

ARBA MINCH UNIVERSITY
COLLEGE OF BUSINESS AND ECONOMICS
DEPARTMENT OF MANAGEMENT



OPERATION RESEARCH (MGMT-2112)

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COURSE DESCRIPTION

Operations Research (OR) as one of the quantitative aids to decision making offers the decision-maker a method of evaluating every possible alternative (act or course of action) by using various techniques to know the potential outcomes. It includes Models and modeling, linear programming, transportation models, assignment models, decision models, project management techniques, and game theory. Operations research quantifies the relevant factors of an issue and uses mathematical techniques to arrive at an optimal decision.

Dear students, Because of Operation Research's multidisciplinary character and application in varied fields, it has a splendid future, provided people dedicated to Operation Research study can meet the needs of society. Operation Research can contribute towards the improvement in the social life and ranges of worldwide need. The aim of this course is to make you a better decision maker by learning the decision-making process. Decision-making is a complicated process. This complication arises from the fact that your present goal (including wants, resources, and abilities) dictates your choices; however, your choices will change your goals. This influential-cycle keeps the decision-maker busy all the time. Selecting your goals and your criteria for success is a dynamic process and changes over time.

On a daily basis a manager has to make many decisions. Some of these decisions are routine and inconsequential, while others have drastic impacts on the operations of the firm for which he/she works. Some of these decisions could involve large sums of money being gained or lost, or could involve whether or not the firm accomplishes its mission and its goals. In our increasingly complex world, the tasks of decision-makers are becoming more challenging with each passing day. The decision-maker (i.e., the responsible manager) must respond quickly to events that seem to take place at an ever-increasing pace. In addition, a decision-maker must incorporate a sometimes-bewildering array of choices and consequences into his or her decision. Routine decisions are often made quickly, perhaps unconsciously without the need for a detailed process of consideration. However, for complex, critical or important managerial decisions it is necessary to take time to decide systematically. Making strategic decisions requires that one takes a structured approach following a formal decision-making process. Otherwise, it will be difficult to be sure that one has considered all the key aspects of the decision.

OBJECTIVES OF THE COURSE

At the end of this course students will be able to:



Understand the significance of operation research in managerial decision making



Understand the different models of operation research



Solving different quantitative problems using operations research models

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CHAPTER ONE

INTRODUCTION TO OPERATIONAL RESEARCH

1.1 Introduction

Many people still remain in the bondage of self-incurred tutelage. Tutelage is a person's inability to make his/her own decisions. Self-incurred is this tutelage when its cause lies not in lack of reason but in lack of resolution and courage to use it without wishing to have been told what to do by something or somebody else. The difficulty in life is the choice. Good decision-making brings about a better life. A bad decision may force you to make another one. A good decision is never an accident; it is always the result of high intention, sincere effort, intelligent direction and skillful execution; it represents the wise choice of many alternatives. One must appreciate the difference between a decision and an objective. A good decision is the process of optimally achieving a given objective.

When decision making is too complex or the interests at stake are too important, quite often we do not know or are not sure what to decide. In many instances, we resort to informal decision support techniques such as tossing a coin, asking an oracle, visiting an astrologer, etc. However formal decision support from an expert has many advantages. Business Science focuses on the formal model-driven decision support techniques such as mathematical programs for optimization, and decision tree analysis for risky decisions. Such techniques are now part of our everyday life. For example, when a bank must decide whether a given client will obtain credit or not, a technique, called credit scoring, is often used. Rational decisions are often made unwillingly, perhaps unconsciously. We may start the process of consideration. It is best to learn the decision-making process for complex, important and critical decisions.

1.2 Objectives of The Chapter

At the end of this chapter students will be able to:



Define the term operations research



Explain the historical development of operations research



Understand the nature and characteristics of operations research



Explain the significance of operations research to managers



Identify the operation research problem solving approach



Understand the meaning of model and its different classifications

1.3. Definition of Operation Research

Operations Research is defined by many authors in different ways however, the following definition is proposed here.

“**Operations Research** is a systematic analysis of a problem through scientific methods, carried out by appropriate specialists, working together as a team, constituted at the instance of management for the purpose of finding an optimum and the most appropriate solution, to meet the given objective under a given set constraints.”

“Operation **research** is the representation of real-world system by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with the view of optimizing.

From the concepts and definitions given above, Operations Research is:

1. The application of scientific methods, techniques and tools to the problem to find an answer
2. A management tool in the hands of a manager to take a decision
3. A scientific approach to decision making process
4. An “**applied research**” aims at finding a solution for an immediate problem facing a society, industry or a business enterprise. This is not “**fundamental research**”
5. A decision-oriented research, using scientific methods, for providing management a quantitative basis for taking decision regarding operations under its control
6. Applied decision theory. It uses scientific, mathematical and logical means to take decisions.

1.4 History of Operations Research

OR is a relatively new discipline Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study OR, indeed the term OR did not exist then. It was only really in the late 1930's that operational research began in a systematic fashion, and it started in the UK.

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the Centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained. It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defense of Britain, was first created. It lacked however any effective fighter aircraft – no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and control system. It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground-based observer data for fighter interception, was the start of OR.

The first of three major pre-war air-defense exercises was carried out in the summer of 1937. The experimental radar station at Bawdsey Research Station was brought into operation and the information derived from it was fed into the general air-defense warning and control system. From the early warning point of view this exercise was encouraging, but the tracking information obtained from radar, after filtering and transmission through the control and display network, was not very satisfactory.

In July 1938 a second major air-defense exercise was carried out. Four additional radar stations had been installed along the coast and it was hoped that Britain now had an aircraft location and control system greatly improved both in coverage and effectiveness. Not so! The exercise revealed, rather, that a new and serious problem had arisen. This was the need to coordinate and correlate the additional, and often conflicting, information received from the additional radar stations. With the outbreak of war apparently imminent, it was obvious that something new - drastic if necessary - had to be attempted. Some new approach was needed. Accordingly, on the termination of the

exercise, the Superintendent of Bawdsey Research Station, A.P. Rowe, announced that although the exercise had again demonstrated the technical feasibility of the radar system for detecting aircraft, its operational achievements still fell far short of requirements. He therefore proposed that a crash program of research into the operational - as opposed to the technical - aspects of the system should begin immediately. The term "operational research" [RESEARCH into (military) OPERATIONS] was coined as a suitable description of this new branch of applied science. The first team was selected from amongst the scientists of the radar research group the same day.

In the summer of 1939 Britain held what was to be its last pre-war air defense exercise. It involved some 33,000 men, 1,300 aircraft, 110 antiaircraft guns, 700 searchlights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defense warning and control system. The contribution made by the OR teams was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore. On May 15th 1940, with German forces advancing rapidly in France, Stanmore Research Section was asked to analyze a French request for ten additional fighter squadrons (12 aircraft a squadron) when losses were running at some three squadrons every two days. They prepared graphs for Winston Churchill (the British Prime Minister of the time), based upon a study of current daily losses and replacement rates, indicating how rapidly such a move would deplete fighter strength. No aircraft were sent and most of those currently in France were recalled. This is held by some to be the most strategic contribution to the course of the war made by OR (as the aircraft and pilots saved were consequently available for the successful air defense of Britain, the Battle of Britain).

In 1941 an Operational Research Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II. Although scientists had (plainly) been involved in the hardware side of warfare (designing better planes, bombs, tanks, etc) scientific analysis of the operational use of military resources had never taken place in a systematic fashion before the Second World War. Military personnel, often by no means stupid, were simply not trained to undertake such analysis. These early OR workers came from many different disciplines, one group consisted of a physicist, two physiologists, two mathematical physicists and a surveyor. What such people brought to their work were "scientifically trained" minds, used to querying assumptions, logic, exploring hypotheses, devising experiments,

collecting data, analyzing numbers, etc. Many too were of high intellectual caliber (at least four wartime OR personnel were later to win Nobel prizes when they returned to their peacetime disciplines).

By the end of the war OR was well established in the armed services both in the UK and in the USA. OR started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations, the objective was to find the most effective utilization of limited military resources by the use of quantitative techniques. Following the end of the war OR spread, although it spread in different ways in the UK and USA. You should be clear that the growth of OR since it began (and especially in the last 30 years) is, to a large extent, the result of the increasing power and widespread availability of computers. Most (though not all) OR involves carrying out a large number of numeric calculations. Without computers this would simply not be possible.

1.5. The Methodology of Operation Research

When OR is used to solve a problem of an organization, the following seven step procedure should be followed:

Step 1: Formulate the Problem

OR analyst first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

Step 2: Observe the System

Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step 4) a mathematical model of the organization's problem.

Step 3: Formulate a Mathematical Model of the Problem

The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

Step 4: Verify the Model and Use the Model for Prediction

The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

Step 5: Select a Suitable Alternative

Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives. Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

Step 6: Present the Results and Conclusions of the Study

In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the decision maker(s) choose the one that best meets her/his/their needs. After presenting the results of the OR study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or 3.

Step 7: Implement and Evaluate Recommendation

If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.



Self-check exercise 1.1

1. What is operations research?
2. What are the main features of operations research?
3. What are basic causes for the rapid development of operations research?

1.6 Operational Research Tools and Techniques

Operations Research uses any suitable tools or techniques available. The common frequently used tools/techniques are mathematical procedures, cost analysis, electronic computation. However, operations researchers given special importance to the development and the use of techniques like linear programming, game theory, decision theory, queuing theory, inventory models and simulation. In addition to the above techniques, some other common tools are non-linear programming, integer programming, dynamic programming, sequencing theory, Markov process, network scheduling (PERT/CPM), symbolic Model, information theory, and value theory. There are many other Operations Research tools/techniques also exists. The brief explanations of some of the above techniques/tools are as follows:

Linear Programming

This is a constrained optimization technique, which optimize some criterion within some constraints. In Linear programming the objective function (profit, loss or return on investment) and constraints are linear. There are different methods available to solve linear programming.

Game Theory

This is used for making decisions under conflicting situations where there are one or more players/opponents. In this the motive of the players are dichotomized. The success of one player tends to be at the cost of other players and hence they are in conflict.

Decision Theory

Decision theory is concerned with making decisions under conditions of complete certainty about the future outcomes and under conditions such that we can make some probability about what will happen in future.

Queuing Theory

This is used in situations where the queue is formed (for example customers waiting for service, aircrafts waiting for landing, jobs waiting for processing in the computer system, etc). The objective here is minimizing the cost of waiting without increasing the cost of servicing.

Inventory Models

Inventory model make a decision that minimize total inventory cost. This model successfully reduces the total cost of purchasing, carrying, and out of stock inventory.

Markov Process

Markov process permits to predict changes over time information about the behavior of a system is known. This is used in decision making in situations where the various states are defined. The probability from one state to another state is known and depends on the current state and is independent of how we have arrived at that particular state.

Network Scheduling:

This technique is used extensively to plan, schedule, and monitor large projects (for example computer system installation, R & D design, construction, maintenance, etc.). The aim of this technique is minimizing trouble spots (such as delays, interruption, production bottlenecks, etc.) by identifying the critical factors. The different activities and their relationships of the entire project are represented diagrammatically with the help of networks and arrows, which is used for identifying critical activities and path. There are two main types of technique in network scheduling, they are:

Program Evaluation and Review Technique (PERT) – is used when activities time is not known accurately/ only probabilistic estimate of time is available.

Critical Path Method (CPM) – is used when activities time is known accurately.

1.7 Applications of Operations Research

Operation Research is mainly concerned with the techniques of applying scientific knowledge, besides the development of science. It provides an understanding which gives the expert/manager new insights and capabilities to determine better solutions in his decision-making problems, with great speed, competence and confidence. In recent years, O.R. has successfully entered many

different areas of research in Defense, Government, Service Organizations and Industry. Some applications of O.R. in the functional areas of management are briefly presented here under.

Finance, Budgeting, and Investment

- i) Cash flow analysis, long range capital requirements, dividend policies, investment portfolios.
- ii) Credit policies, credit risks and adequate account procedures.
- iii) Claim and complaint procedure.

Marketing

- i) Product selection, timing, comparative actions.
- ii) Advertising media with respect to cost and time.
- iii) Number of salesmen, frequency of calling of account, etc.
- iv) Effectiveness of market research.

Physical Distribution

- i) Location and size of warehouses, distribution centers, retail outlets, etc.
- ii) Distribution policy.

Purchasing, Procurement and Exploration

- i) Rules of buying.
- ii) Determine the quality and timing of purchase.
- iii) Bidding policies and vendor analysis.
- iv) Equipment replacement policies.

Personnel

- i) Forecasting the manpower requirement, recruitment policies and job assignments.
- ii) Selection of suitable personnel with due consideration for age and skills, etc.
- iii) Determination of optimum number of persons for each service center.

Production

- i) Scheduling and sequencing the production run by proper allocation of machines.
- ii) Calculating the optimum product mix.
- iii) Selection, location and design of sites for the production plant.

Research and Development

- i) Reliability and evaluation of alternative designs.
- ii) Control of developed projects.
- iii) Co-ordination of multiple research projects.

iv) Determination of time and cost requirements.

Besides the above-mentioned applications of O.R. in the context of modern management, its use has now extended to a wide range of problems, such as the problems of communication and information, socio-economic fields and national planning.

1.8 Opportunities and Shortcomings of Operation Research

The use of O.R. to improve decision-making has become almost universal today. However, this remarkable achievement of O.R. is not totally free from shortcomings. Certain shortcomings result from lack of awareness on the part of managers about their roles, while others result due to avoidance of the behavioral and organizational issues which are a part of every successful application.

1.8.1 Opportunities of Operational Research

1. Using O.R. approach, the decision-maker can determine a solution to his routine or repetitive problem. For obtaining solution of such type of problems, it is necessary to build a model so that future solution can be obtained using the model thus freeing managers to concentrate on more pressing matters.
2. O.R. requires business managers to be quite explicit about their objective, their assumptions and the way of visualizing the constraints.
3. While using O.R. approach, a manager has to consider very carefully all those variables which influence his decisions and the way these variables in a problem interact with each other. He then selects a decision which is best for the organization as whole.
4. O.R. approach allows a decision-maker to solve a complex problem involving multiple variables much more quickly than if he had to compute them using traditional methods. Sometimes it may not be possible to solve such complex problems without using O.R. methods.
5. Using O.R. approach, a decision-maker can examine a situation from various angles by simulating the model which he has constructed for the real problem. He can change various conditions under which decisions are being made, and examine the effect of these changes through appropriate experiments on the model, to determine the best or optimal solution for the problem under consideration. All these experiments can be carried out without causing any series damage to the existing system or incurring excessive cost.

1.8.2 Shortcomings of operational Research

- 1) There are certain problems which a decision-maker may have to solve only once. Constructing a complex O.R. model for solving such problems is often too expensive when compared with the cost of other less sophisticated approaches available to solve them.
- 2) Sometimes O.R. specialists become too much enamored with the model they have built and forget the fact that their model does not represent the “real world problem” in which decisions have to be made.
- 3) Sometimes the basic data are subject to frequent changes. In such cases, modification of O.R. models are a costly affair.
- 4) Many O.R. models are so complex that they cannot be solved without the use of computer. Also, the solutions obtained from these models are difficult to explain to managers and hence fail to gain their support and confidence.
- 5) Magnitude of computation involved, lack of consideration for non-quantifiable factors and psychological issues involved in implementation are some of the other shortcomings of O.R.

