Applied II Mathematics Handout

Chapter One

1. Sequence and Series

1.1 Sequence.

1.1.1 Definition and types of sequence

Definition: A sequence is a list of numbers called terms in a specified order. And denoted by $\{a_n\}$ where a_n is called the n^{th} term or general term of the sequence. or

Simply it is defined as a function whose domain is the set of natural number

A sequence can be finite or infinite. A ***finite sequence*** has a last term and an infinite sequence has no last term

$\{a_n\} = a_1, a_2, a_3, ... a_n, a_{n+1} …$ is called an infinite sequence. Whereas,

$\{a_n\} = a_1, a_2, a_3, ... a_n$. is called finite sequence.

Types of sequence

- **An arithmetic sequence** is a sequence in which the difference between successive terms is a fixed number and each term is obtained by adding a fixed amount to the term before it. This fixed amount is called the ***common difference.*** Arithmetic sequences can be represented by first-degree polynomial expressions.

A finite arithmetic sequence can be expressed as:

$a , a + d , a + 2d , a + 3d , a + 4d , a + 5d,. .., a + (n - l)d$

where **a** is the first term, **d** is the difference between each term, and **a + (n - 1)d** is the last or "n^{th}" term.

Example; {3, 6, 9, 12, 15, 18} is an *arithmetic sequeence with a=3 and d = 3*

- A **geometric sequence** is a sequence in which the ratio of successive terms is a fixed number, and each term is obtained by multiplying a fixed amount to the term before it. This fixed amount is called the **common ratio**.

Terms in a geometric sequence can be represented as:

$$a, ar, ar^2, ar^3, ar^4, ar^5, ..., ar^{n-l}$$

where a is the first term, ar^{n-1} is the last term and the ratio of successive terms is given by **r**

such that:

$$ar/a = r,\ ar^2/ar = r,\ ar^3/ar^2 = r, \text{etc.}$$

Example {2, 4, 8, 16, 32 ,... }, with a = 2 and r = 2.

1.1.2 Convergence properties of sequence.

Definition; A real number L is said to be a limit of a sequence $\{an\}n\epsilon\mathbb{N}$ if and only if,

for all $\epsilon > 0$ *there exists a postive integer N such that*;

$$|a_n - L| < \epsilon \ \ for\ all\ n > N$$

We write as

$$\lim_{n\to\infty} a_n = L$$

And *the sequence* $\{a_n\}$ is a convergence sequence

Note: that this definition holds we have to:

- Guess the value of the limit `
- Assume $\epsilon > 0$ has been given,
- Find $N \epsilon \mathbb{N}$ such that $|a_n - L| < \epsilon$

  i.e $L- \epsilon < a_n < L + \epsilon$ for all $n \geq N$

If $\lim_{n\to\infty} a_n$ doesn't exist, we say that $\{a_n\}$ diverges.

$\lim_{n\to\infty} a_n = \infty$, means that the sequence $\{a_n\}$ diverges to infinity. *i.e* if for every number M, there is an integer N, such that for all n $> N, a_n > M$

Similarly; if for every number m, there is an integer N, such that for all n $> N$, we have $a_n < m$,

Then we say $\{a_n\}$ diverges to negative infinity and we write

$$\lim_{n\to\infty} a_n = -\infty, or \ a_n \to -\infty$$

Generally: A sequence which has a limit is said to be convergent and A sequence with no limit is called divergent.

Theorem 1:1 if the sequence of a real numbers {an}n$\epsilon\mathbb{N}$ has a limit then, this limit is unique.

Proof; assume let {an}n$\epsilon\mathbb{N}$ denote a convergence sequence with two limits say L_1 and L_2

with $L_1 \neq L_2$

Now choose $\epsilon = \frac{1}{3}|L_1 - L_2|$

Since $L_1$is a limit of {an}n$\epsilon\mathbb{N}$, then to find $N_1 \in \mathbb{N}$ such that

$$|a_n - L_1| < \epsilon \ for \ all \ n \geq N_1$$

Similarly;

Since L_2 $is\ a\ limit\ of$ {an}n$\epsilon\mathbb{N}$ then, to find $N_2 \in \mathbb{N}$ such that

$$|a_n - L_2| < \epsilon \ for \ all \ n \geq N_2$$

Choose any n $\geq$ max $\{N_1, N_2\}$ then

$$\begin{aligned} |L_1 - L_2| &= |L_1 - a_n + a_n - L_2| \\ &\leq |L_1 - a_n| + |a_n - L_2| \\ &< \epsilon + \epsilon \\ &= 2\epsilon \quad \text{but from the choice of } \epsilon = \frac{1}{3}|L_1 - L_2| \\ &= \frac{2}{3}|L_1 - L_2| \end{aligned}$$

$|L_1 - L_2| < \frac{2}{3}|L_1 - L_2|$, $L_1 \neq L_2$, This contradicts

Therefore our assumption is false, so the theorem is true.

Limit properties for sequences

If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist, then the following properties hold true;

- $\lim_{n\to\infty} ca_n = c(\lim_{n\to\infty} a_n)\ for\ any\ constant\ c.$
- $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n.$
- $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} a_n).$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}\ if\ b_n \neq 0\ for\ all\ n$

$$and\ \lim_{n\to\infty} b_n \neq 0$$

The next three theorems are often helpful in finding limits of sequences.

Theorem1.2

If $\lim_{n\to\infty} a_n = L$, $and\ f\ is\ a\ function\ whose\ domain\ includes$ $L\ and\ \ a_n\ for\ n \geq N, and\ if\ f\ is\ continuous\ at\ x = L, then$;

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Let $f(x) = x^k for\ k\ a\ postive\ integer, is\ continuous$ $for\ all\ x, we\ have$;

$$\lim\ (_{n\to\infty} a_n)^k = L^k.$$

Provided the sequence $\{a_n\}$ converges to L. similarly,

$$\lim_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{L}.$$

Provided $a_n > 0 and\ L > 0\ for\ even\ ordered\ k^{th}\ roots.$

Theorem 1.3 Let $\{a_n\}$ be a sequence and f a function such that,

$$f(n) = a_n, \qquad n = 1, 2, 3, \dots$$

If

$$\lim_{x\to\infty} f(x) = L.$$

Then also, $\lim_{n\to\infty} a_n = L.$

Example, find the limit of each of the following sequences.

