Chapter 2:orthogonalty

2.1.Inner product

Definition An inner product on real or complex vector spaces V is afunction that associates areal or complex $number < u, v >$ with each pair vector in V in such that thr following axiom s are satisfied for all u v and w in V and for all scalr k in field K

1. $< u, v > = < v, u$ ……………………………………..symmety axiom
2. $< u + v, w > = < u, w > + < v, w$ ……………………. . … additives axioms
3. $< ku, v > = k < u, v$ ………………………………. . …homogenity axiom
4. $< u, u > \geq 0 \; and \; = 0 \; iff \; u = 0.$ ………………………….positvity axioms

2.2Inner product spaces:

Defintion :A real or complex vector space V with an inner product is called an inner product space

Examples: Eucleadean inner product on R^n define u,v in R^n by

$<u, v> = u_1 v_1 + u_2 v_2 + \cdots u_n v_n$ satisfy the axioms of inner productin R^n such that

Solution:

$$Norm\ of\ u = \sqrt{u.u}\ \ and\ d(u,v) = norm(u-v) = \sqrt{(u-v).(u-v)}$$

If norm is one the v is called unit vector

Examples: if u,v in R^2 verify $< u, v >= 3u_1v_1 + 2u_2v_2$satisfy the four inner product axioms

Solution:

Examples. If V is vector spaces of matrices over real number veryfy that

$< A, B >= trace(B^tA)$

Inner products generated by matices

The euclidean inner product and weighet euclidean inear product are special cases of inner product on R^n

Called Matrix inner products define .if u.v is Euclidean inner product on R^n then $< u, v >= \; Au.Av$,A is invertable matrix

Note. If u and v are in column form then $u.v = \; v^T u = (Av)^T Au = v^T A^T Au$ the weighted Euclidean inner product$<u, v >= w1u_1 v_1 + w2u_2v_2 + \cdots wnu_n v_n$ is generated by matrix

$$A = \begin{pmatrix} \sqrt{w_1} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{w_2} & 0 & \dots & 0 \\ 0 & 0 & \sqrt{w_3} & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & \sqrt{w_n} \end{pmatrix}$$

Examples : let $< u, v >= 3u_1v_1 + 2u_2v_2$ the $w1 = 3 \; and \; w2 = 2$

$$A = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

Examples An inner product on M_{nxn} if $u \; and \; v$ are an nxn matrices then $< u, v >= trac(u^T \; v)$

Exercise .Take 3by3 matrices and check by your own

Examples :standard inner product on P_n

If $p(x) = a_0 + a_1\mathrm{x} + a_2\mathrm{x}^2 + \cdots + a_n\mathrm{x}^\mathrm{n}$ and $q(x) = b_0 + b_1\mathrm{x} + b_2\mathrm{x}^2 + \cdots + b_n\mathrm{x}^\mathrm{n}$ are polynomial in $\boldsymbol{P_n}$ define the inner product on this space

$$< p, q > = a_0b_0 + a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

$norm\ of\ p = \sqrt{< p, p >} = \sqrt{a_0a_0 + a_1a_1 + a_2a_2 + \cdots + a_na_n}$

$p(x) = 3x^2 + 2x + 4\ and\ q(x) = 2 + 4x + 2x^2$ compute

a) $< p, q >$
b) $norm\ of\ p\ and\ q$

Examples :Evaulation inner product o n P_n

I f $p(x) = a_0 + a_1\mathrm{x} + a_2\mathrm{x}^2 + \cdots + a_n\mathrm{x}^{\mathrm{n}}$ **and** $q(x) = b_0 + b_1\mathrm{x} + b_2\mathrm{x}^2 + \cdots + b_n\mathrm{x}^{\mathrm{n}}$ are polynomial in $\boldsymbol{P_n}$ and if $x_0,\ x_1, x_2, \ldots x_n$ are distinct real numbers then

$< p, q >$ =$\mathrm{p}(x_0)\mathrm{q}(x_0) + \mathrm{P}(x_1)\mathrm{q}(x_1) + \mathrm{p}(x_2)\mathrm{q}(x_2) + \cdots + \mathrm{p}(x_n)\mathrm{q}(x_n)$ viewed as dot prudct in R^n satisfies the axioms of inner product

$$norm\ of\ p = \sqrt{< p, p >} = \sqrt{\mathrm{p}(x_0)\mathrm{p}(x_0) + \mathrm{P}(x_1)\mathrm{p}(x_1) + \mathrm{p}(x_2)\mathrm{p}(x_2) + \cdots + \mathrm{p}(x_n)\mathrm{p}(x_n)}$$

<u>Examples</u>; working with evaluation inner product

Let $\boldsymbol{P_2}$ have evaluation inner product at the point $x_0 = -2,\ x_1 = 0, x_2 = 2$,

