 2 \end{aligned}$ ii) $\begin{aligned} x + 2y - 3z &= a \\ 2x + 6y - 11z &= b \\ x - 2y + 7z &= c \end{aligned}$ iii) $\begin{aligned} x - 2y + bz &= 3 \\ ax + 2z &= 2 \\ 5x + 2y &= 2 \end{aligned}$

1.7 Eigenvalues and Eigenvectors

Definition 1.16: (Eigenvalue, eigenvector)

Let A be a square matrix. Then if λA, is a real number and $\mathbf{X}$ a non zero column vector satisfying $AX = \lambda X$, we call $\mathbf{X}$ an eigenvector of A, while λ is called an eigenvalue of A. We also say that X is an eigenvector corresponding to the eigenvalue λ.

Example 1.30

Let $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, *then show that* $X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ *is eigenvector of* $\boldsymbol{A}$ *with* $\lambda = 1$.

If λ is an eigenvalue of an $n \times n$ matrix $\mathbf{A}$, with corresponding eigenvector $\mathbf{X}$, then $(A - \lambda I_n)X = 0$, with $X \neq 0$, so $det(A - \lambda I_n) = 0$ and there are at most $\mathbf{n}$ distinct eigenvalues of A. Conversely if $det(A - \lambda I_n) = 0$, then $(A - \lambda I_n)X = 0$ has a nontrivial solution $\mathbf{X}$.

The equation $det(A - \lambda I_n) = 0$ is called the **characteristic equation** of A, while the polynomial $det(A - \lambda I_n)$ is called the **characteristic polynomial** of $\mathbf{A}$. The characteristic polynomial of A is often denoted by $ch_A(\lambda)$.
Hence the eigenvalues of A are the roots of the characteristic polynomial of A.

Example 1.31

Find the eigenvalues for the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Theorem 1.7

if A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.

Example 1.32

Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ *and find all eigenvectors.*

Solution: *The characteristic equation of A is $\lambda^2 - 4\lambda + 3 = 0$, or $(\lambda - 1)(\lambda - 3) = 0$ Hence $\lambda = 1$ or $\lambda = 3$. The eigenvector equation $(A - I_n)X = 0$ reduces to*

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{aligned} (2-\lambda)x + y &= 0 \\ x + (2-\lambda)y &= 0 \end{aligned}$$

Taking $\lambda = 1$ gives

$$\begin{aligned} x + y &= 0 \\ x + y &= 0. \end{aligned}$$

which has solution $x = -y$, and let $y = t$ is arbitrary non zero. Consequently the eigenvectors corresponding to $\lambda = 1$ are the vectors

$$\begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

with $t \neq 0$ which is the scalar multiple of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Taking $\lambda = 3$ gives

$$\begin{aligned} -x + y &= 0 \\ x - y &= 0. \end{aligned}$$

which has solution $x = y$, and let $y = t$ is arbitrary non zero. Consequently the eigenvectors corresponding to $\lambda = 3$ are the vectors

$$\begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with $t \neq 0$ hence the scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Therefore $\lambda_1 = 1$ and $\lambda_2 = 3$ are the eigenvalues of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ *and the corresponding eigenvector are* $X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ *and* $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *respectively.*

Theorem 1.8

If A is an $n \times n$ matrix, the following statements are equivalent.

(a) λ *is an eigenvalue of* A.

(b) *The system of equations* $(A - \lambda I)\mathbf{x} = 0$ *has nontrivial solutions.*

(c) *There is a nonzero vector* $\mathbf{x}$ *such that* $A\mathbf{x} = \lambda\mathbf{x}$.

(d) λ *is a solution of the characteristic equation* $\det(A - \lambda I) = 0$.

1.7.1 Diagonalization

Problem 1 Given an $n \times n$ matrix A, does there exist an invertible matrix P such that

$$P^{-1}AP$$

is diagonal?

Problem 2 Given an $n \times n$ matrix A, does A have n linearly independent eigenvectors?

Theorem 1.9

Let $\mathbf{A}$ *be an* $n \times n$ *matrix having distinct eigenvalues* $\lambda_1,\ \lambda_2, \cdots, \lambda_n$ *and corresponding eigenvectors* $X_1,\ X_2,\ \cdots\ X_n$ *respectively. Let* P *be the matrix whose columns are respectively* $X_1,\ X_2,\ \cdots,\ X_n$. *Then* P *is non singular and*

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Definition 1.17

If A *and* B *are square matrices, then we say that* B *is* **similar** *to* A *if there is an invertible matrix* P *such that*

$$B = P^{-1}AP.$$

Definition 1.18

A square matrix A *is said to be* **diagonalizable** *if it is similar to some diagonal matrix. In other words,* A *is diagonalizable if there exists an invertible matrix* P *such that* $P^{-1}AP$ *is diagonal. In this case the matrix* P *is said to* **diagonalize** A.

Theorem 1.10

If A *is an* $n \times n$ *matrix, the following statements are equivalent.*

(a) A *is diagonalizable.*

(b) A *has n linearly independent eigenvectors.*

Procedure for Diagonalizing a Matrix

1. Confirm that the matrix is actually diagonalizable by finding n linearly independent eigenvectors. One way to do this is by finding a basis for each eigenspace and merging these basis vectors into a single set S. If this set has fewer than n vectors, then the matrix is not diagonalizable.

2. Form the matrix
$$P = [\mathbf{p}_1\ \mathbf{p}_2\ \cdots\ \mathbf{p}_n]$$
that has the vectors in S as its column vectors.

3. The matrix $P^{-1}AP$ will be diagonal and have the eigenvalues
$$\lambda_1,\ \lambda_2 \cdots, \lambda_n$$
corresponding to the eigenvectors
$$\mathbf{p}_1,\ \mathbf{p}_2,\ \cdots, \mathbf{p}_n$$
as its successive diagonal entries.

Example 1.33

In each of the following, determine if the the matrix is diagonalizable

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$