$\{\frac{\ln n}{n}\}$ b. $\{\frac{\ln(2+e^n)}{3n}\}$ c. $\{(1+3n)^{\frac{1}{n}}\}$

Solution,

a. $a_n = \frac{\ln n}{n}$, Let $f(x) = \frac{\ln x}{x}$

$\Rightarrow f(n) = \frac{\ln n}{n} = a_n$. Then by **Theorem 1.3**

$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x}$

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$ (Applying L'hopital's rule)

b. $a_n = \frac{\ln(2+e^n)}{3n}$, $let\ f(x) = \frac{\ln(2+e^x)}{3x}$,

$\Rightarrow f(n) = \frac{\ln(2+e^n)}{3n}$. Then by **Theorem 1.3**

$\lim_{n\to\infty} \frac{\ln(2+e^n)}{3n} = \lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) =$

$\lim_{x\to\infty} \frac{\ln(2+e^x)}{3x} = \lim_{x\to\infty} \frac{e^x/(2+e^x)}{3x} = \lim_{x\to\infty} \frac{1}{6e^{-x}+3}$

$= \frac{1}{3}$(Applying L'hopital's rule)

c. $a_n = (1+3n)^{\frac{1}{n}}$.

let $y = (1+3x)^{\frac{1}{x}} \Rightarrow \ln y = \ln(1+3x)^{\frac{1}{x}} = \frac{\ln(1+3x)}{x}$

$\Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+3x)}{x}$

$= \lim_{x\to\infty} \frac{3/(1+3x)}{1}$

$\ln \lim_{x\to\infty} y = 0$

$\lim_{x\to\infty} y = e^0 = 1$

$\lim_{x\to\infty}(1+3x)^{\frac{1}{x}} = 1.$

Then by **Theorem 1.3**, $\lim_{n\to\infty}(1+3n)^{\frac{1}{n}} = 1$

Theorem 1.4 The Squeeze Theorem for Sequence.

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L\ and\ if\ for\ all\ sufficiently$ Large n the inequality $a_n \le c_n \le b_n$ holds true, then;

$$\lim_{n\to\infty} c_n = L$$

Example; Find the limit of the sequences.

a. $\left\{\frac{nsin\ n}{1+n^2}\right\}$ b. $\left\{\frac{3+(-1)^n}{n^2}\right\}$ c. $\{\sqrt{n+2}-\sqrt{n}\}$.

Solution;

a; $a_n = \frac{nsin\ n}{1+n^2} \Rightarrow \left|\frac{nsin\ n}{1+n^2}\right| \le \frac{n}{1+n^2} < \frac{n}{n^2} = \frac{1}{n}$

$\Rightarrow -\frac{1}{n} \le \frac{nsin\ n}{1+n^2} \le \frac{1}{n}$

$$\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

Then by **Squeeze Theorem**

$$\lim_{n\to\infty} \frac{nsin\ n}{1+n^2} = 0$$

b. $a_n = \frac{3+(-1)^n}{n^2} \Rightarrow 0 < \frac{3+(-1)^n}{n^2} \le \frac{4}{n^2}$

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{4}{n^2} = 0$$

Then by **Squeeze Theorem**

$$\lim_{n\to\infty} \frac{3+(-1)^n}{n^2} = 0$$

c. $a_n = \sqrt{n+2}-\sqrt{n}$. $\Rightarrow (\sqrt{n+2}-\sqrt{n})\left(\frac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n}}\right) =$

$\frac{n+2-n}{\sqrt{n+2}+\sqrt{n}} < \frac{2}{2\sqrt{n}} = \frac{1}{\sqrt{n}}$

$$\Longrightarrow 0 < \sqrt{n+2} - \sqrt{n} < \frac{1}{\sqrt{n}}.$$

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Then by **Squeeze Theorem**

$$\lim_{n\to\infty} \sqrt{n+2} - \sqrt{n} = 0$$

Recursive Definition of Sequence.

Sometimes sequences are defined **recursively** by giving

- The value of the initial term or terms, and
- A rule called a recursion formula, for calculating any later term from terms that precede it.
  i.e the formula giving a_n in terms of a_{n-1} is called recursion formula.

The best known sequence defined recursively is the **Fibonacci sequence,** defined by

$f_1 = 1, f_2 = 1, \ and \ f_n = f_{n-1} + f_{n-2} \quad for \ n \geq 2$

The number f_n are called **Fibonacci numbers**.

Monotonicity and Boundedness

Definition; A sequence $\{a_n\}$ is said to be

- Increasing if $a_n \leq a_{n+1} for \ all \ postive \ integer \ n$
- Decreasing if $a_n \geq a_{n+1} \ for \ all \ postive \ integer \ n.$
- A sequence that is either always increasing or always decreasing is said to be monotone.

Example: show that each of the following sequence is monotone.

a. $\left\{\frac{2n+3}{n}\right\}$ b. $\left\{\frac{n}{\sqrt{1+n^2}}\right\}$ c. $\left\{\frac{n!}{n^n}\right\}$

Solution: a, $a_n = \frac{2n+3}{n}$ and $a_{n+1} = \frac{2(n+1)+3}{n+1}$

$$a_{n+1} - a_n = \frac{2(n+1)+3}{n+1} - \frac{2n+3}{n}$$
$$= \frac{(2n+5)n - (2n+3)(n+1)}{n(n+1)}$$
$$= \frac{2n^2 + 5n - (2n^2 + 5n + 3)}{n(n+1)} = \frac{-3}{n(n+1)} < 0$$

$$a_{n+1} - a_n < 0, \Longrightarrow a_{n+1} < a_n$$

$$\Longrightarrow a_n \ is \ strictly \ decreasing, so \ it \ is \ monotone.$$

b, $a_n = \frac{n}{\sqrt{1+n^2}}$

Consider a function for which $f(n) = a_n$

$$f(x) = \frac{x}{\sqrt{1+x^2}}$$

Taking its derivative, we have $f'(x) = \frac{1+x^2-x^2}{(1+x^2)^{\frac{3}{2}}} = \frac{1}{(1+x^2)^{\frac{3}{2}}} > 0$

$$f'(x) > 0 \ for \ all \ x, \Longrightarrow f \ is \ an \ increasing \ function.$$

Thus since $f(n) = a_n$, we see that $\{a_n\}$ is also increasing, so it is monotone.