CHAPTER SUMMARY

In our increasing complex world, the tasks of decision-makers are becoming more challenging with each passing day. The decision-maker is expected to respond quickly to events that seem to take place at an ever-increasing pace. Operations research attempts to resolve the conflicts of interest among various sections of the organization and seeks the optimal solution which may not be acceptable to one department but is in the interest of the organization as a whole.

It is generally agreed that OR came into existence as a discipline during the second world war when there was a critical need to manage scarce resources. OR has interdisciplinary, methodological, wholistic and objectivistic approach for problem solving. OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing.

The operation research problem solving approach involves five major steps: observation, definition of the problem, model construction and implementation of solution results.

The approximation or abstraction, maintaining only the essential elements of the system, which may be constructed with various forms by establishing relationships among specified variables and parameters of the system, is called a model.

A model is constructed to analyze and understand the given system for the purpose of improving its performance. The reliability of the solution obtained from a model depends on the validity of the model in representing the system under study. There are different classifications of models which are constructed to represent and resolve real world business problem.

CHAPTER TWO

LINEAR PROGRAMMING

2.1 Introduction

Linear programming is one of the most vital and requisite quantitative technique for decision making. The difference between the rich and the poor is the quality of the decisions they make. Those who make good decisions are becoming effective and efficient as the result they are becoming more and more affluent; those who make poor decisions are less and less effective and efficient as a result they are becoming worse and worse. As a manager we always make decisions that have significant bearing on the success of the organization we manage. Hence, we need to know how to make good and informed decisions. Moreover, given complete and certain information managers need to know how to maximize the return they get or how to minimize the cost associated with each and every decision they make. Therefore, student of Business needs to know technique of making a better decision.

2.2 Objectives of the Chapter

After completing this chapter students will be able to:

- ☞ Define what linear programming is
- ☞ Identify application areas of linear programming
- ☞ Formulate linear programming model
- ☞ Solve linear programming model using graphical method
- ☞ Solve linear programming model using simplex method.
- ☞ Formulate dual of a primal model and read solution of primal from dual and vice versa.

2.3 Overview of Linear Programming

Every organization faces the problem of allocation of resources. The resources include men, machine, material, capital, information etc. most of these decisions are made subject to constraints. For example, production from factory is limited due to capacity constraints, organizations face working capital constraint, technical constraints, demand constraint, and others. If the decision is

to be made after long time, some of these resources can be expanded; fund can be raised, capacity can be added, additional resources can be acquired. However, when the decision making is short term the resources cannot be expanded. Therefore, the decision is to go for what is possible. The best solution cannot be achieved, but given the constraint (limitation) the decision makers try to optimize (maximize or minimize) the objective.

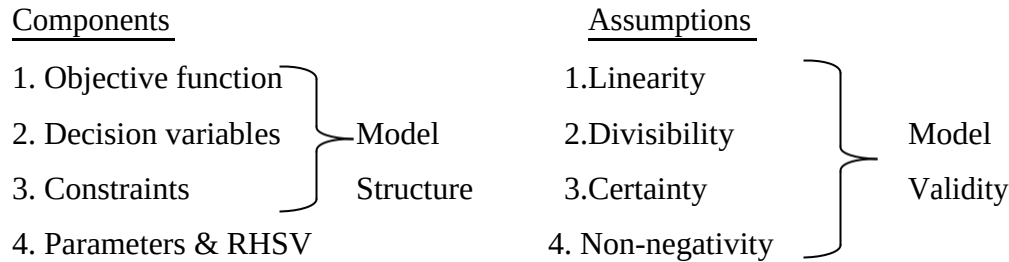
Linear Programming- is an optimization method, which shows how to allocate scarce resources such as money, materials or time and how to do such allocation in the best possible way subject to more than one limiting condition expressed in the form of inequalities and/or equations. It enables users to find optimal solution to certain problems in which the solution must satisfy a given set of requirements or constraints.

Optimization in linear programming implies either maximization (such as profit, revenue, sales, and market share) or minimization (such as cost, time, and distance) a certain objective function. It implies that in LP we cannot max/min two quantities in one model. It involves linearly related multi-variate functions, i.e., functions with more than one independent variable. The goal in linear programming is to find the best solution given the constraints imposed by the problem; hence the term constrained optimization.

2.3. Linear Programming Models (LPM)

Linear Programming (LP) models are mathematical representations of LP problems. Some LP models have a specialized format, whereas others have a more generalized format. Despite this, LP Models have certain characteristics in common. Knowledge of these characteristics enables us to recognize problems that are amenable to a solution using LP models and to correctly formulate an LP model. The characteristics can be grouped into two categories:

Components and **Assumptions**. The *components* relate to the *structure* of a model, whereas the *assumptions* reveal the *conditions under which the model is valid*.



2.3.1 Components of LP Model

1. **The Objective Function-** is the mathematical or quantitative expression of the objective of the company/model. The objective in problem solving is the criterion by which all decisions are evaluated. In LPs a *single* quantifiable objective must be specified by the decision maker. For example, the objective might relate to profits, or costs, or market share, but to only one of these. Moreover, because we are dealing with optimization, the objective will be either maximization or minimization, but not both at a time.

2. **The Decision Variables** - represent unknown quantities to be resolved for. These decision variables may represent such things as the number of units of different products to be sold, the amount of Birr to be invested in various projects, the number of ads to be placed with different media. Since the decision maker has freedom of choice among actions, these decision variables are *controllable variables*.

3. **The constraints** - are restrictions which define or limit the feasibility of a proposed course of action. They limit the degree to which the objective can be pursued. Atypical restriction embodies *scarce resources* (such as labor supply, raw materials, production capacity, machine time, storage space), *legal or contractual requirements* (e.g. Product standards, work standards), or they may reflect other limits based on forecasts, customer orders, company policies etc.

4. **Parameters** - are fixed values that specify the impact that one unit of each decision variable will have on the objective and on any constraint, it pertains to as well as to the numerical value of each constraint.

➤ The components are the building blocks of an LP model. We can better understand their meaning by examining a simple LP model as follows.

Example:

Maximize: $4X_1 + 7X_2 + 5X_3$ (Profit) _____ objective function

Subject to:

$2X_1 + 3X_2 + 6X_3 \leq 300$ labor hrs

$5X_1 + X_2 + 2X_3 \leq 200$ lb raw material A } System constraint

$3X_1 + 5X_2 + 2X_3 \leq 360$

$X_1 = 30$ } Individual
 $X_2 = 40$ } constraints

$X_1, X_2, X_3 \geq 0 \rightarrow$ Non-negativity constraints.

System constraints – involve more than one decision variables.

Individual constraints – involve only one decision variable.

Non-negativity constraints specify that no variable will be allowed to take on a negative value. The non-negativity constraints typically apply in an LP model, whether they are explicitly stated or not.

2.3.2 Assumption of LP Models

1. **Linearity.** The linearity requirement is that each decision variable has a linear impact on the objective function and in each constraint in which it appears. Following the above example, producing one more unit of product 1 adds Br. 4 to the total profit. This is true over the entire range of possible values of X_1 . The same applies to each of the constraints. It is required that the same coefficient (for example, 2 lb. per unit) apply over the entire range of possible value so the decision variable.
2. **Divisibility.** The divisibility requirement pertains to potential values of decision variables. It is assumed that non-integer values are acceptable. For example: 3.5 TV sets/hr would be acceptable $\rightarrow$ 7 TV sets/2hrs.
3. **Certainty.** The certainty requirement involves two aspects of LP models.
 - i) With respect to model parameters (i.e., the numerical values) – It is assumed that these values are known and constant e.g. in the above example each unit of product 1 requires 2 lab his is

known and remain constant, and also the 300 lab/hr available is deemed to be known and constant.

ii) All the relevant constraints identified and represented in the model are as they are.

4. **Non-negativity.** The non-negativity constraint is that negative values of variables are unrealistic and, therefore, will not be considered in any potential solution; only positive values and zero will be allowed.

2.3.3 Formulating LP Models

Once a problem has been defined, the attention of the analyst shifts to formulating a model. Just as it is important to carefully define a problem, it is important to carefully formulate the model that will be used to solve the problem. If the LP model is ill formulated, ill-structured, it can easily lead to poor decisions.

Formulating linear programming models involves the following steps:

1. Define the problem/problem definition
2. Identify the decision variables or represent unknown quantities
3. Determine the objective function

Once the variables have been identified, the objective function can be specified. It is necessary to decide if the problem is maximization or a minimization problem and the coefficients of each decision variable.

Note:

- a. The units of all the coefficients in the objective function must be the same. E.g. If the contribution of type 1 is in terms of Br so does for type 2.
- b. All terms in the objective function must include a variable each term has to have 1 variable.
- c. All decision variables must be represented in the objective function.

4. Identifying the constraints

- System constraints
- Individual constraints
- Non-negative constraints

Examples

Example 1: A firm that assembles computer and computer equipment is about to start production of two new microcomputers. Each type of micro-computer will require assembly time, inspection time and storage space. The amount of each of these resources that can be devoted to the production of microcomputers is limited. The manager of the firm would like to determine the quantity of each microcomputer to produce in order to maximize the profit generated by sales of these microcomputers.

Additional information: In order to develop a suitable model of the problem, the manager has met with design and manufacturing personnel. As a result of these meetings, the manager has obtained the following information:

	Type 1	Type 2
Profit per unit	Birr 60	Birr 50
Assembly time per unit	4hrs	10hrs
Inspection time per unit	2hrs	1hr
Storage space per unit	3cubic ft	3cubic ft

The manager also has acquired information on the availability of company resources. These weekly amounts are:

Resource	Available Resource
Assembly time	100hrs
Inspection time	22hrs
Storage space	39 cubic feet

The manger also meets with the firm's marketing manager and learned that demand for the microcomputers was such that whatever combination of these two types of microcomputer is produced, all of the output can be sold.

Required: Formulate the Linear programming model.

Solution:

Step 1: Problem Definition

- To determine the number of two types of microcomputers to be produced (and sold) per week so as to maximize the weekly profit given the restriction.

Step 2: Variable Representation

- Let X_1 and X_2 be the weekly quantities of type 1 and type 2 microcomputers, respectively.

Step 3: Develop the Objective Function

$$\text{Maximize or } Z_{\max} = 60X_1 + 50X_2$$

Step 4: Constraint Identification

System constraints:	$4X_1 + 10X_2$	100hrs	Assembly time
	$2X_1 + X_2$	22hr	inspector time
	$3X_1 + 3X_2$	39 cubic feet	Storage space

Individual constraint: No

Non-negativity constraint: $X_1, X_2 \geq 0$

In summary, the mathematical model for the microcomputer problem is:

$$Z_{\max} = 60X_1 + 50X_2$$

$$\text{Subject to: } 4X_1 + 10X_2 \leq 100$$

$$2X_1 + X_2 \leq 22$$

$$X_1 + 3X_2 \leq 39$$

$$X_1, X_2 \geq 0$$

Example 2: An electronics firm produces three types of switching devices. Each type involves a two-step assembly operation. The assembly times are shown in the following table:

Assembly time per Unit (in minutes)

	Section #1	Section #2
Model A	2.5	3.0
Model B	1.8	1.6
Model C	2.0	2.2

Each workstation has a daily working time of 7.5 hrs. The manager wants to obtain the greatest possible profit during the next five working days. Model A yields a profit of Birr 8.25 per unit,

Model B a profit of Birr 7.50 per unit and Model C a profit of Birr 7.80 per unit. Assume that the firm can sell all it produces during this time, but it must fill outstanding orders for 20 units of each model type.

Required: Formulate the linear programming model of this problem.

Solution:

Step 1: Problem definition

To determine the number of three types of switching devices to be produced and sold for the next 5 working days so as to maximize the 5 days profit.

Step 2: Variable representation

Let X_1 , X_2 and X_3 be the number of Model A, B and C switching devices respectively, to be produced and sold.

Step 3: Develop objective function

$$Z \text{ max: } 8.25X_1 + 7.50X_2 + 7.80X_3$$

Step 4: Constraint identification

$2.5X_1 + 1.8X_2 + 2.0X_3$	2250 minutes	Ass. time station 1	}	System constraint
$3.0X_1 + 1.6X_2 + 2.2X_3$	2250 minutes	Ass. time station 2		
X_1	20	Model A	}	Individual constraint
X_2	20	Model B		
X_3	20	Model C		
X_1, X_2, X_3	0	Non negativity		

In summary:

$$Z_{\text{max:}} 8.25X_1 + 7.50X_2 + 7.80X_3$$

$2.5X_1 + 1.8X_2 + 2.0X_3$	2250	minutes
$3.0X_1 + 1.6X_2 + 2.2X_3$	2250	minutes
X_1	20	model A
X_2	20	model B
X_3	20	model C
X_1, X_2, X_3	0	non negativity

Example 3: A diet is to include at least 140 mgs of vitamin A and at least 145 Mgs of vitamin B. These requirements are to be obtained from two types of foods: Type 1 and Type 2. Type 1 food contains 10Mgs of vitamin A and 20mgs of vitamin B per pound. Type 2 food contains 30mgs of vitamin A and 15 mgs of vitamin B per pound. If type 1 and 2 foods cost Birr 5 and Birr 8 per pound respectively, how many pounds of each type should be purchased to satisfy the requirements at a minimum cost?

<u>Foods</u>	<u>Vitamins</u>	
	<u>A</u>	<u>B</u>
Type 1	10	20
Type 2	30	15

Solution:

Step 1. Problem definition

To determine the pounds of the two types of foods to be purchased to make the diet at a minimum possible cost within the requirements.

Step 2. Variable representation

Let X_1 and X_2 be the number of pounds of type 1 and type 2 foods to be purchased, respectively.

Step 3. Objective function

$$\text{Minz: } 5X_1 + 8X_2$$

4. Constraints

$$\left. \begin{array}{l} 10X_1 + 30X_2 \geq 140 \\ 20X_1 + 15X_2 \geq 145 \end{array} \right\} \text{System constraints}$$

$$X_1, X_2 \geq 0 \quad \text{non-negativity constraints.}$$

Example 4: A farm consists of 600 hectares of land of which 500 hectares will be planted with corn, barley and wheat, according to these conditions.

- (1) At least half of the planted hectare should be in corn.
- (2) No more than 200 hectares should be barley.
- (3) The ratio of corn to wheat planted should be 2:1

It costs Birr 20 per hectare to plant corn, Birr 15 per hectare to plant barley and Birr 12 per hectare to plant wheat.

- a. Formulate this problem as an LP model that will minimize planting cost while achieving the specified conditions.

Solution:

Step 1. Problem definition

To determine the number of hectares of land to be planted with corn, barley and wheat at a minimum possible cost meeting the requirements.

Step 2. Decision variable representation

Let X_1 be the number of hectares of land to be planted with corn, X_2 be the number of hectares of land to be planted with barley, and X_3 be the number of hectares of land to be planted with wheat.