$p(x) = 3x^2 + 2x + 4\ and\ q(x) = 2 + 4x + 2x^2$ compute

c) $< p, q >$

d) $norm\ of\ p\ and\ q$

Examples : An inner product on collection of continues function on C[a,b]

Let $f(x) = f$ and $g(x) = g$ be thwo continoues function on [a,b] define by

$< f, g > = \int_a^b f(x)g(x)dx$ verify the four inner product axioms

Orthogonality

Two vector u and V in inner product spaces are called orthogonal if $< U, V >= 0$

Examples : euclidean inner product spaces in R^2 and R^3 where the operation is dot product

Take your own particular examples

2.3.Orthogonal and ortho normal sets

Definition : Aset of two or more vectors in areal inner product spaces is said to be orthogonal if for all pairs of distinct vectors in the set are orthogonal

An orthogonal set in which every vectors has norm one is said to be orthonormal

Examples $.\boldsymbol{an\ orthogonal\ set\ in\ R^3}$

let $u = (0,1,0)$, $v = (1,01)$, $w = (1,0,-1)$ and assume that R^3 has the euclidean inner product so each vector are orthogonal to each other . show

Examples :constructing an orthonormal set

Let $s = \{\boldsymbol{u} = (0,1,0)\ , v = (1,01)\ , w = (1,0,-1)\}$ **then** $norm(u) = \sqrt{1}$, $norm(w) = \sqrt{2} = norm(v)$ then unit vector $\boldsymbol{s} = \{(0,1,0), \frac{1}{\sqrt{2}}(1,01), \frac{1}{\sqrt{2}}(1,0,-1)\}$ **are orthogonal to each other**

Theorem If $S = \{v_1, v_2, \dots v_n\}$ is an orthogonal set of nonzero vectors in an inner product spaces then S is lineary independent

Proof:assume that $k_1v_1 + k_2v_2 + \cdots + k_nv_n = 0 \dots \dots \dots (1)$

We want show $k_1, k_2, \dots, k_n = 0$ for each $v_i \in S$ it follow that from equation (1)

$< k_1v_1 + k_2v_2 + \cdots + k_nv_n, v_i >=< 0, v_i >= 0$ from the axiom of inner product spaces

$$< k_1v_1, v_i > +< k_2v_2, v_i > + \cdots +< k_nv_n, v_{i>} > \quad = 0$$

$$k_1 < v_1, v_i > +k_2 < v_2, v_i > + \cdots + k_n < v_n, v_{i>} > \ = 0$$

From orthogonalty of S it follow that $\begin{cases} k_i < v_j, v_i > \ = 0 \ if \ i \neq j \\ k_i < v_j, v_i > \ \neq 0 \ if \ i = j \end{cases}$

$$\Rightarrow k_i < v_j, v_i > \ = 0 \ implies \ k_i \ = 0 \ for \ i = 1,2, \dots n$$

Examples :standard orthonormal basis in in R^n with euclidean inner product

$$e_1 \ = (1,0,0 \dots ,0), e_2 = (0,1,0 \dots ,0) \dots \ e_n = (0,0,0 \dots ,1)$$

Examples : ortho normal basis

The set $s = \{(0,1,0), \frac{1}{\sqrt{2}}(1,01), \frac{1}{\sqrt{2}}(1,0,-1)\}$ are orhonormal then linearly independent sets and s is basis for R^3 by the above theorem

<u>Coordinates relatives to orthonormal bases</u>

We express $u \in R^n$ *as* linear combination of basis vector $\boldsymbol{S} = \{v_1, v_2, \dots v_n\}$ thar means

$$u = k_1v_1 + k_2v_2 + \cdots + k_nv_n \text{ vector equation}$$

so coordinate vector realatives to s is $(u)_S = (k_1, k_2, \dots, k_n)$

theorem : If $\boldsymbol{S} = \{v_1, v_2, \dots v_n\}$ is an orthogonal basis for inner product spaces V and if u is any vector in V then $u = \frac{<u,v_1>}{(||v_1||)^2}v_1 + \frac{<u,v_2>}{(||v_2||)^2}v_2 + \frac{<u,v_3>}{(||v_3||)^2}v_3 + \cdots + \frac{<u,v_n>}{(||v_n||)^2}v_n$

prooof: ***since*** $\boldsymbol{S} = \{v_1, v_2, \dots v_n\}$ is basis for V, every vector in u in V can expreesed in the form of $u = k_1v_1 + k_2v_2 + \cdots + k_nv_n$

so $<\boldsymbol{u}, \boldsymbol{v_i}> =< k_1v_1 + k_2v_2 + \cdots + k_nv_n, v_i >$

$==< k_1v_1 + k_2v_2 + \cdots + k_nv_n, v_i >$

$= k_1 < v_1, v_i > +k_2 < v_2, v_i > + \cdots + k_n < v_n, v_i > = k_i < v_i, v_i > = k_i(||v_i||)^2$