Tests for monotonicity

1. $if \begin{cases} a_{n+1} - a_n \geq 0 \ for \ all \ n, then \ \{a_n\} \ is \ increasing \\ a_{n+1} - a_n \leq 0 \ for \ all \ n, then \{a_n\} \ is \ decreasing \end{cases}$
2. $Let \ f(x) be \ continuous \ function \ with \ f(n) = a_n.$
   $calculate \ f'(x) \ if \ it \ exists.$
3. $If \begin{cases} f'(x) \geq 0 \ on [1, \infty), then \ \{a_n\} \ is \ increasing. \\ f'(x) \leq 0 \ on \ [1, \infty), then \ \{a_n\} \ is \ decreasing \end{cases}$
4. $if \ a_n > 0 \ for \ all \ n, calculete \ the \ ratio \ \frac{a_{n+1}}{a_n}.$

$$if \begin{cases} \frac{a_{n+1}}{a_n} \geq 1 \;\; for\ all\ n, then\ \{a_n\}\ is\ increasing. \\ \frac{a_{n+1}}{a_n} \leq 1 \;\; for\ all\ n, then\ \{a_n\}\ is\ decreasing. \end{cases}$$

Definition:

A sequence $\{a_n\}$ is said to be bounded if there is some positive constant number M such that

$$|a_n| \leq M$$

$for\ all\ positive\ integer\ n.$

A sequence $\{a_n\}$ is said to be bounded from;

- Above, if there is some real number M, such that, $a_n \leq M$ for all n, M is upper bound for $\{a_n\}$ and no number less than M is an upper bound for $\{a_n\}$, then M is the least upper bound for $\{a_n\}$.
- Below, if there is some real number m , such that, $a_n \geq m$ for all n, m is a lower bound for $\{a_n\}$ and no number greater than m is a lower bound for $\{a_n\}$, then m is the greatest lower bound for$\{a_n\}$.
- If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is bounded. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is unbounded sequence.

Note: convergence of a power sequence

If r is fixed number such that

- $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$
- $r = 1$, then $\lim_{n\to\infty} r^n = 1$
- For all other value of r, the sequence diverges.

Definition; A sequence $\{a_n\}$ of real numbers is called a Cauchy sequence if for each $\epsilon > 0$ there is a number $N \in \mathbb{N}$ so that if m; n > $\mathbb{N}$ then $|a_n - a_m| < \epsilon$.

Note; Convergent sequences are Cauchy sequences.

Proof: Suppose that lim a_n = L. Note that

$|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |a_m - L|$. Thus, given any $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that if k > N then $|a_k - L| < \epsilon$. Thus, if m; n > N we have

$|a_n - a_m| \leq |a_n - L| + |a_m - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Thus, $\{a_n\}$ is a Cauchy sequence.

Theorem 1.5: Monotone Bounded Sequence Theorem

If $\{a_n\}$ is a sequence of real numbers that is both monotone and bounded, then it is converges.

Theorem 1.6 Every convergent sequence is bounded. But the converse is not always true.

Proof: Let $\{a_n\}_{n\geq 1}$ converge to a. Then there exists an $N \in \mathbb{N}$ such that $|a_n - a| < 1 = \epsilon$ for $n \geq N$. It follows that $|a_n| < 1 + |a|$ for $n \geq N$. Define $M = \max\{1+|a|, |a_1|, |a_2|, \ldots |a_{n-1}|\}$. Then $|a_n| < M$ for every $n \in \mathbb{N}$.

To see that the converse is not true, it suffices to consider the sequence $\{(-1)^n\}_{n\geq 1}$, which is bounded but not convergent, although the odd terms and even terms both form convergent sequences with different limits.

Example: show that the sequence $\left\{\frac{1.3.5....(2n-1)}{2.4.6....(2n)}\right\}$ converges.

Solution; the first few terms of this sequence are,

$a_1 = \frac{1}{2}$ $\quad a_2 = \frac{1.3}{2.4} = \frac{3}{8}$ $\quad a_3 = \frac{1.3.5}{2.4.6} = \frac{15}{48} = \frac{5}{16}$

$a_4 = \frac{1.3.5.7}{2.4.6.8} = \frac{35}{128} \ldots$

$\frac{1}{2} > \frac{3}{8} > \frac{5}{16} > \frac{35}{128} > \cdots$

$\Rightarrow$ *the sequence is decreasing(i.e it is monotonic)*

Generally;

we can show that, $a_{n+1} < a_n$, $\Rightarrow \frac{a_{n+1}}{a_n} < 1$, $a_n > 0$ *for all n*

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1.3.5.\ldots(2(n+1)-1)}{2.4.6.\ldots(2(n+1))}}{\frac{1.3.5.\ldots(2n-1)}{2.4.6.\ldots(2n)}}$$

$$= \frac{1.3.5.\ldots(2n+1)}{2.4.6.\ldots(2n+2)} \cdot \frac{2.4.6.\ldots(2n)}{1.3.5.\ldots(2n-1)} = \frac{2n+1}{2n+2} < 1$$

$\frac{a_{n+1}}{a_n} < 1 \Rightarrow a_{n+1} < a_n$ *for any* $n > 0$. *hence* $\{a_n\} = \left\{\frac{1.3.5.\ldots(2n-1)}{2.4.6.\ldots(2n)}\right\}$ is a decreasing sequence.

$a_n > 0$ *for all n it follows that* $\{a_n\}$ *is bounded below by* 0 .

Thus by MBCT

$\{a_n\} = \left\{\frac{1.3.5.\ldots(2n-1)}{2.4.6.\ldots(2n)}\right\}$ Converges.

1.1.3 Subsequence;

Definition: Let $\{a_n\}$ be a sequence. When we extract from this sequence only certain elements and drop the remaining ones we obtain a new sequences consisting of an infinite subset of the original sequence. That sequence is called a **subsequence** and denoted by $\{a_{nk}\}$.