Step 3. Objective function

$$\text{Min } z = 20X_1 + 15X_2 + 12X_3$$

Step 4. Constraints

$$X_1 + X_2 + X_3 = 500$$

$$X_1 \leq 250$$

$$X_2 \leq 200$$

$$2X_1 - X_3 = 0$$

$$X_1, X_2, X_3 \geq 0$$

2.4 Solution Approaches to Linear Programming Problems

There are two approaches to solve linear programming problems:

1. The Graphic solution method
2. The Algebraic solution/ simplex algorithm method

2.4.1 Graphical Solution

To use the graphic method, the following steps are needed:

1. Identify the problem

i.e.: The decision variables, the objective function and the constraints

2. Draw a graph including all the constraints and identify the feasible region

3. Obtain a point on the feasible region that optimizes the objective function-Optimal solution

4. Interpret the results

❖ *Graphical LP is a two-dimensional model.*

2.4.1.1 Maximization Problem

Example 1: Consider two models of color TV sets; **Model A** and **B**, are produced by a company to maximize profit. The profit realized is \$300 from A and \$250 from set B. The limitations are

A. availability of only 40hrs of labor each day in the production department.

B. a daily availability of only 45 hrs on machine time

C. ability to sale 12 set of model A.

How many sets of each model will be produced each day so that the total profit will be as large as possible?

Resources used per unit

<i>Constraints</i>	<i>Model A (X_1)</i>	<i>Model B (X_2)</i>	<i>Maximum Available hrs.</i>
<i>Labor hr.</i>	2	1	40
<i>Machine hr.</i>	1	3	45
<i>Marketing hr.</i>	1	0	12
<i>Profit</i>	\$300	\$250	

Solution

1. Formulation of mathematical modeling of LPP

$$\text{Max } Z = 300X_1 + 250X_2$$

$$\text{St: } \left. \begin{array}{l} 2X_1 + X_2 \leq 40 \\ X_1 + 3X_2 \leq 45 \\ X_1 \leq 12 \\ X_1, X_2 \geq 0 \end{array} \right\} \text{LPP Model}$$

2. Convert constraints inequalities into equalities

$$2X_1 + X_2 = 40$$

$$X_1 + 3X_2 = 45$$

$$X_1 = 12$$

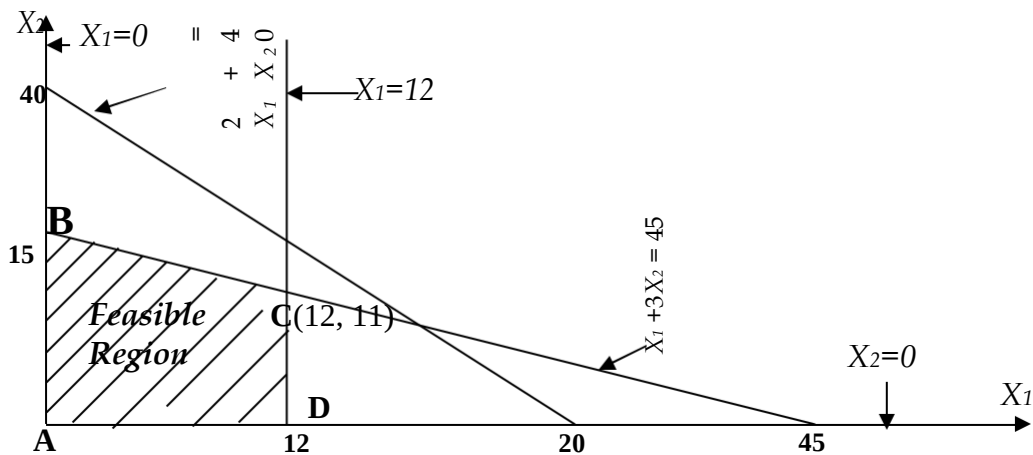
3. Draw the graph by intercepts

$$2X_1 + X_2 = 40 \implies (0, 40) \text{ and } (20, 0)$$

$$X_1 + 3X_2 = 45 \implies (0, 15) \text{ and } (45, 0)$$

$$X_1 = 12 \implies (12, 0)$$

$$X_1, X_2 = 0$$



4. Identify the feasible area of the solution which satisfies all constraints.

5. Identify the corner points in the feasible region

A (0, 0), B (0, 15), C (12, 11) and D (12, 0)

6. Identify the optimal point

<u>Corners</u>	<u>Coordinates</u>	<u>MaxZ=300 X₁ +250X₂</u>
A	(0, 0)	\$0
B	(0, 15)	\$3750
C	(12, 11)	\$6350
D	(12, 0)	\$3600

7. Interpret the result

Interpretation:

12 units of product A and 11 units of product B should be produced so that the total profit will be \$6350.

Exercise 2:

A manufacturer of light weight mountain tents makes two types of tents, **REGULAR** tent and **SUPER** tent. Each **REGULAR** tent requires 1 labor-hour from the cutting department and 3 labor-hours from the assembly department. Each **SUPER** tent requires 2 labor-hours from the cutting department and 4 labor-hours from the assembly department. The maximum labor hours available per week in the cutting department and the assembly department are 32 and 84 respectively. Moreover, the distributor, because of demand, *will not take more than 12 SUPER* tents per week. The manufacturer sales each **REGULAR** tent for \$160 and costs \$110 per tent to make. Whereas **SUPER** tent ales for \$210 per tent and costs \$130 per tent to make.

Required:

- A. Formulate the mathematical model of the problem
- B. Using the graphic method, determine how many of each tent the company should manufacture each tent the company should manufacture each week so as to maximize its profit?
- C. What is this maximum profit assuming that all the tents manufactured in each week are sold in that week?

Solution

<i>Labor hours per tent</i>			
<i>Department</i>	<i>Regular (X₁)</i>	<i>Super (X₂)</i>	<i>Maximum labor hours available per week</i>
Cutting department	1	2	32
Assembly department	3	4	84
<i>Selling price per tent</i>	\$160	\$210	
<i>Cost per tent</i>	<u>\$110</u>	<u>\$130</u>	
<i>Profit per tent</i>	<u>\$50</u>	<u>\$80</u>	

*The distributor will not take more than 12 SUPER tents per week. Thus, the manufacturer should not produce more than 12 SUPER tents per week.

Let X_1 = The N^o of **REGULAR** tents produced per week.

X_2 = The N^o of **SUPER** tents produced per week.

X_1 and X_2 are called the decision variables

$$\begin{array}{l}
 \text{LPP Model} \left\{ \begin{array}{l}
 \text{Max. } Z = 50X_1 + 80X_2 \\
 \text{St :} \\
 X_1 + 2X_2 \leq 32 \quad \text{..... Cutting department constraint} \\
 3X_1 + 4X_2 \leq 84 \quad \text{..... Assembly department constraint} \\
 X_2 \leq 12 \quad \text{..... Demand constraint} \\
 X_1, X_2 \geq 0 \quad \text{..... Non-negativity constraints}
 \end{array} \right.
 \end{array}$$

Corners	Coordinates	MaxZ=50 X_1 +800X_2
A	(0, 0)	\$0
B	(0, 12)	\$960
D	(20, 6)	\$1480
E	(28, 0)	\$1400

Interpretation:

The manufacturer should produce and sale **20 REGULAR** tents and **6 SUPERS** tents to get a maximum weekly profit of \$1480.

2.4.1.2 Minimization Problem

Example 1: Suppose that a machine shop has two different types of machines; machine 1 and machine 2, which can be used to make a single product. These machines vary in the amount of product produced per hr., in the amount of labor used and in the cost of operation. Assume that at least a certain amount of product must be produced and that we would like to utilize at least the regular labor force. How much should we utilize each machine in order to utilize total costs and still meets the requirement?

Resource used			
	Machine1(X_1)	Machine (X_2)	Minimum required hours
Product produced/hr	20	15	100
Labor/hr	2	3	15
Operation Cost	\$25	\$30	

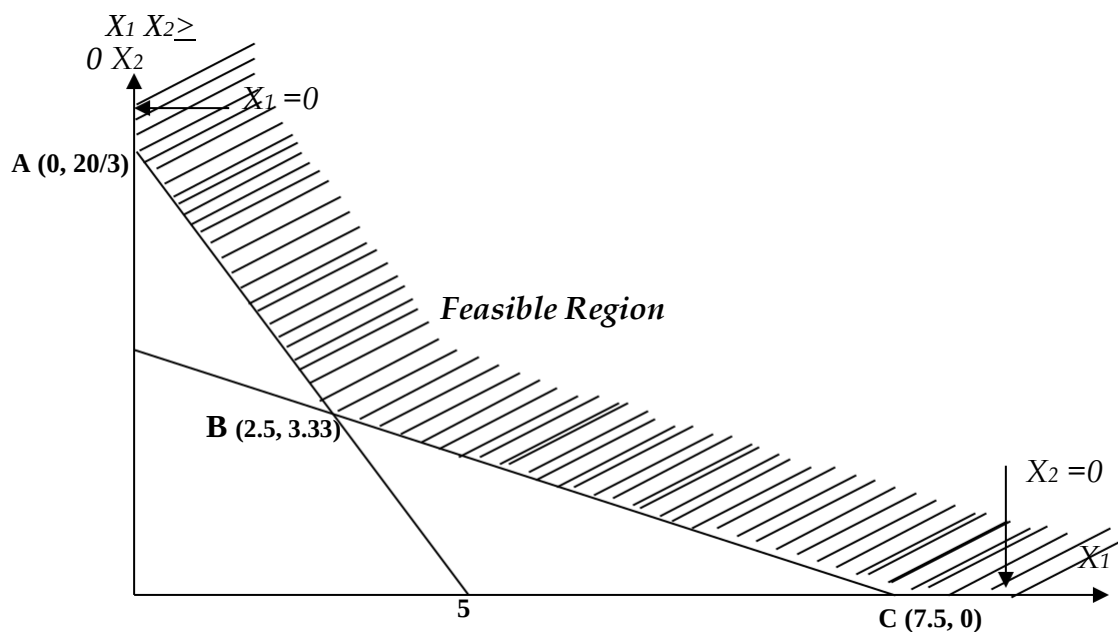
Solution

$$\begin{array}{l}
 \text{Min. } Z = 25X_1 + 30X_2 \\
 \text{St :} \\
 20X_1 + 15X_2 \leq 100 \\
 2X_1 + 3X_2 \leq 15 \\
 X_1, X_2 \geq 0
 \end{array}
 \quad \left. \vphantom{\begin{array}{l} \text{Min. } Z = 25X_1 + 30X_2 \\ \text{St :} \\ 20X_1 + 15X_2 \leq 100 \\ 2X_1 + 3X_2 \leq 15 \\ X_1, X_2 \geq 0 \end{array}} \right\} \text{LPP Model}$$

Constraint equation:

$$20X_1 + 15X_2 = 100 \implies (0, 20/3) \text{ and } (5, 0)$$

$$2X_1 + 3X_2 = 15 \implies (0, 5) \text{ and } (7.5, 0)$$



Corners	Coordinates	MinZ=25 X₁ + 30X₂
A	(0, 20/3)	200
B	(2.5, 3.33)	162.5
C	(7.5, 0)	187.5

$$X_1 = 2.5$$

$$X_2 = 3.33 \text{ and}$$

$$\text{MinZ} = \mathbf{162.5 \text{dollar}}$$

Example 2: A company owns two flour mills (A and B) which have different production capacities for **HIGH**, **MEDIUM** and **Low**-grade flour. This company has entered contract supply flour to a firm every week with 12, 8, and 24 quintals of **HIGH**, **MEDIUM** and **LOW** grade respectively. It costs the Co. \$1000 and \$800 per day to run mill A and mill B respectively. On a day, mill A produces 6, 2, and 4 quintals of **HIGH**, **MEDIUM** and **Low**-grade flour respectively.

Mill B produces 2, 2 and 12 quintals of **HIGH**, **MEDIUM** and **Low**-grade flour respectively. How many days per week should each mill be operated in order to meet the contract order most economically standardize? Solve graphically.

Solution:

	Mil A (X ₁)	Mill B (X ₂)	Minimum flour in quintals
HIGH Capacity (in quintal)	6	2	12
MEDIUM Capacity (in quintal)	2	2	8
LOW Capacity (in quintal)	4	12	24
Profit per unit	<u>\$1000</u>	<u>\$800</u>	

$$\text{Min. } Z = 100X_1 + 800X_2$$

St :

$$6X_1 + 2X_2 \leq 12$$

$$2X_1 + 2X_2 \leq 8$$

$$4X_1 + 12X_2 \leq 24$$

$$X_1, X_2 \geq 0$$

Constraint equation:

$$6X_1 + 2X_2 = 12$$

$$2X_1 + 2X_2 = 8 \quad (0, 6), (2, 0)$$

$$4X_1 + 12X_2 = 24 \quad (0, 4), (4, 0)$$

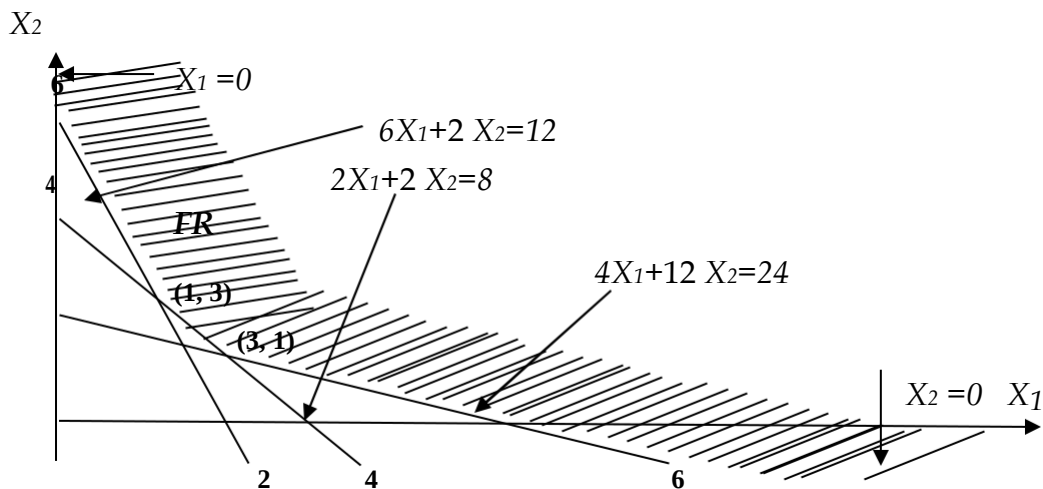
$$X_1, X_2 = 0 \quad (0, 2), (6, 0)$$

Corners	Coordinates	MinZ=\$1000 X₁ + 800X₂
A	(0, 6)	\$4800
B	(1, 3)	\$3400
C	(3, 1)	\$3800
D	(6, 0)	\$6000

$X_1 = 1$

$X_2 = 3$ and

MinZ= **\$3400**



Note:

-In maximization problems, our point of interest is looking the furthest point from the origin.

-In minimization problems, our point of interest is looking the point nearest to the origin.

2.4.2 Special Cases in Graphics Methods

2.4.2.1 Redundant Constraint

If a constraint when plotted on a graph doesn't form part of the boundary making the feasible region of the problem that constraint is said to be redundant.

Example: A firm is engaged in producing two products A and B. Each unit of product A requires 2Kg of raw material and 4 labor-hrs for processing. Whereas each unit of product B requires 3Kg

of raw materials and 3hrs of labor. Every unit of product A needs 4hrs to packaging and every unit of product B needs 3.5hrs for packaging. Every week the firm has availability of 60Kg of raw material, 96 labor-hours and 105 hrs I the packaging department. 1 unit of product A sold yields \$40 profit and 1 unit of B sod yields \$35 profit.