$k_i = \frac{<\boldsymbol{u},\boldsymbol{v_i}>}{<v_i,v_i>} = \frac{<\boldsymbol{u},\boldsymbol{v_i}>}{(||v_i||)^2}$

theorem : If $\boldsymbol{S} = \{v_1, v_2, \dots v_n\}$*is* an orthonormal basis for an inner produc spaces V and u is any vector in V then $u = \frac{<u,v_1>}{(||v_1||)^2}v_1 + \frac{<u,v_2>}{(||v_2||)^2}v_2 + \frac{<u,v_3>}{(||v_3||)^2}v_3 + \cdots + \frac{<u,v_n>}{(||v_n||)^2}v_n$ *and*

$(||v_i||) = 1 \quad , for\ i = 1,2, \dots, n$

Prooof: from the above theorem normality of each vector $k_i = <\boldsymbol{u}, \boldsymbol{v_i}>$

Examples : find coordinate vector relatives to the orthonormal basis $\boldsymbol{s} = \{(\mathbf{0}, \mathbf{1}, \mathbf{0}), \left(-\frac{4}{5}, \mathbf{0}, \frac{3}{5}\right), \left(\frac{3}{5}, \mathbf{0}, \frac{4}{5}\right)\}$ **set** s is an orthonormal basis for R^3 with Euclidean

express the vector $\boldsymbol{u} = (\mathbf{1}, \mathbf{1}, \mathbf{1}) \in \boldsymbol{R}^3$ **as LC of vector in S and find coordinate vector** $(u)_S$

examples : An orthogonal and orthonormal basis

a) Show that the vector $w_1 = (0,2,0)$ $\boldsymbol{W_2} = (3,0,3)$ *and* $w_3 = (-4,0,4)$ form orthogonal Basis for R^3 with respect to Euclidean inner product

b) Expess vector $u = (1,2,4)$ as LC of orthonormal basis vector in part (a)

Orthogonal projection

Projection theorem: If W is a finte dimensional subspaces of inner product spaces V, then every vector u in V can be expressed in exactly one way as

U= w_1+w_2 where w_1 *is W and* w_2 *is in* $W^{\perp}$ and $w_1 = proj_W^u$ and $w_2 = projproj_{W^{\perp}}^u$

$$w_1 = proj_W^u = \frac{< u, W > W}{< W, W >} \quad and \; w_2 = projproj_{W^{\perp}}^u = \frac{< u, W^{\perp} >}{< W^{\perp}, W^{\perp} >}$$

$$u = proj_W^u + (u - proj_W^u) \ldots \ldots \ldots (1)$$

Calculating orthogonal projection

Theorem :let W be finite dimensional subspaces of an inner product spaces V

a) If $S = \{v_1, v_2, \dots v_r\}$ *is* an orthogonal basis for W and u is any vector in V then

$$proj_W^u = \frac{< u, v_1 >}{(||v_1||)^2} v_1 + \frac{< u, v_2 >}{(||v_2||)^2} v_2 + \frac{< u, v_3 >}{(||v_3||)^2} v_3 + \cdots + \frac{< u, v_r >}{(||v_r||)^2} v_r$$

b) If $S = \{v_1, v_2, \dots v_r\}$ *is* orthonormal basis for w and u is any vector in V then

$$proj_W^u = \frac{< u, v_1 >}{1} v_1 + \frac{< u, v_2 >}{1^2} v_2 + \frac{< u, v_3 >}{1} v_3 + \cdots + \frac{< u, v_r >}{1^2} v_r$$

$proof : u = w_1 + w_2$ and where w_1 *is* W *and* w_2 *is in* $W^\perp$

$: proj_W^u = w_1$ can be expressed interms of basis vector for W as

$$proj_W^u = w_1 = \frac{< w_1, v_1 >}{(||v_1||)^2} v_1 + \frac{< w_1, v_2 >}{(||v_2||)^2} v_2 + \frac{< w_1, v_3 >}{(||v_3||)^2} v_3 + \cdots + \frac{< w_1, v_r >}{(||v_r||)^2} v_r$$

Since w_2 is orthoigonal to W it follow that $< w_2, v_1 >, =< w_2, v_2 >= \cdots =< w_2, v_r >= 0$

So $proj_W^u = w_1 = \frac{<w_1+w_2,v_1>}{(||v_1||)^2} v_1 + \frac{<w_1+w_2,v_2>}{(||v_2||)^2} v_2 + \frac{<w_1+w_2,v_3>}{(||v_3||)^2} v_3 + \cdots + \frac{<w_1+w_2,v_r>}{(||v_r||)^2} v_r$

$$proj_W^u = w_1 = \frac{< u, v_1 >}{(||v_1||)^2} v_1 + \frac{< u, v_2 >}{(||v_2||)^2} v_2 + \frac{< u, v_3 >}{(||v_3||)^2} v_3 + \cdots + \frac{< u, v_r >}{(||v_r||)^2} v_r$$