Theorem 1.6;

- If $\{a_n\}$ is a convergent sequence, then every subsequence of that sequence converges to the same limit.
- If is a sequence such that every possible subsequence extracted from that sequences converges to the same limit, then the original sequence also converges to that limit.
- Let $\{a_n\}$ be a sequence of real numbers that is bounded. Then there exists a subsequence $\{a_{nk}\}$ that converges.

1.2 Infinite Series.

Definition; given a sequence of numbers a_n, an expiration of the form

$$a_1 + a_2 + a_3 + a_4 + \ldots$$

is an infinite series. The number a_n is the nth term of the series. The sequence $\{s_n\}$ defined by

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$
$$\vdots$$
$$s_n = a_1 + a_2 + a_3 + a_4 \ldots a_n = \sum_{k=1}^{n} a_k$$
$$\vdots$$

is the sequence of partial sums of the series, the number s_n being the nth partial sum.

- If the sequence of partial sums converges to a limit L (i.e; $\lim_{n\to\infty} s_n = L$), we say that the series converges and that its sum is L. we also write

$$a_1 + a_2 + a_3 + a_4 \ldots + a_n = \sum_{k=1}^{n} a_k = L$$

- If the sequence of partial sums of the series does not converge, (i.e; $\lim_{n\to\infty} s_n = \infty$ *or does not exist*) , we say that the series diverges
- In general

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}(a_1 + a_2 + a_3 + a_4 \ldots a_n)$$

Provided the limit on the right exist, i.e $\lim_{n\to\infty} s_n = s$
Given any positive number ϵ, there is a positive number N such that for all n > N, $|s_n - s| < \epsilon$

Geometric Series

A geometric series is an infinite series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots,$$

in which **a** is its first term with a$\neq 0$ and **r** is called the common ratio

If r = 1 the n[th] partial sum of the geometric series is,

$$s_n = a + a(1) + a(1)^2 + a(1)^2 + \cdots + a(1)^{n-1} = na.$$

And the series diverge because;

$$\lim_{n\to\infty} s_n = \pm\infty, \textit{depend on } \text{the sign of a.}$$

If r = -1 the series diverges because the n[th] partial sums alternate between **a** and **0**

If r $\neq 1$ we can determine the convergence or divergence of the series in the following way;

$$s_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$
$$rs_n = ra + ar^2 + ar^3 + ar^4 + \cdots + ar^n$$
$$s_n - rs_n = a - ar^n$$
$$s_n(1-r) = a(1-r^n)$$
$$s_n = \frac{a(1-r^n)}{(1-r)} \qquad r \neq 1$$

If $|r| < 1$, the geometric series $a + ar + ar^2 + ar^3 + \cdots$ converges to $\frac{a}{1-r}$ *since* $r^n \to 0$ *as* n→ ∞, and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If $|r| > 1$, the geometric series diverges.

Example; determine whether each of the following series is convergent or divergent. If convergent find the sum.

a. $2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$

b. $\sum_{n=1}^{\infty} (\frac{5}{4})^n$

Solution:

a. The series is geometric with a=2 and r = $-1 \div 2 = -\frac{1}{2}$,

$\left|-\frac{1}{2}\right| = \frac{1}{2} < 1$

Therefore the series is converges to

$$\frac{a}{1-r} = \frac{2}{1 + 1/2} = 4/3$$

b. $\sum_{n=1}^{\infty} (\frac{5}{4})^n = \sum_{n=1}^{\infty} \frac{5}{4} (\frac{5}{4})^{n-1}$ the series is geometric with a = $\frac{5}{4}$ and r = $\frac{5}{4}$

$$\left|\frac{5}{4}\right| = \frac{5}{4} > 1$$

Therefore the series is diverges and it has no sum.

Example: Find the rational number represented by the repeating decimal 0.784784784 . . .

Solution. We can write
0.784784784... = 0.784 + 0.000784 + 0.000000784 +. ..
so the given decimal is the sum of a geometric series with a =0.784 and r = 0.001. Thus.

$$0.784784784\ldots = \frac{a}{1-r} = \frac{0.784}{1-0.001} = \frac{784}{999}$$

Theorem 1.7

1. If $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$, but not the converse.

   Proof: let s_n *the* n^{th} *partial sum of* $\sum_{n=1}^{\infty} a_n$ that is,

   $$s_n = a_1 + a_2 + a_3 + a_4 \ldots + a_n \text{ then},$$

   if $n > 1$, *we also have*,

   $$s_{n-1} = a_1 + a_2 + a_3 + a_4 \ldots + a_{n-1}$$

   $$s_n - s_{n-1} = a_n$$

   since the series converges $\lim_{n\to\infty} s_n = s$. *but* $n \to \infty$, *we also* $n - 1 \to \infty$, *so* $\lim_{n\to\infty} s_{n-1} = s$. Thus

   $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

2. $\sum_{n=1}^{\infty} a_n$ *diverge, if* $\lim_{n\to\infty} a_n \neq 0$ *or does not exist*.

Example;

- The series $\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{k}{k+1} + \ldots$ diverges since

  $$\lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{1}{1+1/k} = 1 \neq 0$$

- The series $\sum_{n=1}^{\infty} (-1)^n$ *diverges, since* $\lim_{n\to\infty} (-1)^n$ *does not exist*.
- The series $\sum_{n=1}^{\infty} n^2$ *diverges, since* $\lim_{n\to\infty} n^2 = \infty$.
- The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \ldots$ diverges. This is an example of a series where $\lim_{n\to\infty} a_n = 0$, *but n=1∞andiverges*.

Property of convergent series

If $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent series*, and if c is any constant, then

- $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ *converges*.
- $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$ *converges*.
- *if* $\sum_{n=1}^{\infty} a_n$ *converges and* $\sum_{n=1}^{\infty} b_n$ *diverges then* $\sum_{n=1}^{\infty} (a_n \pm b_n)$ *diverges*.
- If $\sum_{n=1}^{\infty} a_n$ *diverges and* $c \neq 0$ *then* $\sum_{n=1}^{\infty} ca_n$ *diverges*.
- *if* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\sum_{n=k}^{\infty} a_n$ *converges for any* k > 1, and $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 \ldots + a_{k-1} + \sum_{n=k}^{\infty} a_n$. Conversely,
  if $\sum_{n=k}^{\infty} a_n$ *converges for any*, k > 1 then $\sum_{n=1}^{\infty} a_n$ *converges*.