Required:

- a. Formulate this problem as a LPP
- b. Find the optimal solution

Solution

Resources	Products		Resource available per week
	A	B	
Raw materials (Kg)	2	3	60
Labor (hr)	4	3	96
Packaging (hr)	4	3.5	105
Profit per unit	<u>\$40</u>	<u>\$35</u>	

Let X_1 =The N^0 of units of product A produced per week

X_2 =The N^0 of units of product B produced per week

A. LPP Model

$$\text{Max. } Z = 40X_1 + 35X_2$$

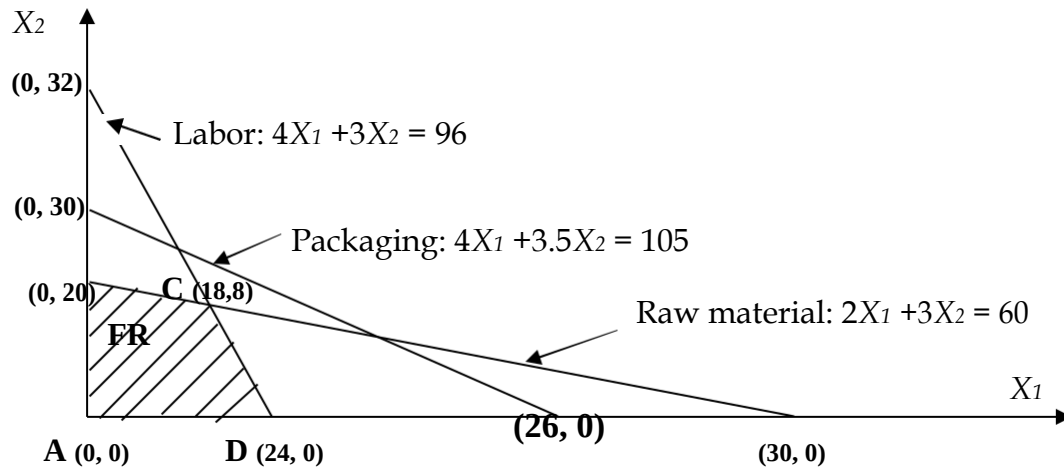
St :

$$2X_1 + 3X_2 \leq 60$$

$$4X_1 + 3X_2 \leq 96$$

$$4X_1 + 3.5X_2 \leq 105$$

$$X_1, X_2 \geq 0$$



The packaging hr is redundant.

<i>Corners</i>	<i>Coordinates</i>	<i>MinZ=40 X₁ + 35X₂</i>
A	(0, 0)	0
B	(0, 20)	700
C	(18, 8)	1000
D	(24, 0)	960

$X_1 = 18$

$X_2 = 8$ and

$\text{MinZ} = 1000$

Interpretation:

The company should produce and sale 18 units of product A and 8 units of product B per week so as to get a maximum profit of 1000.



By this production plan the entire raw material will be consumed.

$$2X_1 + 3X_2 \leq 60$$

$$2(18) + 3(8) = 60$$

$$\underline{60=60} \implies \text{No idle or unused raw material}$$



$$4X_1 + 3X_2 \leq 96$$

$$4(18) + 3(8) \leq 96$$

$$\underline{96=96} \implies \text{the entire labor hour will be consumed}$$



$$4X_1 + 3.5X_2 \leq 105$$

$100 \leq 105 \Rightarrow$ There is to be idle or unused capacity of 5hrs in the packaging department.

Note:

The packaging hour's constraint does not form part of the boundary making the feasible region. Thus, this constraint is of no consequence and is therefore, redundant. The inclusion or exclusion of a redundant constraint does not affect the **optimal solution** of the problem.

2.4.2.2 Multiple optimal Solutions

/Alternative optimal solutions/

-This is a situation where by a LPP has more than one optimal solution.

Multiple optimal Solutions will be found if two corners give optimal solution, then the line segment joining these points will be the solution.

$\Rightarrow$ We have unlimited number of optimal solutions without increasing or decreasing the objective function.

Example: The information given below is for the products A and B.

<i>Machine hours per week</i>			<i>Maximum available per week</i>
<i>Department</i>	<i>Product A</i>	<i>Product B</i>	
Cutting	3	6	900
Assembly	1	1	200
<i>Profit per unit</i>	<u><i>\$8</i></u>	<u><i>\$16</i></u>	

Assume that the company has a marketing constraint on selling products B and therefore it can sale a maximum of 125units of this product.

Required:

- Formulate the LPP of this problem
- Find the optimal solution

Solution:

Let X_1 = The N^o of units of product A produced per week

X_2 = The N^o of units of product B produced per week

a. The LPP Model of the problem is:

$$\text{Max. } Z = 8X_1 + 16X_2$$

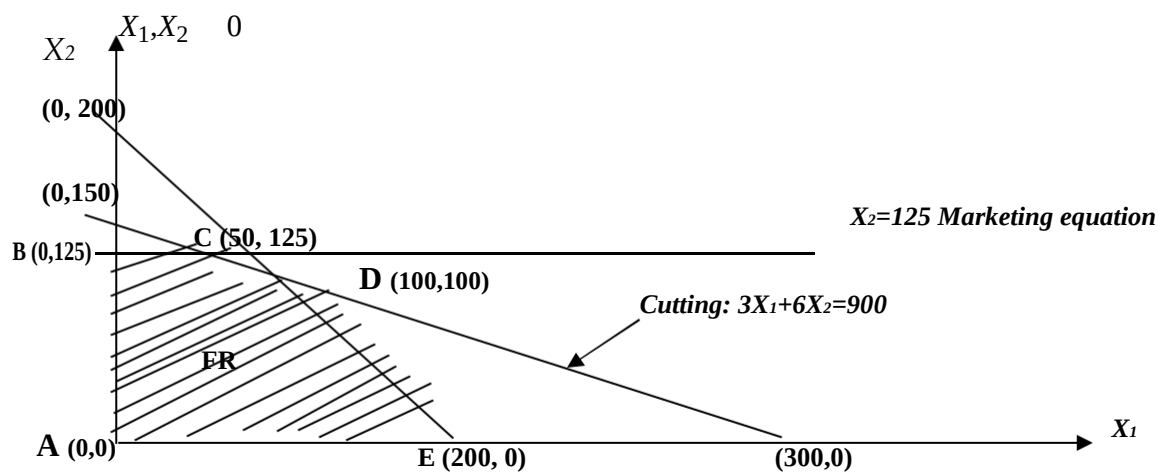
St :

$$3X_1 + 6X_2 \leq 900$$

$$X_1 + X_2 \leq 200$$

$$X_2 \leq 125$$

$$X_1, X_2 \geq 0$$



Corners	Coordinates	MaxZ=8 X₁ + 16X₂
A	(0, 0)	0
B	(0, 125)	2000
C	(50, 125)	2400
D	(100, 100)	2400
E	(200, 100)	1600

Interpretation:

Both **C** and **D** are optimal solutions. Any point on the line segment **CD** will also lead to the same optimal solution.



Multiple optimal solutions provide more choices for management to reach their objectives.

2.4.2.3 Infeasible Solution

A solution is called feasible if it satisfies all the constraints and the constraints and non-negativity condition. However, it is sometimes possible that the constraints may be inconsistent so that there is no feasible solution to the problem. Such a situation is called **infeasibility**.

Example:

$$\text{Max } Z = 20X_1 + 30X_2$$

St:

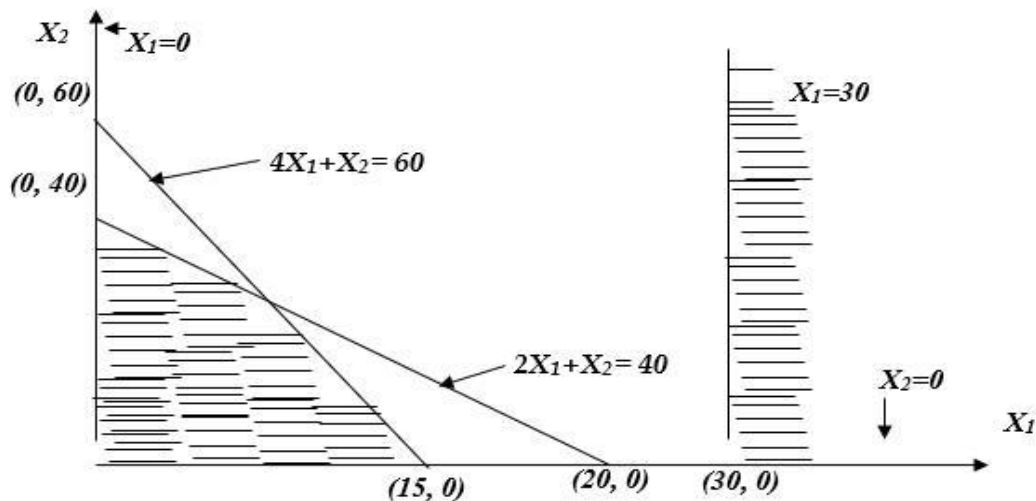
$$2X_1 + X_2 \leq 40$$

$$4X_1 + X_2 \leq 60$$

$$X_1 \geq 30$$

$$X_1, X_2 \geq 0$$

Solution:



Note:

- In the above graph, there is no common point in the shaded area.
- All constraints cannot be satisfied simultaneously and there is no feasible solution to the problem.

2.4.2.4 Mix of constraints

EXAMPLE: ABC gasoline company has two refineries with different production capacities. refinery a can produce 4,000gallons per day of *super unleaded gasoline*, 2000 gallons per day of *regular unleaded gasoline* and 1000 gallons per day of *leaded gasoline*. on the other hand, refinery b can produce 1000 gallons per day of *super unleaded*, 3000 gallons per day of *regular unleaded* and 4,000 gallons per day of *leaded*.

The company has made a contract with an automobile manufacturer to provide 24000 gasolines of *super unleaded*, 42000 gallons of *regular unleaded* and 36000 gallons of *leaded*. the automobile manufacturer wants delivery in not more than 14 days. the cost of running refinery a is \$1500 per day and refinery b is \$2400 per day.

Required:

- a) Formulate this problem as a LPP
- b) Determine the number of days the gasoline company should operate each refinery in order to meet the terms of the above contract most economical. (i.e. At a minimum running cost)
- c) Which grade of gasoline would be over produced?

Solution:

Grade of gasoline	Production per day (in gallons)		Contract with an automobile manufacturer
	A	B	
Super Unleaded	4000	1000	24,000
Regular Unleaded	2000	3000	42,000
Leaded	<u>1000</u>	<u>4000</u>	36,000
Running cost per day	<u>\$1,500</u>	<u>\$2,400</u>	



The automobile manufacturer wants delivery in not more than 14 days.

Let X_1 = The N^o of days refinery A should work.

X_2 = The N^o of days refinery B should work.

a. LPP of the problem

$$\text{Min } Z = 1500X_1 + 2400X_2$$

St:

$$4000X_1 + 1000X_2 \geq 24000$$

$$2000X_1 + 3000X_2 \geq 42000$$

$$1000X_1 + 2000X_2 \geq 36000$$

$$X_1 \leq 14$$

$$X_2 \leq 14$$

$$X_1, X_2 \geq 0$$

==> To simplify the problem, divide by 1000 the constraints

$$\text{Min } Z = 1500X_1 + 2400X_2$$

St:

$$4X_1 + X_2 \geq 24$$

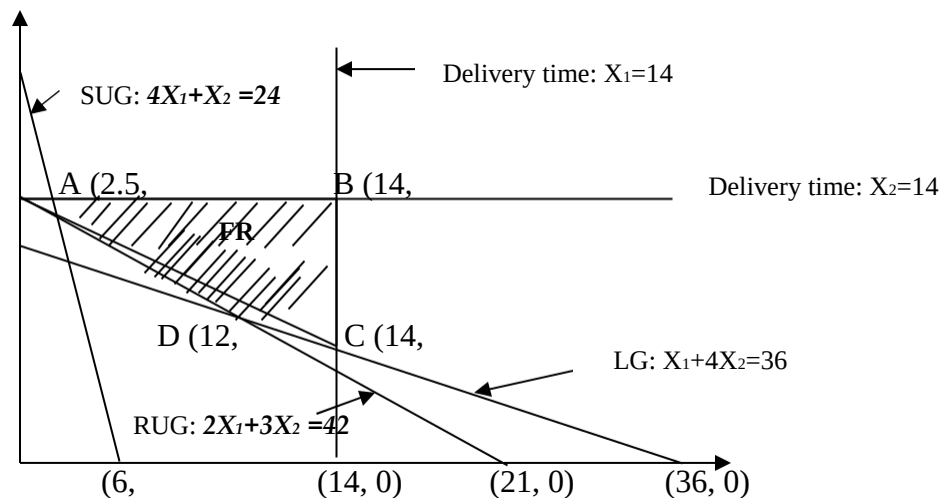
$$2X_1 + 3X_2 \geq 42$$

$$X_1 + 4X_2 \geq 36$$

$$X_1 \leq 14$$

$$X_2 \leq 14$$

$$X_1, X_2 \geq 0$$



Note: Point A, B, C, and D are solved by elimination-substitution method

Corners	Coordinates	MaxZ=1500X₁ + 2400X₂
A	(2.5, 4)	\$37350
B	(14, 14)	54600
C	(14, 5.5)	34200
D	(12, 6)	32400
E	(3, 12)	33300

Interpretation:

The oil company should operate refinery A for 12 days and refinery B for 6 days at a minimum operating cost of \$32,400.

c. Is there any over production

SUG: $4000X_1 + 1000X_2 \geq 24000$

$$4000(12) + 1000(6) \geq 24000$$

$$\underline{54000} > 24000$$

Therefore, 30,000 gallons over production

RUG: $2000X_1 + 3000X_2 \geq 42000$

$$2000(12) + 3000(6) \geq 42000$$

$$\underline{42000} > \underline{42000}$$

Therefore, there is no over production of RUG

LG: $1000X_1 + 4000X_2 \geq 36000$

$$1000(12) + 1000(6) \geq 36000$$

$$\underline{36000} > \underline{36000}$$

Therefore, No over production of LG

2.4.2.5 Unbounded Solution

When the value of decision variables in LP is permitted to increase infinitely without violating the feasibility condition, then the solution is said to be unbounded. Here, the objective function value can also be increased infinitely. However, an unbounded feasible region may yield some definite value of the objective function.

Example:

Use the graphical method to solve the following LPP.