Proof of (b) in this case $norm(v_i) = 1\ for\ i = 1,2, \dots r$

$$proj_W^u = w_1 = \frac{< u, v_1 >}{1} v_1 + \frac{< u, v_2 >}{1^2} v_2 + \frac{< u, v_3 >}{1} v_3 + \cdots + \frac{< u, v_r >}{1^2} v_r$$

Examples :calculting projection :let R^3have the inner product and let W be the subspaces spanned by orthonormal basis $S = \left\{(0,1,0), \left(-\frac{4}{3}, 0, \frac{3}{5}\right), (\frac{3}{5}, 0, -\frac{4}{5})\right\}$ **if** $\boldsymbol{u} = (\mathbf{1}, \mathbf{1}, \mathbf{2})$ **find** $proj_W^u = \boldsymbol{w_1}$

And $\boldsymbol{w_2} = \boldsymbol{u} - proj_W^u$

2.4.Gram –schmidt orthogonalization process

Theorem :every non zero finite dimensional inner product has an orthonormal basis

Proof:let W be non zero finte dimensional subspaces of an innerproduct spaces .supposes that

$\boldsymbol{S} = \{u_1, u_2, \dots, u_r\}$ is any bss for W.T.S that W has an orthogonal basis $\{\{v_1, v_2, v_3, \dots vr\}\}$

Step1:$v_1 = u_1$

Step2:construct $v_2$orthogonal to v_1 so $v_2 = u_2 - proj_{w_1}^{u_2} = u_2 - \frac{<u_2,v_1>}{(||v_1||)^2} v_1$

Step3: construct $v_3$orthogonal to v_1 and v_2:we compute component of $u3$ orthogonalto space spanned by v_1 and v_2 $v_3 = u_3 - proj_{w_2}^{u_3} = u_3 - \{\frac{<u_3,v_1>}{(||v_1||)^2} v_1 + \frac{<u_3,v_2>}{(||v_2||)^2} v_2\}$

Step4: to determine $v_4$that is orthogonal to v_1, v_2 and $v_3$we compute the component of u_4 orthogonal to the spaces w_3 spanned by v_1, v_2, v_3

$$v_4 = u_4 - proj_{w_3}^{u_4} = u_4 - \frac{<u_4,v_1>}{(||v_1||)^2} v_1 + \frac{<u_4,v_2>}{(||v_2||)^2} v_2 + \frac{<u_4,v_3>}{(||v_3||)^2} v_3$$

Continuing this way for r –step

$v_r = u_r - proj_{w_{r-1}}^{u_r}$ where $w_{r-1} = span\{v_1, v_2, \dots, v_{r-1}\}$

$$v_r = u_r - \{\frac{< u_r, v_1 >}{(||v_1||)^2} v_1 + \frac{< u_r, v_2 >}{(||v_2||)^2} v_2 + \frac{< u_r, v_3 >}{(||v_3||)^2} v_3 + \dots + \frac{< u_r, v_{r-1} >}{(||v_{r-1}||)^2} v_{r-1}\}$$

To construct orthonormal basis $\{q_1, q_2, \dots, q_r\}$ is $q_i = \frac{v_i}{||v_i||}$ for i=1,2…r

Examples: using gram -schmdit process tranform the basis vectors

a) $s = \{(1,1,1), (0,1,1), (0,0,1\}$ to orthogonal basis$\{v_1, v_2, v_3\}$.

Example.calculate the set of orthonormal polynomial w.r.t inner product define by

$$< f, g >= \int_{-1}^{1} f(x)g(x)dx$$

$s = \{1, x, x^2, \ldots x^n\}$ to orthogonal and orthonormal basis$\{v_1, v_2, v_3, \ldots v_n\}$
$s = \{1, x, x^2, \ldots x^n\}$ to orthogonal and orthonormal basis$\{v_1, v_2, v_3, \ldots v_n\}$
Solution: set ing $X_j(x) = x^j \; for \; j = 0,1,2, \ldots$, our othogonal set $\{\psi_j\}, j = 0,1,2, \ldots$and setorthonormal set by$\{\varphi_j\}, j = 0,1,2, \ldots$

$$\psi_0(x) = X_j(x) = 1, for \; j = 0$$

$$\|\psi_0(x)\| = \sqrt{< 1,1 >} = \sqrt{\int_{-1}^{1} 1dx} = \sqrt{1/2}$$

$$\varphi_0(x) = \frac{\psi_0(x)}{\|\psi_0(x)\|} = \sqrt{1/2}$$

$$\psi_1(x) = X_1(x) - < X_1(x), \varphi_0(x) > \varphi_0(x) = x - \sqrt{1/2}\int_{-1}^{1} \sqrt{1/2}xdx = x$$