1.2.1 **Test of Convergence.**

The integral test.

The *series* $\sum_{n=1}^{\infty} a_n$ *of nonnegative terms converges*, iff *its partial sum is bounded from above*.

Theorem 1.8: the integral test.

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, Where f *is continuos, postive , decreasing function of* x *for all* $x \geq N$ $(N > 0)$.

Then the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both converge or both diverge

Example: show that the p – series

$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} +$ ... converges if p > 1, and diverges if p ≤ 1.

Solution: if p > 1, then $f(x) = \frac{1}{x^p}$ is a positive decreasing function for x > 1. Since

$$\int_1^{\infty} \frac{1}{x^p} dx = \int_1^{\infty} x^{-p} dx = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \frac{1}{1-p} \lim_{b\to\infty} \left(\frac{1}{b^{p-1}} - 1\right) = \frac{1}{p-1}$$

the improper integral converges.

Then the series converges by the integral test. But it does not tell the sum of the p- series.

If p < 1, then 1- p > 0 and

$\int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{1-p} \lim_{b\to\infty} (b^{1-p} - 1) = \infty$. diverge.

Then the series diverges by integral test

If p = 1 we have the divergence harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \ ...$$

Therefore, p – series is convergence series for p > 1 but divergence for all other values of p.

Example: show that $\sum_{n=1}^{\infty} \left(\frac{1}{n^2+1}\right)$ convergent.

Solution: let $f(x) = \frac{1}{x^2+1}$ is continues, positive, and decreasing for x > 1,and

$\int_1^{\infty} \frac{1}{x^2+1} dx = \lim_{b\to\infty} [arctan\, x]_1^b = \lim_{b\to\infty} [arctan\, b - arctan\, 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$. Convergent.

Then, the series converges by the integral test. But we do not know the value of its sum.

Theorem 1.10; Comparison test.

$\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series* of non negative terms, with $a_n \leq b_n$ for all n.

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Limit comparison test

Suppose that $a_n > 0$ and $b_n > 0$ for all n$\geq N$(N an integer)

- If $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ *then* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *both* converge or both diverges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
- If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverge, then $\sum_{n=1}^{\infty} a_n$ diverge.

Example: test each of the following series for convergence or divergence.

a. $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$ b. $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n+4}$ c. $\sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$

Solution:

a. Let $a_n = \frac{1}{n^2+2n} < \frac{1}{n^2} = b_n$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is a convergent p − series*, then

$\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$ *is convergent by comparition test*

b. Let $a_n = \frac{\sqrt[3]{n}}{n+4}$ for large n is like $\frac{\sqrt[3]{n}}{n} = \frac{1}{\sqrt[3]{n^2}} = b_n$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\sqrt[3]{n}}{n+4}}{\frac{1}{\sqrt[3]{n^2}}} = \lim_{n\to\infty}\frac{n}{n+4} = 1,$$

Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^2}}$ diverges p- series with p = $\frac{2}{3}$.

$\sum_{n=1}^{\infty}\frac{\sqrt[3]{n}}{n+4}\, diverges\ by\ the\ limit\ comparition\ test$,

c. Let $a_n = \frac{1+nln\,n}{n^2+5}$ for large n we expect a_n to behave like $\frac{nln\,n}{n^2} = \frac{ln\,n}{n} > \frac{1}{n}\ for\ n \geq 3$

So let $b_n = \frac{1}{n}$. since,

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n}\, diverges$, and

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1+nln\,n}{n^2+5}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n+n^2\ln n}{n^2+5} = \infty$$

Therefore by limit comparison test

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1+nln\,n}{n^2+5}\ diverges.$

The ratio and root tests

The ratio test.

Let $\sum_{n=1}^{\infty} a_n\ be\ a\ series$ of non negative terms, and suppose that

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = p$. Then

a. the series converges if p < 1

b. The series diverges if p > 1

c. The test is inconclusive if p = 1

Example: investigates the convergence of the following series.

a. $\sum_{n=1}^{\infty}\frac{n}{4^n}$ b. $\sum_{n=1}^{\infty}\frac{(2n)!}{n!n!}$ c. $\sum_{n=1}^{\infty}\frac{4^n n!n!}{(2n)!}$

Solution:

a. Let $a_n = \frac{n}{4^n} \implies a_{n+1} = \frac{n+1}{4^{n+1}}$.

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{n+1}{4^{n+1}}}{\frac{n}{4^n}} = \lim_{n\to\infty}\frac{n+1}{4n} = \frac{1}{4} < 1$. Thus

By ratio test $\sum_{n=1}^{\infty}\frac{n}{4^n}$ converges.

b. Let $a_n = \frac{(2n)!}{n!n!} \implies a_{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!}$

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{(2n+2)!}{(n+1)!(n+1)!}}{\frac{(2n)!}{n!n!}} =$

$\lim_{n\to\infty}\frac{n!n!(2n+2)(2n+1)(2n)!}{n!n!(n+1)(n+1)(2n)!}$.

$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim_{n\to\infty}\frac{4n+2}{n+1} = 4 > 1.$

Thus,

By ratio test $\sum_{n=1}^{\infty}\frac{(2n)!}{n!n!}$ is diverges.

c. Let $a_n = \frac{4^n n!n!}{(2n)!} \implies a_{n+1} = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{4^{n+1}(n+1)!\,(n+1)!}{(2n+2)!}}{\frac{4^n n!\,n!}{(2n)!}}$$

$$= \lim_{n\to\infty}\frac{4^{n+1}(n+1)!\,(n+1)!}{(2n+2)(2n+1)!\,(2n)!}\cdot\frac{(2n)!}{4^n n!\,n!}$$

$= \lim_{n\to\infty}\frac{4(n+1)(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty}\frac{2(n+1)}{(2n+1)} = 1.$

Thus

We cannot decide by ratio test.