1. $\text{Max. } Z = 3X_1 + 4X_2$

St:

$$X_1 - X_2 \leq -1 \implies -X_1 + X_2 \geq 1 \text{ since the quantity solution is positive}$$

$$-X_1 + X_2 \leq 0$$

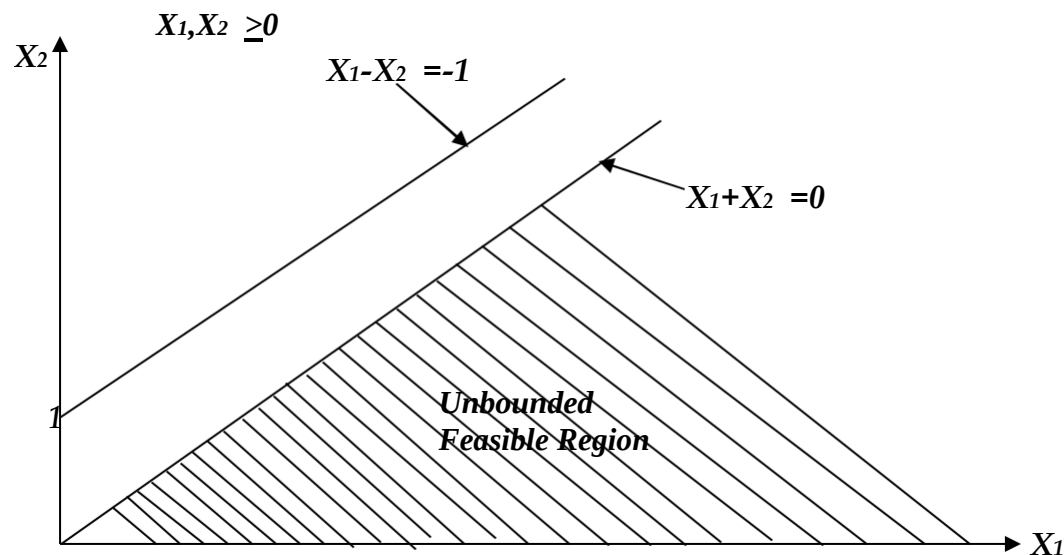


Fig: Unbounded Solution

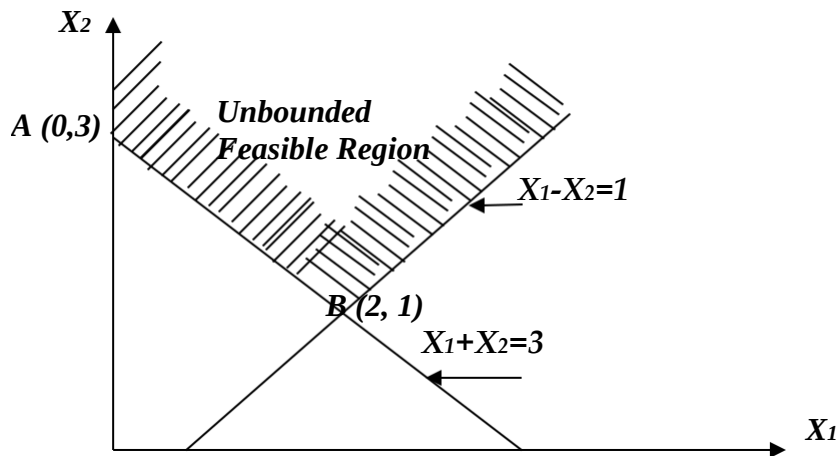
2. $\text{Max. } Z = 3X_1 + 2X_2$

St:

$$X_1 - X_2 \leq 1$$

$$X_1 + X_2 \leq 3$$

$$X_1, X_2 \geq 0$$



Note:

Here that the two corners of the region are A (0,3) and B (2,1). The value of $\text{Max}Z(A)=6$ and $\text{Max}Z(B)=8$. But there exist number of points in the shaded region for which the value of the objective function is more than 8. For example, the point (10, 12) lies in the region and the function value at this point is 70 which is more than 8.

Remark:

An unbounded solution does not mean that there is no solution to the given LPP, but implies that there exists an infinite number of solutions.



Self-check exercise 2.1

Use graphical method to solve the following LPP.

1. **Max.** $Z = 7/4X_1 + 3/2X_2$

St:

$$8X_1 + 5X_2 \leq 320$$

$$4X_1 + 5X_2 \leq 20$$

$$X_1 \geq 15$$

$$X_2 \geq 10$$

$$X_1, X_2 \geq 0$$

3. **Max.** $Z = 3X_1 + 2X_2$

St:

$$X_1 - X_2 \leq 1$$

$$X_1 + X_2 \geq 3$$

$$X_1, X_2 \geq 2$$

2. **Max.** $Z = 3X_1 + 2X_2$

St:

$$-2X_1 + 3X_2 \leq 9$$

$$X_1 - 5X_2 \geq -20$$

$$X_1, X_2 \geq 0$$

4. **Max.** $Z = X_1 + X_2$

St:

$$X_1 + X_2 \leq 1$$

$$-3X_1 + X_2 \geq 3$$

$$X_1, X_2 \geq 0$$

2.5 Simplex Method

The graphical method to solving **LPPs** provides fundamental concepts for fully understanding the LP process. However, the graphical method can handle problems involving only two decision variables (say X_1 and X_2).

In 1940's **George B. Dantzig** developed an algebraic approach called the **Simplex Method** which is an efficient approach to solve applied problems containing numerous constraints and involving many variables that cannot be solved by the graphical method.

The simplex method is an **ITERATIVE** or “**step by step**” method or repetitive algebraic approach that moves automatically from one basic feasible solution to another basic feasible solution improving the situation each time until the optimal solution is reached at.

Note: The simplex method starts with a corner that is in the solution space or feasible region and moves to another corner. The solution space improving the value of the objective function each time until optimal solution is reached at the optimal corner.

Scope of solution of LPP by simplex method

Following types of problems are solved by simplex method:

- **Maximize Z** with inequalities of constraints in “ $\leq$ ” form.
- **Minimize Z** with inequalities of constraints in “ $\geq$ ” form.
- **Maximize Z** with inequalities of constraints in “ $\geq$ ” form.
- **Maximize Z or Minimize Z** with inequalities of constraints in “ $\leq$ ”, “ $\geq$ ” or “ $=$ ” form.

2.5.1 Maximization Problems

- Maximize Z with inequalities of constraints in “ $\leq$ ” form

Example:

Solve the problem using the simplex approach

$$\text{Max. } Z = 300x_1 + 250x_2$$

Subject to:

$$2x_1 + x_2 \leq 40 \text{ (Labor)}$$

$$x_1 + 3x_2 \leq 45 \text{ (Machine)}$$

$$x_1 \leq 12 \text{ (Marketing)}$$

$$x_1, x_2 \geq 0$$

Solution

Step 1; Formulate LPP Model

Step 2; Standardize the problem. i.e Convert constraint inequality into equality form by introducing a variable called **Slack variable**.

Slack Variables: A slack variable(s) is added to the left-hand side of a $\leq$ constraint to convert the constraint inequality into equality. The value of the slack variable shows unused resource.

A slack variable emerges when the LPP is a maximization problem.

Slack variables represent unused resource or idle capacity. Thus, they don't produce any product and their contribution to profit is **zero**.

Slack variables are added to the objective function with **zero coefficients**.

Let that s_1 , s_2 , and s_3 are unused labor, machine and marketing hrs respectively.

$$\begin{array}{lcl}
 \text{Max. } Z = 300x_1 + 250x_2 + 0s_1 + 0s_2 + 0s_3 & & \\
 \text{St:} & & \\
 2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 40 & & \\
 x_1 + 3x_2 + 0s_1 + s_2 + 0s_3 = 45 & & \\
 x_1 + 0s_1 + 0s_2 + s_3 = 12 & & \\
 x_1, x_2, s_1, s_2, s_3 \geq 0 & &
 \end{array}
 \left. \vphantom{\begin{array}{l} \\ \\ \\ \\ \end{array}} \right\} \text{Standard form}$$

Step 3: Obtain the initial simplex tableau

To represent the data, the simplex method uses a table called the **simplex table** or the **simplex matrix**.

⇒

In constructing the initial simplex tableau, the search for of the optimal solution begins at the origin. Indicating that nothing can be produced; Thus, first assumption, no production implies that $x_1 = 0$ and $x_2 = 0$

→

$$2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 40$$

$$2(0) + 0 + s_1 + 0s_2 + 0s_3 = 40$$

$$\underline{s_1 = 40} - \text{Unused labor hrs.}$$

→

$$x_1 + 3x_2 + 0s_1 + s_2 + 0s_3 = 45$$

$$0 + 3(0) + 0s_1 + s_2 + 0s_3 = 45$$

$$\underline{s_2 = 45} - \text{Unused machine hrs.}$$

→

$$x_1 + 0s_1 + 0s_2 + s_3 = 12$$

$$0 + 0s_1 + 0s_2 + s_3 = 12$$

$$\underline{s_3 = 12} - \text{Unused Marketing hrs.}$$

$$\begin{aligned}\text{Therefore, } \mathbf{Max.Z} &= 300x_1 + 250x_2 + 0s_1 + 0s_2 + 0s_3 = 300(0) \\ &+ 250(0) + 0(40) + 0(45) + 0(12) \\ &= \underline{0}\end{aligned}$$

Note: In general, whenever there are n variables and m constraints (excluding the non-negativity), where m is less than n ($m < n$), $n-m$ variables must be **set equal to zero** before the solution can be solved algebraically.

a) Basic variables are variables with non-zero solution values.

Or: basic variables are variables that are in the basic solution. Basic variables have 0 values in the $C_j - Z_j$ row.

b) Non-basic variables are variables with zero solution values.

Or: non-basic variables are variables that are out of the solution.



$n=5$ variables (x_1 , x_2 , s_1 , s_2 , and s_3) and $m=3$ constraints (Labor, machine and marketing constraints), excluding non-negativity.

Therefore, $n-m=5-3=2$ variables (x_1 and x_2) are set equal to zero in the 1st simplex tableau. These are non-basic variables. 3 Variables (s_1 , s_2 , and s_3) are basic variables (in the 1st simplex tableau) because they have non-zero solution values.

Step 4: Construct the initial simplex tableau

Initial simplex tableau

Profit per unit column		Basic or Solution variable column Real or decision variables column					Slack variables column		Solution quantity column	
C_j		300	250	0	0	0				
CBV	BV	X_1	X_2	S_1	S_2	S_3			Q	
0	S_1	2	1	1	0	0			40	R_1
0	S_2	1	3	0	1	0			45	R_2
0	S_3	1	0	0	0	1			12	R_3
	Z_j	0	0	0	0	0			0	
	$C_j - Z_j$	300	250	0	0	0				

Profit per unit row

Constraint equation rows

Gross Profit row

Net Profit row /Indicator row/

Step 5: Choose the “incoming” or “entering” variables

Note: The entering variable is the variable that has **the most positive value** in the $C_j - Z_j$ row also called as **indicator row**. Or the entering variable is the variable that has **the highest contribution to profit per unit**.

- X_1 in our case is the entering variable
- The column associated with the entering variable is called **key** or **pivot column** (X_1 column in our case)

Step 6: Choose the “leaving” or “outgoing” variable

==> In this step, we determine the variable that will leave the solution for X_1 (or entering variable)

Note: The row with **the minimum** or **lowest positive (non-negative) replacement ratio** shows the variable to leave the solution.

$$\text{Replacement Ratio (RR)} = \frac{\text{Solution Quantity (Q)}}{\text{Corresponding values in pivot column}}$$

Note: $RR > 0$

- The variable leaving the solution is called **leaving variable** or **outgoing variable**.
- The row associated with the leaving variable is **called key or pivot row** (s3 column in our case)
- The element that lies at the intersection of the pivot column and pivot row is called pivot element (N^0 1 in our case)

Step 7: Repeat step 3-5 till optimum basic feasible solution is obtained.

Or: repeat step 3-5 till no positive value occurs in the $C_j - Z_j$ row.

Note: → Divide each element of the pivot row by the pivot element to find new values in the key or pivot row.

→ Perform row operations to make **all other entries** for **the pivot column equal to zero**.

2nd simplex tableau

C_j		300	250	0	0	0	
CBV	SV	X_1	X_2	S_1	S_2	S_3	Q
0	S_1	0	1	1	0	-2	16
0	S_2	0	3	0	1	-1	33
300	X_1	1	0	0	0	1	12
	Z_j	300	0	0	0	300	
	$C_j - Z_j$	0	250	0	0	-300	

$$R2 \text{ NEW} = R2 \text{ OLD} - R3 \text{ NEW}$$

$$R1 \text{ NEW} = R1 \text{ OLD} - 2 R3 \text{ NEW}$$

$$R3 \text{ NEW} = R3 \text{ OLD}$$

3rd simplex tableau

C_j		300	250	0	0	0	
CBV	BV	X_1	X_2	S_1	S_2	S_3	Q
0	S_1	0	0	1	-1/3	-5/3	17/3
250	X_2	0	1	0	1/3	-1/3	31/3
300	X_1	1	0	0	0	1	12
	Z_j	300	250	0	250/3	650/3	6350
	$C_j - Z_j$	0	0	0	-250/3	-650/3	

$$R2 \text{ NEW} = R2 \text{ OLD} / 3$$

$$R1 \text{ NEW} = R1 \text{ OLD} - R2 \text{ NEW}$$

$$R3 \text{ NEW} = R3 \text{ OLD}$$

Since all the $C_j - Z_j \leq 0$ optimal solution is reached at.

Therefore, $X_1=12$, $X_2=11$, $S_1=5$ and $\text{Max } Z=6350$

Example 2;

A Juice Company has available two kinds of food Juices: **Orange Juice** and **Grape Juice**. The company produces two types of punches: **Punch A** and **Punch B**. One bottle of punch A requires 20 liters of Orange Juice and 5 liters of Grape Juice. 1 Bottle of punch B requires 10 liters of Orange Juice and 15 liters of Grape Juice.

From each of bottle of Punch A a profit of \$4 is made and from each bottle of Punch B a profit of \$3 is made. Suppose that the company has 230 liters of Orange Juice and 120 liters of Grape Juice available

Required:

- a) Formulate this problem as a LPP
- b) How many bottles of Punch A and Punch B the company should produce in order to maximize profit? (Using the simplex method)
- c) What is this maximum profit?

Solution

<i>Juice</i>	<i>Juice needed for one bottle of</i>		Juice Available
	<i>Punch A</i>	<i>Punch B</i>	
<i>Orange Juice (lt)</i>	20	10	230
<i>Grape Juice (lt)</i>	5	1	120
Profit per tent	<u>\$4</u>	<u>\$3</u>	

Let X_1 = the N^o of bottles of punch A produced.

X_2 = the N^o of bottles of punch B produced.

• **LPP Model**

$$\text{Max } Z = 4X_1 + 3X_2$$

St:

$$20X_1 + 10X_2 \leq 230 \quad \text{Orange Constraint}$$

$$5X_1 + 15X_2 \leq 120 \quad \text{Grape Constraint}$$

$$X_1, X_2 \geq 0 \quad \text{Non-negativity constraint}$$

• **Standard form**

$$\text{Max. } Z = 4x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$$

St:

$$20x_1 + 3x_2 + s_1 + 0s_2 = 230$$

$$5x_1 + 15x_2 + 0s_1 + s_2 = 120$$

$$x_1, x_2, s_1, s_2, > 0$$

Standard form

Where, s_1 = *Unused orange juice*

s_2 = *Unused grape juice*

Initial simplex tableau

C_j		4	3	0	0	
CBV	BV	X_1	X_2	S_1	S_2	Q
0	S_1	20	10	1	0	230
0	S_2	5	15	0	1	120
	Z_j	0	0	0	0	0
	$C_j - Z_j$	4	3	0	0	

$$R1 \text{ NEW} = R1 \text{ OLD} / 20$$

$$R2 \text{ NEW} = R2 \text{ OLD} - 5R1 \text{ NEW}$$

2nd simplex tableau

C_j		4	3	0	0	
CBV	bV	X_1	X_2	S_1	S_2	Q
4	X_1	1	1/2	1/20	0	11.5
0	S_2	0	25/2	-1/4	1	62.5
	Z_j	4	2	1/5	0	46
	$C_j - Z_j$	0	1	-1/5	0	

$$R2 \text{ NEW} = R2 \text{ OLD} * 2/25$$

$$R1 \text{ NEW} = R1 \text{ OLD} - 1/2 * R2 \text{ NEW}$$

C_j		4	3	0	0	
	SV	X_1	X_2	S_1	S_2	Q
4	X_1	1	0	3/50	-1/25	9
3	X_2	0	1	-1/50	2/25	5
	Z_j	4	3	0.18	0.08	51
	$C_j - Z_j$	0	0	-0.18	-0.08	

Optimal solution with all $c_j - z_j$ values less than or equal to zero

$X_1 = 9$ bottles of punch A

$X_2 = 5$ bottles of punch B

$s_1 = 0$

$s_2 = 0$

$Max Z = \$51$



Self-check exercise 2.2

Solve the following LPPs using the simplex method

1. $Max. Z = 3x_1 + 5x_2$

Subject to:

$$x_2 \leq 6$$

$$3x_1 + 2x_2 \leq 18$$

$$x_1, x_2 \geq 0$$

2. $Max. Z = 20x_1 + 10x_2$

Subject to:

$$5x_1 + 4x_2 \leq 250$$

$$2x_1 + 5x_2 \leq 150$$

$$x_1, x_2 \geq 0$$

2.5.2 Minimization Problems



Minimize Z with inequalities of constraints in " $\geq$ " form There are two methods to solve minimization LP problems:

1. Direct method/Big M-method/



Using artificial variables

2. Conversion method



Minimization by maximizing the dual



Surplus Variable (-s):

- A variable inserted in a greater than or equal to constraint to create equality. It represents the amount of resource usage above the minimum required usage.
- Surplus variable is subtracted from a $\geq$ constraint in the process of converting the constraint to standard form.
- Neither the slack nor the surplus is negative value. They must be positive or zero.