$$\|\psi_1(x)\| = \sqrt{< x, x >} = \sqrt{\int_{-1}^{1} 1x^2dx} = \sqrt{2/3}$$

$$\varphi_1(x) = \frac{\psi_1(x)}{\|\psi_1(x)\|} = \sqrt{3/2}x$$

$$\psi_2(x) = X_2(x) - < X_2(x), \varphi_0(x) > \varphi_0(x) - < X_2(x), \varphi_1(x) > \varphi_1(x)$$

$$= x^2 - < x^2, \sqrt{\tfrac{1}{2}} > \sqrt{1/2} - < x^2, \sqrt{\tfrac{3}{2}}x > \sqrt{\tfrac{3}{2}}x = x^2 - \tfrac{1}{3}$$

$$\|\psi_2(x)\| = \sqrt{< x^2 - \frac{1}{3}, x^2 - \frac{1}{3} >} = \sqrt{\int_{-1}^{1} (x^2 - \frac{1}{3})(x^2 - \frac{1}{3}) dx} = \sqrt{\frac{28}{45}} =$$

$$= \frac{2}{3}\sqrt{7/5}$$

$$\varphi_2(x) = \frac{\psi_2(x)}{\|\psi_2(x)\|} = \frac{\sqrt{5/7}}{1}(\frac{3}{2}x^2 - \frac{1}{2})$$

Continiung this procese we obtaion

$$\varphi_3(x) = \frac{\psi_3(x)}{\|\psi_3(x)\|} = \frac{\sqrt{7/2}}{1}(5/2x^3 - \frac{3}{2}x) =$$

In genneral

$\varphi_n(x) = \frac{\psi_n(x)}{\|\psi_n(x)\|} = \sqrt{\frac{2n+1}{2}} P_n(x)$ where $P_n(x)$ is legendre polynomial

What is legender polynomial ?

2.5 **<u>Theorem : cauchy –schwarz in equality</u>**

<u>Angle and orthogonallity in inner product</u>

Recall : angle θ *between* u and vin R^n, $cos\theta = \frac{v.u}{abs(v).abs(u)}$ *and*

$$abs(cos\theta) \leq 1$$

Implies taking absolute both sides $abs < u, v > \leq norm(u)norm(v)$

<u>Theorem : cauchy –schwarz in equality</u>

If u and v are vector in real inner product spaces , then $abs < u, v > \leq norm(u)norm(v)$

Theorem : if u,v and w are vector in real inner product spaces V and if k is any scalar then

a) $norm(v+w) \leq norm(v)+norm(w)$ triangle in equalty
b) $d(u,v) \leq d(u,w)+d(v,w)$, where d is distance by $d(u,v)=abs(u-v)$
   proof:
   2.6 The Dual spaces

Let V= avector spaces over field K and let $V^*=L(V,K)=$ the set of all linear maps from V to K, V^* is vectors spaces over K

Definition: the vector spaces $V^*$is called the dual spaces of V . an element of V^* are called linear fuctional on V.

Notation.Let $\varphi\epsilon V^*$ or $\varphi\epsilon L(V,K)$ we will use the notation $<\varphi,v>=\varphi(v)$

1. $<\varphi_1+\varphi_2,v>=<\varphi_1,v> + <\varphi_2,v>$
2. $<\varphi,v_1+v_2>=<\varphi,v>+<\varphi,v>$
3. $<\lambda\varphi,v>=\lambda<\varphi,v>$
4. $<\varphi,\lambda v>=\lambda<\varphi,v>$
   Defition 2: $\{v^*{}_1,v^*{}_2,v^*{}_3,\dots v^*{}_n\}$ is called the dual basis of $\{v_1,v_2,v_3,\dots,v_n\}$
   Let V= an inner product spaces to each $v\in V$ we can associate linear function $L_v\in V^*$ given by

$$L_v(w)=<v,w> \quad for\ all\ w\ inV$$

   Note. $L_v(w+w)=<w_1+w_2,v>=<w_1,v> + <w_2,v>=L_v(w_1)+L_v(w_2)$

$$L_v(\lambda w)=<\lambda w,V>=\lambda<W,V>=\lambda L_v<w>$$

   Theorem:Let V= a finite dimensional inner product spaces. Let $\{v_1,v_2,v_3,\dots,v_n\}$ be b asis for V then $\{L_{v_1},L_{v_2},L_{v_3},\dots,L_{v_n}\}$ is basis of V^*

   Proof:let

Examples of dual spaces:

1. π_i:$R^n \to R$ be projection of ith component define by $\pi_i(a_1, a_2, a_3, \dots, a_n) = a_i$. then π_i is linear so it linear funtional on R^n . for every $\gamma_1, \gamma_2 \in IR$ and $u, v \in R^n$
   Solution:
2. Let $V = M_{nxn}$ matrices over K (field) $T: V \to K$ defined by T(A)= trace(A) where $A \in M_{nxn}$
   T is linear so it is Linear functional on V
3. $\phi: R^n \to R$ by $\phi(x_1, x_2, \dots x_n) = a_1x_1 + a_2x_2 + \cdots a_nx_n$ is linear functional on R^n