Root test.

Let $\sum_{n=1}^{\infty} a_n$ *be a series* of non negative terms for n $\geq N$. and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = p. \text{ Then}$$

a. the series converges if p < 1

b. The series diverges if p > 1

c. The test is inconclusive if p = 1

Example: Investigates the convergence of the following series

a. $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$ b. $\sum_{n=1}^{\infty} (\frac{4n+3}{3n-5})^n$.

Solution:

a. Let $a_n = \frac{4^n}{(3n)^n}$ $\Rightarrow \sqrt[n]{a_n} = \sqrt[n]{\frac{4^n}{(3n)^n}} = \frac{4}{3n}$.

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{4}{3n} = 0 < 1$. Thus by root test, $\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$ *converges.*

b. Let $a_n = (\frac{4n+3}{3n-5})^n$ $\Rightarrow \sqrt[n]{a_n} = \sqrt[n]{(\frac{4n+3}{3n-5})^n} = \frac{4n+3}{3n-5}$.

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{4n+3}{3n-5} = \frac{4}{3} > 1$. Thus by root, Test the series, $\sum_{n=1}^{\infty} (\frac{4n+3}{3n-5})^n$ *diverges.*

1.3 Alternating series, absolute and conditional convergence.

Alternating series.

Definition;

A series in which the terms are alternately positive and negative is an **alternating series**.

Example; the n^th term of an alternating series is of the form, $a_n = (-1)^{n+1} u_n$ or $a_n = (-1)^n u_n$ where $u_n = |a_n|$ is a positive number.

Alternating series test

If $a_n > 0$, for all n and the following two conditions are satisfied

- $a_{n+1} \leq a_n$ and,
- $\lim_{n\to\infty} a_n = 0$, then;
- The series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.

Example; show that the alternating harmonic series, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ Converges.

Solution: $a_n = \frac{1}{n}$ let the series is alternating series in which,

- $\frac{1}{n+1} < \frac{1}{n} \Rightarrow a_{n+1} \leq a_n$
- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Therefore by the alternate series test $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Absolute and conditional convergence.

Definition:

- the series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent, if $\sum_{n=1}^{\infty} |a_n|$ converges.
- If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n$ is said to be conditionally convergent.
- If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Example: Determine whether each converges conditionally, converges absolutely or diverges.

a. $\sum_{n=1}^{\infty} \frac{\sin n + \cos n}{n^3}$. b. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Solution:

a. Let $a_n = \frac{\sin n + \cos n}{n^3}$, since $\sin n + \cos n \leq 2$ is both positive and negative.

$\frac{|\sin n+\cos n|}{n^3} \le \frac{2}{n^3}$ and $\sum_{n=1}^{\infty} \frac{2}{n^3}$ converges. Since it is a p – series with p =3. Thus ,

$\sum_{n=1}^{\infty} \left|\frac{\sin n+\cos n}{n^3}\right| = \sum_{n=1}^{\infty} \frac{|\sin n+\cos n|}{n^3}$ Converges, by comparison test.

Therefore $\sum_{n=1}^{\infty} \frac{\sin n+\cos n}{n^3}$ is absolutely convergent.

b. Let $a_n = \frac{(-1)^{n-1}}{\sqrt{n}}$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \left|\frac{1}{\sqrt{n}}\right|$ is diverges absolutely. But

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$.

$let\ b_n = \frac{1}{\sqrt{n}}, \quad b_{n+1} = \frac{1}{\sqrt{n+1}}, \quad \Longrightarrow b_{n+1} > b_n$ and,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Then by alternating series test, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$. Converges.

Chapter Two

2. Power Series

2.1 Definition of Power series.

Definition: A power series about x = 0 is a series of the form,

$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n$ ..., and

A power series about x = a is a series of the form,

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$ ... in which the center a and the coefficients $c_0,\ c_1,\ c_2,\ c_3 \ldots c_n, \ldots$ are constants.

Example: consider a geometric series,

$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n +$ ... with first term 1 and ratio x. it converges to $\frac{1}{1-x}$ for $|x| < 1$.

We express this fact by writing.

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n +$ ..., $-1 < x < 1$. it is also called a power series with all the coefficients equal to 1 of the first form.

Example: consider the power series of the second form.

$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \ldots + (-\frac{1}{2})^n (x-2)^n + \cdots$, $0 < x < 4$.

With a= 2, c_0= 1, $c_1 = -1/2$, $c_2 = 1/4$, … $c_n = (-\frac{1}{2})^n$. this is a geometric series with the first term 1 and ratio r = $-(\frac{x-2}{2})$ the series converges for $\left|-(\frac{x-2}{2})\right| < 1$ or $0 < x < 4$. Then the sum is,

$\frac{1}{1-r} = \frac{1}{1+\frac{x-2}{2}} = \frac{2}{x}$. Therefore,

$\frac{2}{x} = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \ldots + (-\frac{1}{2})^n (x-2)^n + \cdots,$

$0 < x < 4$.

Theorem 2.1. The convergence theorem for power series.

If the power series,

$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ converges at $x = a \neq 0$, then it converges absolutely for all x with $|x| < |c|$.

If the series diverges at x = d then it diverges for all x with $|x| > |d|$

2.2. **Radius of Convergence and Interval of Convergence.**

The convergence of the series.

$\sum_{n=0}^{\infty} c_n (x-a)^n$ is described by one of the following three cases

- The series converge absolutely for every x (R = ∞)
- There is a positive number R such that the series diverges for x with $|x-a| > R$ but converge absolutely for x with $|x-a| < R$. The series may or may not converge at either of the end points x = a – R and x = a+R
- The series converge at x = a and diverge all the rest (R = 0)

Where, the number R in each case is called the **radius of convergence** of the series. For convenience, if the first case holds we agree to call the radius of convergence is R= ∞, if the second case holds R = $x - a$, and the last case holds R = 0.

If $|x-a| < R$. Then the series converges on the intervals (a-R, a+R), [a-R, a+R], [a-R, a+R) or (a-R, a+R] depends on the series converges at a-R or a+R and these intervals are called **intervals of convergence**. When R=0 the interval of convergence degenerates to the single point x = 0, and if R= ∞, it is the entire real line $(-\infty, \infty)$.