Example:

1. $2x_1 + x_2 \leq 40$ $\Rightarrow$ is a constraint inequality
 $x_1 = 12$ and $x_2 = 11 \Rightarrow 2x_1 + x_2 + s = 40 \Rightarrow 2(12) + 11 + s = 40$
 $\Rightarrow \underline{s=5}$ **unused resource**
2. $5x_1 + 3x_2 \leq 45$
 $x_1 = 12$ and $x_2 = 11 \Rightarrow 5x_1 + 3x_2 + s = 45 \Rightarrow 5(12) + 3(11) + s = 45$
 $\Rightarrow \underline{s=0}$ **unused resource (No idle resource)**
3. $5x_1 + 2x_2 \geq 20$
 $x_1 = 4.5$ and $x_2 = 2 \Rightarrow 5x_1 + 2x_2 - s = 20 \Rightarrow 5(4.5) + 2(2) - s = 20$
 $\Rightarrow \underline{s=6}$ **unused resource**
4. $2x_1 + x_2 \geq 40$
 $x_1 = 0$ and $x_2 = 0$ (No production) $\Rightarrow 5x_1 + 2x_2 - s = 20 \Rightarrow 5(4.5) + 2(2) - s = 20$
 $\Rightarrow \underline{s=-6}$ **(This is mathematically unacceptable)**

Thus, in order to avoid the mathematical contradiction, we have to add artificial variable (A)

❖ **Artificial variable (A):**

Artificial variable is a variable that has no meaning in a physical sense but acts as a tool to create an initial feasible LP solution.

Note:

Type of constraint	To put into standard form
$\leq$	Add a slack variable
$=$	Add an artificial variable
$\geq$ -----	Subtract a surplus variable and add artificial variable

2.5.2.1 Big M-method

/Charnes Penalty Method/

The Big-M Method is a method which is used in removing artificial variables from the basis. In this method; we assign coefficients to artificial variables, undesirable from the objective function point of view. If objective function Z is to be minimized, then a very large positive price (called **penalty**) is assigned to each artificial variable. Similarly, if Z is to be maximized, then a very large negative price (also called **penalty**) is assigned to each of these variables.

Following are the characteristics of Big-M Method:

- a. High penalty cost (or profit) is assumed as M
- b. M is assigned to artificial variable A in the objective function Z .
- c. Big-M method can be applied to minimization as well as maximization problems with the following distinctions:
 - i. **Minimization problems:** -Assign $+M$ as coefficient of artificial variable A in the objective function Z
 - ii. **Maximization problems:** -Here $-M$ is assigned as coefficient of artificial variable A in the objective function Z
- d. Coefficient of S (slack/surplus) takes zero values in the objective function Z
- e. For minimization problem, the incoming variable corresponds to the **highest negative** value of $C_j - Z_j$.
- f. Solution is optimal when there is no negative value of $C_j - Z_j$. (For minimization case)

Example:

1. Minimize $Z = 25x_1$

$+30x_2$ Subject to:

$$20x_1 + 15x_2 \geq 100$$

$$2x_1 + 3x_2 \geq 15$$

$$x_1, x_2 \geq 0$$

Solution

Step 1: Standardize the problem

$$\text{Minimize } Z = 25x_1 + 30x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$$

Subject to:

$$20x_1 + 15x_2 - s_1 + A_1 = 100$$

$$2x_1 + 3x_2 - s_2 + A_2 = 15$$

$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

Step 2: Initial simplex tableau

The initial basic feasible solution is obtained by setting $x_1 = x_2 = s_1 = s_2 = 0$

No production, $x_1 = x_2 = s_1 = 0 \Rightarrow 20(0) + 15(0) - 0 + A_1 = 100 \Rightarrow A_1 = 100$

$$x_1 = x_2 = s_2 = 0 \Rightarrow 0(0) + 3(0) - 0 + A_2 = 15 \Rightarrow A_2 = 15$$

Initial simplex tableau

C_j		25	30	0	0	M	M	Q	<u>RR</u>
CBV	BV	x_1	x_2	s_1	s_2	A_1	A_2		
M	A_1	20	15	-1	0	1	0	100	100/20=5
M	A_2	2	3	0	-1	0	1	15	15/2=7.5
Z_j		22M	18M	-M	-M	M	M	115 M	
$C_j - Z_j$		25 - 22M	30 - 18M	M	M	0	0		

$$R1 \text{ NEW} = R1 \text{ OLD} / 20$$

$$R2 \text{ NEW} = R2 \text{ OLD} - 2R1 \text{ NEW}$$

Note: Once an **artificial variable** has left the basis, it has served its purpose and can therefore be removed from the simplex tableau. An artificial variable is never considered for re-entry into the basis.

2nd Simplex Tableau

C_j	25	~~30~~	0	0	M	
CBV BV	X_1	X_2	S_1	S_2	A_2	Q
25 X_1	1	3/4	-1/20	0	0	5
M A_2	0	3/2	1/10	-1	1	5
Z_j	25	75/4+3/2M	-5/4+1/10M	-M	M	125+5M
$C_j - Z_j$	0	45/4-3/2M	5/4-1/10M	M	0	

$$R1 \text{ NEW} = R1 \text{ OLD} - 3/4 R2 \text{ NEW}$$

$$R2 \text{ NEW} = R2 \text{ OLD} * 2/3 \text{ OR } R2/3/2$$

3rd Simplex Tableau

C_j	25	30	0	0	
SV	X_1	X_2	S_1	S_2	Q
25 X_1	1	0	-1/10	1/2	5/2
30 X_2	0	1	1/15	-2/3	10/3
Z_j	25	30	-1/2	-15/2	162.5
$C_j - Z_j$	0	0	1/2	15/2	

All $C_j - Z_j \geq 0$ Optimal solution is reached

$$X_1 = 5/2$$

$$X_2 = 10/3 \text{ and Min } Z = 162.5$$

Note: As long as an “A” variable is available in the solution variable column, the solution is **infeasible**.

2. Use the penalty (Big-M) method to solve the following LPP

$$\text{Min } Z = 5x_1 + 3x_2$$

Subject to:

$$2x_1 + 4x_2 \leq 12$$

$$2x_1 + 2x_2 = 10$$

$$5x_1 + 2x_2 > 10$$

$$x_1, x_2 \geq 0$$

Solution

$$\text{Min } Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$$

Subject to:

If no production

$$2x_1 + 4x_2 + s_1 = 12 \implies x_1 = x_2 = 0 \implies s_1 = 12 \text{ (Solution Value in the initial simplex tableau)}$$

$$2x_1 + 2x_2 + A_1 = 10 \implies x_1 = x_2 = 0 \implies A_1 = 10 \text{ (Solution Value in the initial simplex tableau)}$$

$$5x_1 + 2x_2 - s_2 + A_2 = 10 \implies x_1 = x_2 = s_2 = 0 \implies A_2 = 10 \text{ (Solution Value in the initial simplex tableau)}$$

$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

Initial Simplex tableau

C_j		5	3	0	0	M	M	Q	RR
CBV	SV	x_1	x_2	s_1	s_2	A_1	A_2		
0	s_1	2	4	1	0	0	0	12	6
M	A_1	2	2	0	0	1	0	10	5
M	A_2	5	2	0	-1	0	1	10	2
Z_j		7M	4M	0	-M	M	M	20M	
$C_j - Z_j$		5-7M	3-4M	0	M	0	0		

R1 NEW=R1 OLD-2 R3 NEW

R2 NEW=R2 OLD-2 R3 NEW

R3 NEW=R3 OLD/5

2nd simplex tableau

C_j		5	3	0	0	M	
CBV	BV	X_1	X_2	S_1	S_2	A_1	Q
0	S_1	0	16/5	1	2/5	0	8
M	A_1	0	6/5	0	2/5	1	6
5	X_1	1	2/5	0	-1/5	0	2
	Z_j	5M	6/5M +2	0	2/5M -1	M	10+6 M
	$C_j - Z_j$	0	-6/5M +1	0	-2/5M +1	0	

RR

2.5

5

5

R1 NEW =R1 OLD*5/16

R2 NEW=R2 OLD-6/5*R1 NEW

R3 NEW=R3 OLD-2/5*R1 NEW

3rd simplex tableau

C_j		5	3	0	0	M	
SV		X_1	X_2	S_1	S_2	A_1	Q
3	X_2	0	1	5/16	1/8	0	2.5
M	A_1	0	0	-3/8	1/4	1	3
5	X_1	1	0	-1/8	-1/4	0	1
	Z_j	5	3	-3/8M	+5/16 M	4-7/8 M	12.5+3 M
	$C_j - Z_j$	0	0	3/8M	-5/16 M	-M/4+7/8 M	0

R1 NEW=R1 OLD-1/8 R2 NEW

R2 NEW= R1 OLD*4

R3 NEW= R3 OLD+1/4*R2 NEW

4th Simplex tableau (OPTIMAL SOLUTION)

C_j		5	3	0	0	
	SV	X_1	X_2	S_1	S_2	Q
3	X_2	0	1	1/2	0	1
0	S_2	0	0	-3/2	1	12
5	X_1	1	0	-1/2	0	4
	Z_j	5	3	-1	0	23
	$C_j - Z_j$	0	0	1	0	

$$X_1=4$$

$$S_2=12$$

$$X_2=1$$

$$\text{Min} Z=23$$

$$S_1=0$$

Note: To get an initial feasible solution

Types of constraint

Presence of variables in the initial solution mix

1. $\leq$ (Slack)

Yes

2. $\geq$ *(Surplus)

No

*(Artificial)

Yes

3. $=$ (Artificial)

Yes



Self-check exercise 2.3

Find the optimal solution using simplex method

1. Min $Z=10x_1 + 5x_2$

2. Min $Z=4x_1 + 5x_2$

Subject to:

Subject to:

$$2x_1 + 5x_2 \geq 150$$

$$x_1 + 2x_2 \geq 80$$

$$3x_1 + x_2 \geq 120$$

$$3x_1 + x_2 \geq 75$$

$$x_1, x_2 \geq 0$$

$$x_1, x_2 \geq 0$$

2.5.3 Special Cases in Simplex Method

1. Mixed constraints

Example: Max $Z=6x_1 + 8x_2$

Subject to:

$$x_2 \leq 4$$

$$x_1 + x_2 = 9$$

$$6x_1 + 2x_2 \geq 24$$

$$x_1, x_2 \geq 0$$

• **Standard form**

$$\text{Max. } Z = 6x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3 - M A_2 - M A_3$$

St:

$$x_2 + s_1 = 4$$

$$x_1 + x_2 + A_2 = 9$$

$$6x_1 + 2x_2 - s_3 + A_3 = 24$$

$$\text{All Variables } \geq 0$$

Standard form

Initial simplex tableau

C_j		6	8	0	0	-M	-M	
CBV	SV	X_1	X_2	S_1	S_3	A_2	A_3	Q
0	S_1	0	1	1	0	0	0	4
-M	A_2	1	1	0	0	1	0	9
-M	A_3	6	2	0	-1	0	1	4
	Z_j	-7M	-3M	0	+M	-M	-M	24
	$C_j - Z_j$	7M + 6	3M + 8	0	-M	0	0	

Ans: At the 4th tableau: $X_1 = 5$, $X_2 = 4$, $S_3 = 14$ and $\text{Max } Z = 62$

Note: For the initial basis, use **artificial** variables for constraints that have them. Otherwise, use a constraint **slack** variable. Hence, **surplus** variables will not appear in an initial solution.

2. Two incoming variables

/ Or Tie for entering variables/

In order to break this tie, the selection for the key column (entering variable) can be made arbitrary. However; the number of solutions can be minimized by adopting the following rules:

1. *If there is a tie between two decision variables, then the selection can be made arbitrary.*
2. *If there is a tie between a decision variable and a slack (or surplus) variable, then select the decision variable to enter into basis first.*
3. *If there is a tie between slack or surplus variable, then selection can be made arbitrary.*

Example:

If the equation is max Z:

C_j					Q
SV	X_1	X_2	S_1	S_3	
Z_j					
$C_j - Z_j$	5	2	5	0	

In such a case, X_1 is the entering variable

3. Infeasibility

A situation with no feasible solution may exist if the problem was formulated improperly. Infeasibility comes about when there is **no solution** that satisfies all of the problem's constraints. In the simplex method, an infeasible solution is indicated by looking at the final tableau. In it, all $C_j - Z_j$ row entries will be the proper sign to imply **optimality**, but an **artificial variable (A)** will still be in the solution mix.

Example:

Minimization case

C_j		5	8	0	0	M	
	SV	X_1	X_2	S_1	S_2	A_2	Q
5	X_1	1	1	-2	3	0	200
8	X_2	0	1	1	2	0	100
M	A_2	0	0	0	-1	1	20
	Z_j	5	8	-2	31-M	M	1,800+200M
	$C_j - Z_j$	0	0	2	M-31	0	

Even though all $C_j - Z_j$ are positive or 0 (i.e the criterion for an optimal solution in a minimization case), **no feasible solution** is possible because **an artificial variable (A_2)** remains in the solution mix.

4. Unbounded Solutions

No finite solution may exist in problems that are not bounded. This means that a variable can be infinitely large without violating a constraint.

In the simplex method, the condition of unboundedness will be discovered prior to reaching the final tableau. We will note the problem when trying to decide which variable to remove from the solution mix.

The procedure in unbounded solution is to divide each quantity column number by the corresponding pivot column number. The row with the smallest **positive** ratio is replaced. But if the entire ratios turn out to be **negative** or **undefined**, it indicates that the problem is unbounded.

Example:

Maximization case

C_j		6	9	0	0	
	SV	X_1	X_2	S_1	S_2	Q
9	X_2	-1	1	2	0	30
0	S_2	-2	0	-1	1	10
	Z_j	-9	9	18	0	270
	$C_j - Z_j$	15	0	-18	0	

RR

30/-1=-30

10/-2=-5

}

Unacceptable RRs



Pivot Column

The solution in the above case is not optimal because not all $C_j - Z_j$ entries are 0 or negative, as required in a maximization problem. The next variable to enter the solution should be X_1 . To determine which variable will leave the solution, we examine the ratios of the quantity column numbers to their corresponding numbers in the X_1 or pivot column. Since both pivot column numbers are negative, an unbounded solution is indicated.