Theorem:dual basis. Suppose $\{v_1, v_2, v_3, \dots v_n\}$ is basis for V^* be linear functional defined by $\phi_i(v_j) = \delta_{ij} = \begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases}$ then $\{\varphi_1, \varphi_2, \varphi_3, \dots \varphi_n\}$ is basis for V^*

Proof:

Example or exercise :dual basis

1. **Consider basis of R^2:** $\{\boldsymbol{v1} = (\mathbf{2}, \mathbf{1}), \boldsymbol{v2} = (\mathbf{3}, \mathbf{1})\}$**Define** $\varphi_1(x, y) = ax + by$ and $\varphi_2(x, y) = cx + dy$ find dual basis by the above theorem
2. **Let** $\phi_1: R^2 \to R$ $and \phi_2: R^2 \to R$ be linear functional define by $\phi_1(x, y) = x + 2y$ $\phi_2(x, y) = 3x - y$ $find$ $\phi_1 + \phi_2$;and $3\phi_1 + 5\phi_2$
3. Given basis for R^3 $\{v1 = (1, -1, 3), v2 = (0, 1, -1), (0, 3, -2)\}$find dual basis $\{\varphi_1, \varphi_2, \varphi_3\}$
4. Let V be a vector spaces of polynomial over R of deegree≤ 2 let $\varphi_1, \varphi_2 and$ φ_3 be linear functional on V defined by

$$\varphi_1(f(t)) = \int_0^2 f(t)dt$$

$\varphi_2(f(t)) = f'(1)$ and $\varphi_3(f(t)) = f(1)$ here f$(t) = a + bt + ct^2 \in V$ find basis

$\{f(t)_1, f(t)_2, f(t)_3\}$ of V which is dual to $\{\varphi_1, \varphi_2, \varphi_3\}$

2.7 Ad joint of linear operators

Definition: let $V =$ a finite dimensional inner product spaces. Let T= a linear operator on V then there exist a unique liear operator T^* on V such that

$< T(v), w > \; = \; < v, T^*(w) >$ for all $u, w \in V$. T^* is called ad joint of T

Theorem: let $V =\ a$ finite dimensional inner product spaces. Let T,S= a linear operator on V.let $\lambda \in K[field]$ then

i. $(T + S)^* = T^* + S^*$
ii. $(\lambda T)^\wedge * = \bar{\lambda} T^*$
iii. $(ToS)^* = S^* oT^*$
iv. $(T^*)^* = T$

Proof:

Example: Let $T: R^3 \longrightarrow R^3$ be defined by $T(v) = (x + 2y, 3x - 4z, y)$ clearly T is linear operator on R^3.find $T^*(x, y, z)$

solution:

$T(1,0,0) = (1,3,0)$, $T(0,1,0) = (2,0,1)$ *and* $T(0,0,1) = (0, -4,0)$

hence A=[T]= $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & -4 \\ 0 & 1 & 0 \end{bmatrix}$ and $A^* = [T^*] = A^T = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & -4 & 0 \end{bmatrix}$

$$T^*(x, y, z) = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x + 3y, 2x + z, -4y)$$

Examples 2;let $T: R^n \longrightarrow R^n$ by $T(v) = Av$ and $T^*(w) = Bw$ A and B are matrix ,$v, w \in V = R^n$.Define inner product

$< T(V), w > = (Av)w = vA^Tw$ and $< v, T^*(w) > = v(Bw)$. If what T^* is ad joint Linear operator on V

Solution: clearly T is linear operator on V . T^* is ad joint Linear operator on V by definition if

$< T(v), w > = (Av)w = vA^Tw = < v, T^*(w) > = v(Bw)$.

$$\Longrightarrow B = A^t$$

2.8 Self-ad joint linear operators

Definition: let $V = \ a\,finite$ dimensional inner product spaces. A linear operator T on V is called self-ad joint linear operator if $T^* = T$

Note: If V is Euclidean spaces and T is a self-ad joint linear operator on V then T is called symmetric

Theorem:let $V = \ a\,finite$ dimensional inner product spaces. A linear operator T on V is self-ad joint linear operator on V then

i. Each eigen values of T is real
ii. Eigen vector of T associated with distance Eigen values are orthogonal
Proof:

Example: Self-ad joint linear operators:

let $T: R^n \longrightarrow R^n$ by $T(v) = Av$,A are matrix ,$v, w \in V = R^n$.Define inner product

$< T(V), w > = (Av)w = vA^T w$ and $< v, T^*(w) > = v(Aw)$ and T* is self adjoint then A is symmetric matrix

Solution: $< T(V), w > = (Av)w = vA^T w \ = < v, T^*(w) > = v(Aw)$ by definition.