Using the Ratio Test to Find the Radius of Convergence.

When $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ $exists$, the radius of convergence can be found using the ratio test.

Examples: find the radius and intervals of convergence of the series.

a. $\sum_{n=0}^{\infty} \frac{x^n}{2n+1}$ b. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{(n+1)^2 3^n}$ c. $\sum_{n=0}^{\infty} n!\,(2x-1)^n$

Solution;

a. $a_n = \frac{x^n}{2n+1}$, ratio test is applicable only to series of positive terms, and since x can be either positive or negative , so we must consider in absolute value.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{2(n+1)+1}}{\frac{x^n}{2n+1}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x^{n+1}}{2n+3} . \frac{2n+1}{x^n}\right|$$

$$= \lim_{n\to\infty} \frac{2n+1}{2n+3} |x| = |x|.$$

The series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. Therefore the radius of convergence is R = 1.

If x = 1 $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ = 1+1/3+1/5+…

$a_n = \frac{1}{2n+1}$ $let\ b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2n+1} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$

$\lim_{n\to\infty} \frac{1}{n}$ is divergent harmonic series, then by limit Comparison test $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ is diverges.

If x = -1, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ this series is an alternating series, and $\frac{1}{2n+1}$ decrease monotonically to 0 thus the series converges.

Therefore the complete interval of convergence of the original series is $-1 \le x < 1$.

b. Consider the limit

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x-2)^{n+1}}{(n+2)^2 3^{n+1}} * \frac{(n+1)^2 3^n}{(-1)^n (x-2)^n}\right|$$

$= \lim_{n\to\infty} \frac{1}{3}(\frac{n+1}{n+2})^2|x-2| = \frac{|x-2|}{3}$. Thus by the ratio test the series converges absolutely

If $\frac{|x-2|}{3} < 1, \Longrightarrow |x-2| < 3$ and diverge

If $|x-2| > 3$. Now we test the value (x-2) = ± 3

If (x-2) =3 $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{(n+1)^2 3^n} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$ this series is converges absolutely, since it is p – series with p=2.

If (x-2) = -3 $\sum_{n=0}^{\infty} \frac{(-1)^n(-3)^n}{(n+1)^2 3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n 3^n}{(n+1)^2 3^n}$

$= \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$. is a convergence

P series with p = 2.

Therefore the complete interval of convergence is defined as

$|x-2| \le 3, \Longrightarrow -3 \le x-2 \le 3$

$\Longrightarrow -1 \le x \le 5$ Thus the interval of Convergence is $[-1,5]$

c. Let $a_n = n!\,(2x-1)^n$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right|$

$= \lim_{n\to\infty}(n+1)|(2x-1)|$

$= \infty$

Now for all x $\neq \frac{1}{2}$ the series diverges, so R=0 and interval of Convergence is a single point $\{\frac{1}{2}\}$

d. If $a_n = \frac{nx^n}{1.3.5\ldots.(2n-1)}$ then

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^{n+1}}{1.3.5\ldots.(2n+1)} \cdot \frac{1.3.5\ldots.(2n-1)}{nx^n}\right|$

$=|x| \lim_{n\to\infty} \frac{n+1}{2n+1} = 0$ for all x.

The series converges $\Longrightarrow R = \infty$, and interval of convergence $=(-\infty, \infty)$

2.3. Arithmetic Operations on Convergent Power Series.

If A(x) = $\sum_{n=0}^{\infty} a_n x^n$ and B(x) = $\sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$, then A(x) = $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to A(x)B(x) for $|x| < R$

$$(\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n) = \sum_{n=0}^{\infty} c_n x^n.$$

Similarly,

$(\sum_{n=0}^{\infty} a_n x^n) \pm (\sum_{n=0}^{\infty} b_n x^n) = \sum_{n=0}^{\infty}(a_n \pm b_n)x^n$. Converges to A(x) ± B(x) for $|x| < R$

Note If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then

$$\sum_{n=0}^{\infty} a_n (f(x))^n$$

Converges absolutely for any continuous function f on $|f(x)| < R$

Example; since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ converges absolutely for $|x| < 1$

Then $\frac{1}{1-3x^4} = \sum_{n=0}^{\infty}(3x^4)^n$ converges absolutely for $|3x^4| < 1$,or $|x| < 1/3$

2.4. Differentiation and integration of power series

let $\sum_{n=0}^{\infty} a_n (x-a)^n$ have nonzero radius of convergence R and for $a-R < x < a+R$, we write,

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n. \text{ Then,}$$

1. f is continuous on the interval (a-R, a+R).
2. f is differentiable on the interval (a-R, a+R) and

   $$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}(a_n(x-a)^n) = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}.$$

   the series on the right also has radiuse of convergence R
3. f integrable over any interval $[a, b]$contained in (a-R, a+R),

   $$\int_a^b f(x)dx = \sum_{n=0}^{\infty} \int_a^b a_n (x-a)^n dx.$$

   furthermore, f has an antiderevatives in $(a-$R, a+R) given by,

$$\int f(x)dx = \sum_{n=0}^{\infty} \int a_n x^n dx = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} + C.$$

$$\textit{the series on the right also has radius of convergence } R.$$

Example; let $f(x) = \frac{1}{1-x}$, then find series for $f'(x)$

Solution; $f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$

$$= \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

Differentiate f term by term gives,

$$f'(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$$

$$= \sum_{n=1}^{\infty} nx^{n-1} \qquad |x| < 1.$$

Example; find a power series for ln $(1+x^2)$.

Solution; let f(x) = $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$

$|x| < 1$, then

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots + x^{2n} - \cdots$$

$$\frac{2x}{1+x^2} = 2x - 2x^3 + 2x^4 - \cdots + x^{2n+1} - \cdots$$

$$= \sum_{n=0}^{\infty} 2(-1)^n x^{2n+1},$$

Integrating both sides with respect to x gives,

$$\int \frac{2x}{1+x^2} dx = \int \sum_{n=0}^{\infty} 2(-1)^n x^{2n+1} \text{ dx.}$$

$$= \sum_{n=0}^{\infty} 2(-1)^n \int x^{2n+1} dx$$

$$= \sum_{n=0}^{\infty} 2(-1)^n \frac{x^{2n+2}}{2n+2}.$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n+1}$$

Since ln $(1+x^2) = \int \frac{2x}{1+x^2} dx$

ln $(1+x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n+1}$ is convergence on (-1, 1)

more over converge at the two end points, so it is converge on the interval $[-1,1]$.