- No unbounded solutions, no outgoing variable will exist.

5. Degeneracy

/Tie for leaving basic variable (key row)/

If there is a tie for the smallest ratio, this is a signal that degeneracy exists. Degeneracy can occur right in the first (initial tableau). This normally happens when the number of constraints is less than the number of variables in the objective function. Problem can be overcome by **trial and error**

method.

C_j		5	8	2	0	0	0		
CBV	BV	X_1	X_2	X_3	S_1	S_2	S_3	Q	RR
8	X_2	1/4	1	1	-2	0	0	10	$10/1/4=40$
0	S_2	4	0	1/3	-1	1	0	20	
0	S_3	2	0	2	2/5	0	1	10	$20/4=5$
	Z_j	2	8	8	16	0	0	80	
	$C_j - Z_j$	3	0	-6	-16	0	0		

$\left. \begin{matrix} 10/1/4=40 \\ 20/4=5 \end{matrix} \right\} \text{ Tie for the smallest ratio indicates degeneracy.}$

Degeneracy could lead to a situation known as cycling, in which the simplex algorithm alternatives back and forth between the same non-optimal solutions, i.e, it puts a new variable in, then takes it out in the next tableau, puts it back in, and so on.

One simple way of dealing with the issue is to select either row (**S_2 or S_3** in this case) arbitrary. If we are unlucky and cycling does occur, we simply go back and select the other row.

Remark

When there is a tie between a slack and artificial variable to leave the basis, the preference shall be given to artificial variable to leave the basis and there is no need to apply the procedure for resolving such cases.

6. Multiple Optimal Solutions

Multiple optimal solutions exist when non-basic variable contains **zero** on its $C_j - Z_j$ row.

Example:

Maximization problem

C_j		3	2	0	0	
CBV	SV	X_1	X_2	S_1	S_2	Q
2	X_2	3/2	1	1	0	6
0	S_2	1	0	1/2	1	3
	Z_j	3	2	2	0	12
	$C_j - Z_j$	0	0	-2	0	

$$\text{Max} Z = 3X_1 + 2X_2$$

$X_1=0, X_2=6, S_2=3$ and $\text{Max} Z=12$ or: $X_1=3, X_2=3/2$ and $\text{Max} Z=12$

The $C_j - Z_j$ value of the Non-basic variable (X_1) is 0. Thus, there is alternative optimal solution.

Exercise:

1. Solve the following LPP by the simplex algorithm

$$\text{Min } Z = 6x_1 + 8x_2$$

Subject to:

$$x_1 + 2x_2 \geq 80$$

$$3x_1 + x_2 \geq 75$$

$$x_1, x_2 \geq 0$$

What are the values of the basic variables at each iteration?

Which are the non-basic variables at each iteration?

1. At the 3rd iteration of a particular LP maximization problem, the following tableau is established:

C_j								Q
CBV	BV	X_1	X_2	X_3	S_1	S_2	S_3	
5	X_3	0	1	1	-2	0	0	5
6	X_1	1	-3	0	0	0	1	12
0	S_2	0	2	0	1	1	-1	10
Z_j		6	-13	5	5	0	21	97
$C_j - Z_j$		0	16	0	-5	0	-21	

What special condition exists as you improve the profit and move to the next iteration? Proceed to solve the problem for optimal solution



Self-check exercise 2.4

1. Solve the following LPP to show that it has unbounded solution.

a. Max. $Z = -2x_1 + 3x_2$

b. Max $Z = 3x_1 + 6x_2$

Subject to: $x_1 \leq 5$

Subject to: $3x_1 + 4x_2 \geq 12$

$2x_1 - 3x_2 \leq 6$

$-2x_1 + x_2 \leq 4$

$x_1, x_2 \geq 0$

$x_1, x_2 \geq 0$

2. Solve the following LPP to show that it has alternative optimal

solutions. a. Max $Z = 6x_1 + 3x_2$

Subject to:

$2x_1 + x_2 \leq 8$

$3x_1 + 3x_2 \leq 18$

$x_2 \leq 3$

$x_1, x_2 \geq 0$

2.6 Sensitivity Analysis and Duality

2.6.1 Sensitivity Analysis

Sensitivity Analysis is concerned with the study of 'Sensitivity' of the optimal solution of an LPP with discretion variables (changes) in parameters. The degree of sensitivity of the solution due to those variations can range from no change at all to a substantial change in the optimal solution of the given LPP. Thus, insensitivity analysis, we determine the range over which the LP model parameters can change without affecting the current optimal solution. The process of studying the sensitivity of the optimal solution of an LPP is called **post-optimal analysis**.

The two sensitivity analysis approaches are:

- I. Trial and error approach and
- II. Analytical approach

II. Analytical approach

Five types of discrete changes in the original LP model may be investigated during the sensitivity analysis:

- A. ***Changes of the coefficients of the objective function (c_j)***
- B. ***Changes of the RHS Quantity(b_j)***
- C. ***Changes of the input-output coefficient***
- D. ***Add/delete constraints***
- E. ***The addition of a new variable to the problem***

A. Changes of the coefficients of the objective function (c_j)

Decision variables can be:

- i. ***Basic (in the solution)***
- ii. ***Non-basic (out-of the solution)***

Note: Instead of resolving the entire problem as a new problem with new parameters, we may take the original optimal solution table as an initial solution table for the purpose of knowing ranges both lower and upper within which a parameter may assume value.

a. Range for the coefficients of basic decision variables

The range of optimality is the range over which a basic decision variable coefficient in the objective function can change without changing the optimal solution mix. However, this change will change only the optimal value of the objective function.

Example:

$$\text{Max. } Z = 5x_1 + 4.5x_2 + x_3$$

Subject to:

$$15x_1 + 15.8x_2 \leq 150$$

$$5x_1 + 6.4x_2 + 15x_3 \leq 77$$

$$2.8x_2 + 11.8x_3 \leq$$

$$36 \quad x_1, x_2, x_3 \geq 0$$

The optimal tableau for this solution is:

C_j		5	4.5	1	0	0	0	
CBV	SV	X_1	X_2	X_3	S_1	S_2	S_3	Q
5	X_1	1	1.053	0	0.067	0	0	10
1	X_3	0	0.67	1	-0.022	0.067	0	1.8
0	S_3	0	1.924	0	0.258	-0.773	1	15.12
Z_j		5	5.342	1	0.311	0.067	0	51.8
$C_j - Z_j$		0	-0.842	0	-0.311	-0.067	0	

Determine the range of optimality for the coefficients of the basic-decision variables.

Solution: Analysis of basic decision variables

	X_2	S_1	S_2 ←	Non-basic variables
$C_j - Z_j$ row	-0.842	-0.311	-0.067	
X_1 row	1.053	0.067	0	
<u>$C_j - Z_j$ row</u>	-0.8	-4.64	$-\infty$	

The analysis will be conducted on products on X_1 and X_3 which are in the basic solution. Divide each $C_j - Z_j$ row entry for variables not in the solution (for instance, by X_2 , S_1 and S_2 values) by the associated variable a_{ij} from X_1 or X_3 row.

I. Analysis of X_1

Steps:

- a. Take the $C_j - Z_j$ row of the optimal solution of the non-basic variables
- b. Take the X_1 row of the non-basic variables
- c. $C_j - Z_j$ row
 X_1 row



Upper Limit

The ***smallest positive number*** in the $C_j - Z_j$ row tells how much the profit X_1

of X_1 can be increased before the solution is changed.

$$\text{Upper Limit} = C_j \text{ (for } X_1) + \text{the smallest positive value of } \underline{C_j - Z_j} \text{ row}$$

$$= 5 + \infty = \infty$$

X_1 row

Note: C_j (for X_1) = 5 (Look in the OF of the LP problem)



Lower Limit

The ***largest negative number closest*** (negative amount closest to 0)

$$\begin{aligned}\text{Lower Limit} &= C_j \text{ (for } X_1) + \text{The largest negative value of } \underline{C_j - Z_j \text{ row}} \\ &= 5 + (-0.8) = \underline{4.2} \quad X_1 \text{ row}\end{aligned}$$

Therefore, the range of optimality for the coefficient of X_1 is $4.2 \leq C_j \text{ (for } X_1) < \infty$ (The coefficient of X_1 in the objective function can change between 4.2 and ∞ without changing the optimal solution mix $X_1=10$, $X_3=1.8$ and $S_3=15.12$)

II. Analysis of X_2

$$\begin{aligned}\text{Upper Limit} &= C_j \text{ (for } X_3) + \text{The smallest positive value of } \underline{C_j - Z_j \text{ row}} \\ &= 1 + (14.13) = \underline{15.13} \quad X_1 \text{ row}\end{aligned}$$

Note: $C_j \text{ (for } X_3) = 5$ (Look in the OF of the LP problem)

$$\begin{aligned}\text{Lower Limit} &= C_j \text{ (for } X_3) + \text{The largest negative value of } \underline{C_j - Z_j \text{ row}} \\ &= 1 + (-1) = \underline{0} \quad X_1 \text{ row}\end{aligned}$$

Therefore, the range of optimality for the coefficient of X_3 is $0 \leq C_j \text{ (for } X_3) < 15.13$ (The coefficient of X_3 in the objective function can change between 0 and 15.13 without changing the optimal solution mix $X_1=10$, $X_3=1.8$ and $S_3=15.12$). However, this change will change only the optimal value of the objective function (i.e. MaxZ will change)

Exercise:

$$\text{Max. } Z = 50x_1 + 120x_2$$

Subject to:

$$x_1 + 4x_2 \leq 80$$

$$3x_1 + x_2 \leq 60$$

$$x_1, x_2 \geq 0$$

	X_2	S_1	S_2 ← Non-basic variables
$C_j - Z_j \text{ row}$	-0.842	-0.311	-0.067
$X_3 \text{ row}$	0.67	-0.022	0.067
$\underline{C_j - Z_j \text{ row}}$ $X_1 \text{ row}$	-1.26	14.13	-1

Determine the range of optimality for the coefficient of the basic variables.

Optimal Solution

C_j						
CBV	BV	X_1	X_2	S_1	S_2	Q
5	X_1	1/2	1	1/4	0	20
1	S_2	5/2	0	-1/4	1	40
Z_j		60	120	30	0	\$2,400
$C_j - Z_j$		\$-10	\$0	\$-30	\$0	

Ans: The range of optimality for X_2 's profit coefficient is: $\$100 \leq C_j \text{ (for } X_2) < \infty$

b. The range for the non-basic variables

If there is a variable C_j , not participating in the **optimal basis**, then, in order for the variable to be included in **the optimal solution**, its coefficient in the objective function will have to change from the existing C_j to a new level $C_j(\text{new})$.

$$C_j(\text{new}) > Z_j$$

The range of insignificance is the range over which C_j rates for non-basic variables can vary without causing a change in the optimal solution mix (variable) is called the range of insignificance.

Example:

$$\text{Max. } Z = 5x_1 + 4.5x_2 + x_3$$

Subject to:

$$15x_1 + 15.8x_2 \leq 150$$

$$5x_1 + 6.4x_2 + 15x_3 \leq 77$$

$$2.8x_2 + 11.8x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$

The optimal tableau for this solution is:

C_j		5	4.5	1	0	0	0	
CBV	SV	X_1	X_2	X_3	S_1	S_2	S_3	Q
5	X_1	1	1.053	0	0.067	0	0	10
1	X_3	0	0.67	1	-0.022	0.067	0	1.8
0	S_3	0	1.924	0	0.258	-0.773	1	15.12
Z_j		5	5.342	1	0.311	0.067	0	51.8
$C_j - Z_j$		0	-0.842	0	-0.311	-0.067	0	

Calculate the range of insignificance for the coefficient of non-basic variable(X_2)

Solution

$C_j(\text{for } X_2)=4.5 \text{ and } Z_j(\text{ for } X_2) =5.342$

$C_j(\text{new for } X_2)>5.342 \Rightarrow C_j(\text{new for } X_2)$. If the profit contribution of X_2 is greater than 5.342, then X_2 will be included in the solution.

Thus, $-\infty < C_j(\text{new for } X_2) \leq 5.342$ is the range of insignificance for X_2 .

$C_j(\text{new for } X_2)$ can vary within the given range without causing a change in the optimal solution mix ($X_1=10, X_2=0, X_3=1.8, S_1=0, S_2=0 \text{ and } S_3=15.12$).

B. Change in the Right Hand—Side Quantity (RHS)

Or

Change in the availability of resource (Capacity) (b_i)



Shadow prices:



How much should a firm be willing to pay to make additional resources available?

Shadow prices signify the change in the optimal value of the objective function for **1 unit increases** in the value of the RHS of the constraint that represent the availability of scarce resources.

The negative of the number of $C_j - Z_j$ row in **its slack variable columns** provide as with shadow prices. **Or:** shadow prices are found in the Z_j row of the final simplex tableau in the slack variable columns.

❖ **RHS ranging** is the range over which shadow prices remain valid.

Example:

$$\text{Max. } Z = 3x_1 + 4x_2$$

Subject to:

$$3x_1 + 5x_2 \leq 15$$

$$2x_1 + x_2 \leq 8$$

$$x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

The optimal tableau is given as:

C_j		3	4	0	0	0	
SV		X_1	X_2	S_1	S_2	S_3	Q
3	X_1	1	0	-0.143	0.714	0	3.57
0	S_3	0	0	-0.286	0.428	1	1.143
4	X_2	0	1	0.286	-0.428	0	0.857
$C_j - Z_j$		0	0	-0.714	-0.428	0	

Required:

1. Determine the shadow price for each constraint
2. Determine the RHS ranges over which the shadow prices are valid

❖ Analysis of the 1st constraint (S₁)

<u>Q</u>	<u>S₁</u>	<u>Q/ S₁</u>
3.57	-0.143	-24.96
1.143	-0.286	-3.99
0.857	0.286	3.00

Lower Limit=b_j-the smallest positive number in the Q/ S_j column
Upper Limit=b_j-the largest negative number in the Q/ S_j column

Lower Limit=b₁-the smallest positive number in the Q/ S₁column

$$=15-3=\underline{12}$$

Upper Limit=b₁-the largest negative number in the Q/ S₁ column

$$=15-(-3.99) =\underline{18.99}$$

Therefore, $12 \leq b_1 \leq 18.99$ (The range of resource 1 over which the shadow price \$0.714 per unit is valid).

❖ Analysis of the 2nd constraint (S₂)

<u>Q</u>	<u>S₂</u>	<u>Q/ S₂</u>
3.57	0.714	5
1.143	0.428	2.67
0.857	-0.428	-2

Lower Limit=b₂-the smallest positive number in the Q/ S₂ column

$$=8-(2.67) =\underline{5.33}$$

Upper Limit= b_2 -the largest negative number in the Q/ S_2 column

$$=8-(-2) =\underline{10}$$

Therefore, $5.33 \leq b_1 \leq 10$ (The range of resource 2 over which the shadow price \$0.428 per unit is valid).

❖ Analysis of the 3rd constraint (S_3)

Q	S_3	Q/ S_3
3.57	0	-
1.143	1	1.143
0.857	0	-

Lower Limit= b_3 -the smallest positive number in the Q/ S_3 column

$$=2-(1.143) = \underline{0.857}$$

Upper Limit= b_3 -the largest negative number in the Q/ S_3 column

$$=2- (-\infty) =\underline{\infty}$$

Therefore, $0.857 \leq b_3 \leq \infty$ (The range of resource 3 over which the shadow price \$0 per unit is valid).