$(Av)w = vA^T w \ = v(Aw) \ \Longrightarrow A = A^t$ hence A is symmetric matrix

2.9 Isometric

Definition: let $V = \ a\, finite$ dimensional inner product spaces and T is linear operator on V. the following are equivalent:

i. $\boldsymbol{T^* = T^{-1}}$
ii. $\boldsymbol{T}$ **preserves inner products I i.e**$\boldsymbol{< T(v), T(w) > \ = < v, w >}$
iii. $\boldsymbol{T}$ **preserves length .i.e.**$\boldsymbol{\|T(v)\| = \|v\| \ \forall v \in V}$
   $\boldsymbol{T}$**is called an isometric if it satisfies any of the three equivalent conditions**

Theorem: let $V = \ a\, finite$ dimensional inner product spaces and T is linear operator on V. let $\beta = \{v_1, v_2, v_3, \dots v_n\}$ be orthonormal basis of V let $A = (a_{ij})_{nxn} = [T)_\beta$ be the matrix of T w.r.t β then $(a_{ij}) =< v_j, v_i >$

Proof:

Theorem: let $V =\ a\,finite$ dimensional inner product spaces and T is linear operator on V. let $A = (a_{ij})_{nxn} = [T]_\beta$ be the matrix of T w.r.t orthonormal basis then T is isomeric iff $A^* = A^{-1}$

Proof:

Examples: rotation in $\boldsymbol{R^2}$ *and* $\boldsymbol{R^3}$

1. **Let** Let $T: R^2 \longrightarrow R^2$ be defined by $T(v) = Av = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \theta \in [0,2\pi]$ then clearly T is linear operator and isometry. Show?

2. $\boldsymbol{Rotation\ matrix\ in\ R^3}$,

i. $\boldsymbol{R_X(\alpha)} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{COS\alpha} & \boldsymbol{sin\alpha} \\ \mathbf{0} & \boldsymbol{-sin\alpha} & \boldsymbol{cos\alpha} \end{bmatrix}$

ii. $\boldsymbol{R_y(\beta)} = \begin{bmatrix} \boldsymbol{cos\beta} & \mathbf{0} & \boldsymbol{-sin\beta} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \boldsymbol{sin\beta} & \mathbf{0} & \boldsymbol{cos\beta} \end{bmatrix}$

iii. $\boldsymbol{R_z(\gamma)} = \begin{bmatrix} \boldsymbol{cos\gamma} & \boldsymbol{-sin\gamma} & \mathbf{0} \\ \boldsymbol{sin\gamma} & \boldsymbol{COS\gamma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

$\boldsymbol{define}\ T(v) = (\boldsymbol{R_X(\alpha)})v$,
$T(v) = (\boldsymbol{R_y(\beta)})v$ and $T(v) = \left(\boldsymbol{R_z(\gamma)}\right)v$ is linear operator and isometric?. Show

2.10 Normal operators

Definition: let $V = a\, finite$ dimensional inner product spaces and T is linear operator on V.

T is called normal operator if $TT^* = T^*T$

Theorem: let $V = a\, finite$ dimensional inner product spaces and T is normal linear operator on V then for any $\lambda \in K, T - \lambda I$ is normal operator

Proof:

Examples of normal operator.

1. **Let** Let $T: R^2 \longrightarrow R^2$ be defined by $T(v) = Av = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \theta \in [0,2\pi]$ and θ is fixed then $TT^* = T^*T$ .show?

   Exercise. Are the following are normal operator ?

$$\boldsymbol{Rotation\ matrix\ in\ R^3},$$

$$\boldsymbol{i)\ R_X(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & COS\alpha & sin\alpha \\ 0 & -sin\alpha & cos\alpha \end{bmatrix}}$$

$$\boldsymbol{ii) R_y(\beta) = \begin{bmatrix} cos\beta & 0 & -sin\beta \\ 0 & 1 & 0 \\ sin\beta & 0 & cos\beta \end{bmatrix}}$$

$$\boldsymbol{R_z(\gamma) = \begin{bmatrix} cos\gamma & -sin\gamma & 0 \\ sin\gamma & COS\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\boldsymbol{define}$ $T(v) = (\boldsymbol{R_X(\alpha)})v$,
$T(v) = (\boldsymbol{R_y(\beta)})v$ and $T(v) = \left(\boldsymbol{R_z(\gamma)}\right)v$. $v \in R^3$

Definition: let V be avector spaces over a afield K .T= a .l.o. on V, W= subspaces of V we say that w is T in variant if for each . $w \in W$ the vector $T(w) \subseteq W$

Theorem: let V= an inner product spaces .T= a.l.o. on V. W= a T invariant subspaces of V then $W^{\perp}$ is T^{*} invariant

Proof: let. $w \in W and$. $v \in W^{\perp}$ then $T(w) \subseteq W$

$< w, T^{*}(v) =< T(w), v > = 0 \Rightarrow w \perp T^{*}(v) \Rightarrow T^{*}(v) \in W^{\perp}$

Hence $W^{\perp}$ is T^{*} invariant.