Example; identify the function.

f(x) = $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots + \frac{x^{2n+1}}{2n+1} - \cdots$

for $[-1,1]$.

Solution; differentiating f term by term, we get

$f'(x) = 1 - x^2 + x^4 - \cdots + x^{2n} - \cdots$ for $|x| < 1$

The series is geometric with first term 1 and common ratio $-x^2$. Thus;

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$$

Integrating both sides, gives

$$\int f'(x)dx = \int \frac{1}{1+x^2} dx$$

$$f(x) = tan^{-1}x + C$$

2.5. Taylor and Maclaurin Series

If a function f(x) has derivatives of all orders on the interval I, it can be represented as a power series on I about a is called the **Taylor Series**. (If a = 0 it is called the **Maclaurin Series**). If f(x) is represented by a power series centered at a; then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This can be written out the long way as,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots.$$

Where the coefficient of the n^th term is, $a_n = \frac{f^{(n)}(a)}{n!}$, and

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The function $p_n(x)$ generated by f at x = a is called **Taylor polynomials of order n**

Example; Find the power series and Taylor polynomials $p_3(x)$, $p_4(x)$ *and* $p_5(x)$ for

a. f(x) = e^x centered at x = 0:

Solution: $f(x) = e^x \quad \Longrightarrow f(0) = 1$

$f'(x) = e^x \quad \Longrightarrow f'(0) = 1$

$f''(x) = e^x \quad \Longrightarrow f'(0) = 1$

$\vdots \qquad\qquad \vdots$

$f^{(n)}(x) = e^x \quad \Longrightarrow f^{(n)}(0) = 1$. So

$$f(x) = e^x = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \cdots$$

$$= 1 + 1(x) + \frac{1}{2!}(x)^2 + \frac{1}{3!}(x)^3 \ldots$$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$p_n(x) = 1 + 1(x) + \frac{1}{2!}(x)^2 + \frac{1}{3!}(x)^3 + \cdots + \frac{x^n}{n!}.$$

$$p_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3,$$

$$p_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

$$p_5(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$$

A special limit

$\lim_{n\to\infty} \frac{x^n}{n!} = 0$, since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is a convergence series

b. f(x) = ln x centered at x = 1:

Solution: $f(x) = \ln x \quad \Longrightarrow f(1) = 0.$

$f'(x) = \frac{1}{x} \quad \Longrightarrow f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \quad \Longrightarrow f(1) = -1$

$f'''(x) = \frac{2}{x^3} \quad \Longrightarrow f(1) = 2$

$\vdots \qquad\qquad \vdots$

$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n} \quad \Longrightarrow f^{(n)}(1) = (-1)^{n-1}(n-1)!.$

so the Taylor Series

$$f(x) = \ln x = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots$$

$$f(x) = \ln x = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$$

$$p_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + \frac{(-1)^{n+1}}{n}(x-1)^{n-1}$$

$$p_3(x) = (x-1) - \frac{1}{2}(x-1$$

$$p_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

$$p_5(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \frac{1}{4}(x-1)^4$$

Taylor formula with remainder

If a function f(x) have derivatives up through the (n+1)^st order in an open interval I centered at x = a. then for each x in I there is a number c between a and x such that,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$,is called lagrange form of the remainder. And Taylor formula can be written more briefly,

$$f(x) = P_n(x) + R_n(x)$$

Example; let f(x) = lnx, then find a Taylor's formula with the remainder for arbitrary n about x = 1.

Solution, from the previous example,

$$f(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n - \cdots$$

The Taylor formula with the remainder is,

$$f(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n + R_n(x)$$

Where $R_n(x) = \frac{(-1)^n}{n+1}(x-1)^{n+1}$and

$$P_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n$$

Theorem let f have derivatives of all orders in an open interval I centered at x = a. then the Taylor series for f about x =a converges to f(x) for x in I if and only if,

$$\lim_{n\to\infty} R_n(x) = 0$$

Where $R_n(x)$ is the remainder term in the Taylor formula.

Example, show that the Taylor series for f(x) = e^x about x =0 converges to e^x for all x.

Solution,

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{x^n}{n!}$$

And that

$$R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}, \text{ where } 0 < c < x$$

If $0 < c < x$ then $e^c < e^x$ since f(x) = e^x is an increasing function.

$$|R_n(x)| \le \frac{e^x}{(n+1)!}x^{n+1}.$$

By special limit

$$\lim_{n\to\infty} \frac{e^x}{(n+1)!}x^{n+1} = e^x \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = e^x(0) = 0..$$

Thus, for x > 0

$$\lim_{n\to\infty} R_n(x) = 0$$

If $0 < c < x$ then $e^c < e^0 = 1$. Thus,

$$|R_n(x)| \le \left|\frac{e^x}{(n+1)!}x^{n+1}\right|$$

By special limit

$$\lim_{n\to\infty} \frac{e^x}{(n+1)!}x^{n+1} = e^x \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = e^x(0) = 0.$$

Thus, for all x < 0

$$\lim_{n\to\infty} R_n(x) = 0$$

Therefore the Taylor series for e^x about x = 0 converges to e^x for all real number x.

Taylor series for f about x = 0 (Maclaurin series)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Basic List of Power Series

- $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{x^n}{n!} + \cdots \quad -\infty < x < \infty$
- $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots \quad 0 < x \leq 2$
- $\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \ldots \quad 0 < x < 2$
- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \quad |x| < 1$
- $(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{} \ldots \quad |x| < 1$
- $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{x^{2n+1}}{(2n+1)!} - \cdots \quad -\infty < x < \infty$
- $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + \frac{x^{2n}}{(2n)!} - \cdots \quad -\infty < x < \infty$