Worksheet# 2

1. Compute inner product of the following vectors

a) $u = \begin{pmatrix} 3 & -2 \\ 4 & 8 \end{pmatrix}, v = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$ use Euclidean inner product on square matrix
b) $p = 3 - x + 2x^3 + 3x^2, q = 2 + 3x - 4x^2 + 4x^3$ such that $a = 2, b = 3, c = 4, d = 5$
c) $compte\ norm\ of\ p\ and\ q\ using\ standard\ and\ evaluation\ inner\ prduct\ on\ p_n$

2. Use inner product defined by $< f, g > = \int_0^1 f(x)g(x)dx$. Compute inner product for the following function

a) $f(x) = \cos(2\pi x)\ , and\ g(x) = \sin(2\pi x);$
b) $f(x) = 3 - x + 2x^3 + 3x^2 and\ g(x) = 2 + 3x - 4x^2 + 4x^3$

3. find a basis for orthogonal complement of the subspace of R^n spanned by the vectors

a) $A = (2,1,3)\ ; b = (-1,-4,2); c = (4,-5,13)$
b) $A = (0,2,1), b = (4,0,-3); c = (6,-1,4)$
c) $A = (3,0,1,-2); b = (-1,-2,-2,1); c = (4,2,3,-3)$
d) $A = (1,4,5,6,9); b = (3,-2,1,4,-1), c = (-1,0,-1,-2,-1); d = (2,3,5,7,8)$

4. let the vector space p_3 have inner product $< p, q > = \int_{-1}^{1} p(x)q(x)dx$ apply grams Schmidt process to transform the standard basis$\{1, x, x^2, x^3\}$for p_3 into orthogonal basis

5. Verify that the vectors $\boldsymbol{a} = (\mathbf{1}, -\mathbf{1}, \mathbf{2}, -\mathbf{1}); \boldsymbol{b} = (-\mathbf{2}, \mathbf{2}, \mathbf{3}, \mathbf{2}); \boldsymbol{c} = (\mathbf{1}, \mathbf{2}, \mathbf{0}, -\mathbf{1}); \boldsymbol{d} = (\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1})$ form an orthogonal basis $for R^4$ with Euclidean inner product. Then express each of the following vectors as linear combination of $a, b, c\&, d$ and find coordinate vectors for each vector's

i. $(1,-1,3,5)$
ii. $(3,4,2,6)$
iii. $(2,4,6,3)$
iv. $(2,2,3,3)$
v. $(-2,-3,4,5)$
vi. $(1,3,4,5)$
vii. $(0,3,-2,-3)$
viii. $(2,0,-3,6)$
ix. $(-5,-4,2,1)$
x. $(7,3,1,-3)$
xi. $(2,0,-3,6)$
xii. $(-5,-4,2,1)$
xiii. $(7,3,1,-3)$
xiv. $(2,0,-3,6)$

xv. $(-5,-4,2,1)$ xvi. $(7,3,1,-3)$

6. from the Q5. If w is subspace spanned by the vectors of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \&, \boldsymbol{d}$ find projection of each vector on w

7. Find the orthogonal projection of **u** on subspace of R^4 *spanned by* $\boldsymbol{a}, \boldsymbol{b}$ ***and c***

a) $U = (1,-1,3,1); a = (1,2,1,1), b = (0,1,1,0), c = (2,1,2,1)$
b) $u = (-2,0,2,4); a = (1,1,3,0), b = (-2,-1,-2,1), c = (-3,-1,1,3$

8 Let $\phi_1 : R^2 \to R$ $and \phi_2 : R^2 \to R$ be linear functional define by $\phi_1(x, y) = x + 2y$

$\phi_2(x, y) = 3x - y$ $find$ $\phi_1 + \phi_2$;and $3\phi_1 + 5\phi_2$

9 Given basis for R^3 $\{v1 = (1,-1,3), v2 = (0,1,-1), (0,3,-2)\}$find dual basis $\{\varphi_1, \varphi_2, \varphi_3\}$

10Let V be a vector spaces of polynomial over R of deegree≤ 2 let φ_1, φ_2 and φ_3 be linear functional on V defined by

$$\varphi_1(f(t)) = \int_0^2 f(t)dt$$

$\varphi_2(f(t)) = f'(1)$ and $\varphi_3(f(t)) = f(1)$ here f$(t) = a + bt + ct^2 \in V$ find basis $\{f(t)_1, f(t)_2, f(t)_3\}$ of V which is dual to $\{\varphi_1, \varphi_2, \varphi_3\}$