Exercise:

1. $\text{Max. } Z = 50x_1 + 40x_2$

Subject to:

$$3x_1 + 5x_2 \leq 150 \text{ (Assembly time)}$$

$$x_2 \leq 20 \text{ (Portable display)}$$

$$8x_1 + 5x_2 \leq 300 \text{ (Warehouse space)}$$

$$x_1, x_2 \geq 0$$

The final simplex tableau is:

C_j		50	40	0	0	0	
	SV	X_1	X_2	S_1	S_2	S_3	Q
40	X_2	0	1	8/25	0	-3/25	12
0	S_2	0	0	-8/25	1	3/25	8
50	X_1	1	0	-5/25	0	5/25	30
	Z_j	50	40	14/5	0	26/5	1980
	$C_j - Z_j$	0	0	-14/5	0	-26/5	

Determine the shadow prices for the three constraints for the High-Tech Co.

Answer:

The Z_j values for the three slack variables are 14/5, 0 and 26/5, respectively.

Thus, the shadow price for the assembly **time constraint is 14/5 (i.e.1 additional assembly time over the existing 150 is \$2.8)**

The shadow price for the **portable display constraint is 0.**

The shadow price for the **Warehouse space constraint is 26.5**

Therefore, we see that obtaining more warehouse space will have the biggest positive impact on High Tech's profit.

Note: With a greater than or equal to constraint, the value of the shadow price will be less than or equal to zero because a 1 unit increase in the value of the RHS cannot be helpful; it makes it more difficult to satisfy the constraint. As a result, for a maximization problem the optimal value of the objective function can be expected to decrease when the RHS side of a greater than or equal to constraint is increased.

2. Solve the following LPP using simplex method. A firm that manufactures both lawn mowers and snow blowers:

X_1 =the number of lawn mowers

X_2 =the number of snow blowers

$$\text{Max. } Z = \$30x_1 + \$80x_2$$

Subject to:

$$2x_1 + 4x_2 \leq 1000 \text{ Labor hours available}$$

$$6x_1 + 2x_2 \leq 1,200 \text{ lb of steel available}$$

$$x_2 \leq 20 \text{ snow blower engine available}$$

$$x_1, x_2 \geq 0$$

- a) What is the best product mix? What is the optimal profit?

Answer: $x_1=100$, $x_2=200$ and profit = \$19,000

- b) What are the shadow prices? When the optimal solution has been reached, which resource has the highest marginal value?

Answer: The shadow price for 1 additional labor = \$15

The shadow price for 1 additional pound of steel = 0

The shadow price for 1 additional snow blower engine made available = \$20

Thus, snow blower engine has the highest marginal value at the optimal solution.

- c) Over what range in each of the RHS values are these shadows valid?

Answer: The shadow price for labor hours is valid from 800 hours to 1,066.66 hours. The shadow price for pounds of steel is valid from 1,000 pounds up to an infinite number of pounds. The shadow price for snow blower engines ranges from 180 engines up to 250 engines

- d) What are the ranges over which the objective function coefficients can vary for each of the two decision variables?

Answer: Without changing the current solution mix, the profit coefficient for the mowers can range from \$0 to 40, while the coefficient for the blowers can range from \$60 to infinity.

2.6.2 Duality

Every LPP has another LPP associated with it, which is called its **dual**. The first way of starting a linear problem is called the **primal** of the problem. The second way of starting the same problem is called the **dual**. The optimal solutions for the primal and the dual are equivalent, but they are derived through alternative procedures.

Note:

- *The dual contains economic information useful to management, and it may also be easier to solve, in terms of less computation, than the primal problem.*
- *Corresponding to every LPP, there is another LPP.*
- *The given problem is called the primal. The related problem to the given problem is known as the dual.*
- *The dual of a dual is the primal*
- *If the primal has optimal solution, the dual will have optimal solution*
- *If the primal has no optimal solution, the dual will not have optimal solution.*
- *Whether we follow the dual or primal system, the optimal solution will remain equal.*

Primal-Dual Relationship

<i>Primal</i>	<i>Dual</i>
<i>Objective is minimization</i>	<i>Objective is maximization and vice versa</i>
<i>$\geq$ type constraints</i>	<i>$\leq$ type constraints</i>
<i>N^0 of columns</i>	<i>N^0 of rows</i>
<i>N^0 of rows</i>	<i>N^0 of columns</i>
<i>N^0 of decision variables</i>	<i>N^0 of constraints</i>
<i>N^0 of constraints</i>	<i>N^0 of decision variables</i>
<i>Coefficient of Object function</i>	<i>RHS value</i>
<i>RHS value</i>	<i>Coefficient of Object function</i>

Duality Advantage

1. The dual form provides an alternative form

2. The dual reduces the computational difficulties associated with some formulation
3. The dual provides an important economic interpretation concerning the value of scarce resources used.

Example:

Write the duals to the following problems

a) $\text{Max. } Z = 5x_1 + 6x_2$

Subject to:

$2x_1 + 3x_2 \leq 3000$ (**Labor constraint**)

$5x_1 + 7x_2 \leq 1000$ (**Machine constraint**)

$x_1 + x_2 \leq 500$ (**Market constraint**)

$x_1, x_2 \geq 0$

Solution

Represent primal in the conventional table as follows

<i>Dual variables</i>	x_1	x_2	<i>Constraints</i>
u_1	2	3	≤ 3000
u_2	5	7	≤ 1000
u_3	1	1	≤ 500
MaxZ	5	6	

By referring the above table, dual for this can be stated as:

$\text{Min } Z^* = 3000 u_1 + 1000 u_2 + 500 u_3$

st:

$2u_1 + 5u_2 + u_3 \geq 5$

$3u_1 + 7u_2 + u_3 \geq 6$

$u_1, u_2, u_3 \geq 0$

Note:

1. For maximizing, all constraints must be brought to “ $\leq$ ” form
2. For minimizing, all constraints must be brought to “ $\geq$ ” form
3. If they are not, use multiplication factor -1
4. “ $=$ ” is an intersection of “ $\geq$ ” and “ $\leq$ ”

b. $\text{Max. } Z = 60x_1 + 50x_2$

Subject to:

$$2x_1 + 4x_2 \leq 80$$

$$3x_1 + 2x_2 \leq 60$$

$$x_1 \leq 15$$

$$2x_2 \leq 36$$

$$x_1, x_2 \geq 0$$

Solution

Primal is represented in the table as follows:

<i>Dual variables</i>	<i>x₁</i>	<i>x₂</i>	<i>Constraints</i>
<i>u₁</i>	2	4	≤ 80
<i>u₂</i>	3	2	≤ 60
<i>u₃</i>	1	0	≤ 15
<i>u₄</i>	0	2	≤ 36
<i>MaxZ</i>	60	50	

The dual form is:

$$\text{Min } Z^* = 80u_1 + 60u_2 + 15u_3 + 36u_4$$

St:

$$2u_1 + 3u_2 + u_3 \geq 60$$

$$4u_1 + 2u_2 + u_4 \geq 50$$

$$u_1, u_2, u_3, u_4 \geq 0$$

c. Obtain the dual problem of the following primal LPP

$$\text{Min. } Z = x_1 + 2x_2$$

Subject to:

$$2x_1 + 4x_2 \leq$$

$$160 \quad x_1 - x_2 =$$

$$30 \quad x_1 \geq 10$$

$$x_1, x_2 \geq 0$$

Solution

Rule:

1. For a maximization primal with all $\leq$ type constraints there exists a minimization dual problem with all $\geq$ type constraints and vice versa. Thus, the inequality sign is reversed in all the constraints except the non-negativity conditions.
2. Transform the $\leq$ type constraint to a $\geq$ type constraint by multiplying the constraint by -1.
3. Write the $=$ type constraint equivalent to two constraints of the type $\geq$ and $\leq$.

The standard primal LPP so obtained is:

$$\text{Min. } Z = x_1 + 2x_2$$

Subject to:

$$-2x_1 - 4x_2 \geq -160$$

$$x_1 - x_2 \geq 30$$

$$-x_1 + x_2 \geq -30$$

$$x_1 \geq 10$$

$$x_1, x_2 \geq 0$$

Let u_1, u_2, u_3 , and u_4 be the dual variables corresponding to the four constraints in given order, then the dual of the given primal problem can be formulated as follows:

$$\text{Max } Z^* = -160 u_1 + 30 u_2 - 30 u_3 + 10$$

St:

$$-2u_1 + u_2 - u_3 + u_4 \leq 1$$

$$-4u_1 - u_2 + u_3 \leq 2$$

$$u_1, u_2, u_3, u_4 \geq 0$$

Let $u = u_2 - u_3$, then the above dual problem reduces to the form:

$$\text{Max } Z^* = -160u_1 + 30u_2 - 30u_3 + 10u_4$$

St:

$$-2u_1 + u + u_4 \leq 1$$

$$-4u_1 - u \leq 2$$

$$u_1, u_4 \geq 0, u \text{ unrestricted in sign}$$

Here it may be noted that the second constraint in the primal is equality. Therefore, the corresponding dual u_2 should be unrestricted in sign.



Self-check exercise 2.5

1. Write the dual of the following primal LP problems

a. Min. $Z = 3x_1 - 2x_2 + 4x_3$

St:

$$3x_1 + 5x_2 + 4x_3 \geq 7$$

$$6x_1 + x_2 + 3x_3 \geq 4$$

$$7x_1 - 2x_2 - x_3 \leq 10$$

$$4x_1 + 7x_2 - 2x_3 \geq 2$$

$$x_1, x_2, x_3 \geq 0$$

c. Min. $Z = 7x_1 + 3x_2 + 8x_3$

St:

$$8x_1 + 2x_2 + x_3 \geq 3$$

$$3x_1 + 6x_2 + 4x_3 \geq 4$$

$$4x_1 + x_2 + 5x_3 \geq 1$$

$$x_1 + 5x_2 + 2x_3 \geq 7$$

$$x_1, x_2, x_3 \geq 0$$

e. Min. $Z = 8x_1 + 3x_2$

St:

$$x_1 - 6x_2 \geq 2$$

$$5x_1 + 7x_2 = -4$$

$$x_1, x_2 \geq 0$$

b. Max. $Z = 2x_1 + 5x_2 + 6x_3$

St:

$$5x_1 + 6x_2 - x_3 \leq 3$$

$$-2x_1 + x_2 + 4x_3 \leq 4$$

$$x_1 - 5x_2 + 3x_3 \leq 1$$

$$-3x_1 - 2x_2 + 7x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0$$

d. Max. $Z = 2x_1 + 3x_2 + x_3$

St:

$$4x_1 + 3x_2 + x_3 = 6$$

$$x_1 + 2x_2 + 5x_3 = 4$$

$$x_1, x_2, x_3 \geq 0$$

f. Min. $Z = 3x_1 + x_2 + x_3 - x_4$

St:

$$2x_1 - x_2 + 3x_3 + x_4 = 1$$

$$x_1 + x_2 - x_3 + x_4 = 3$$

$$x_1, x_2, x_3, x_4 \geq 0$$

CHAPTER SUMMARY

Linear Programming is a branch of mathematical programming which is designed to solve *optimization problems* where all the constraints as well as the objectives are expressed as *linear function*. It was developed by George B. Dantzig in 1940. Its earlier application was solely related to the activities of the second World War. However soon its importance was recognized and it came to occupy a prominent place in the industry and trade.

The basic steps in formulating a linear programming model are presented as follows:

Step I Identification of the decision variables. The decision variables (parameters) having a bearing on the decision at hand shall first be identified, and then expressed or determined in the form of linear algebraic functions or in equations.

Step II Identification of the constraints. All the constraints in the given problem which restrict the operation of a firm at a given point of time must be identified in this stage.

Step III formulate the objective function. In the last stage, the objective which is required to be optimized (i.e., maximized or minimized) must be clearly identified and expressed in terms of the pre-defined decision variables.

There are two types of finding a solution for Linear programming problems these are Graphic solution and Simplex solution method.

The graphic solution procedure is one of the methods of solving two variable linear programming problems. Steps in graphic solution method: -

Step I: Defining the problem. Formulate the problem mathematically. Express it in terms of several mathematical constraints & an objective function.

Step II: Plot the constraints graphically. Each inequality in the constraint equation has to be treated as an equation.

Step III: Locate the solution space. *Solution space or the feasible region is the graphical area which satisfies all the constraints at the same time. Such a solution point (x, y) always occurs at the corner Points of the feasible Region the feasible region is determined as follows:*

For "greater than" & "greater than or equal to" constraints, the feasible region or the solution space is the area that lays above the constraint lines.

For "Less Than" & "Less than or equal to" constraint, the feasible region or the solution space is the area that lays below the constraint lines.

Step IV: Selecting the graphic solution technique. *Select the appropriate graphic technique to be used for generating the solution.*

Some of the important terms commonly used in linear programming are:

Solution: Values of the decision variable (= 1, 2, 3 ...) satisfying the constraints of a general linear programming model is known as the solution to that linear programming model.

Feasible solution: Out of the total available solution a solution that also satisfies the non-negativity restrictions of the linear programming problem is called a feasible solution.

Basic solution: For a set of simultaneous equations in Q unknowns (P, Q) a solution obtained by setting (P - Q) of the variables equal to zero & solving the remaining P equation in P unknowns is known as a basic solution. The variables which take zero values at any solution are detained as non-basic variables & remaining are known as-basic variables, often called basic solution.

Basic feasible solution: A feasible solution to a general linear programming problem which is also basic solution is called a basic feasible solution.

Optimal feasible solution: Any basic feasible solution which optimizes (i.e. maximize or minimizes) the objective function of a LP models is known as the optimal feasible solution to that linear programming model.

Degenerate Solution: A basic solution to the system of equations is termed as degenerate if one or more of the basic variables become equal to zero.

The second method to find a solution for LPP is using the Simplex method. It is an **iterative or “step by step”** method or repetitive algebraic approach that moves automatically from one basic feasible solution to another basic feasible solution improving the situation each time until the optimal solution is reached at.

The summary of the extra variable to be added in the given LP problem to convert it in to standard form is given in the following table.

Types of Constraint	Extra variables to be added	Coefficient of extra variables <u>in the objective function</u>		Presence of variables in the initial solution mix
		MaxZ	MinZ	
$\leq$	Add only slack variable	0	0	Yes
$\geq$	Subtract surplus variable and	0	0	No
	Add artificial variable	-M	+M	Yes
=	Add artificial variable	-M	+M	Yes

CHAPTER THREE

TRANSPORTATION AND ASSIGNMENT MODEL

3.1 Introduction

One important application of linear programming has been in the area of the physical distribution (transportation) of resources, from one place to another, to meet a specific set of requirements. This chapter describes two special-purpose algorithms: the transportation model and the assignment model. Model formulation and manual solution are covered for each of these classes of problems. Both transportation and assignment problems are members of a category of linear programming techniques called network flow problems.

All linear programming can be solved by the Simplex method. However due to their specialized structure, certain classes of linear programming problems lend themselves to solutions by other techniques that are computationally more efficient than the simplex methods. These types of linear problems can be represented by the so-called transportation and assignment model. Let us first describe the transportation model and the assignment model will be discussed later in this chapter.

3.2 Objective of The Chapter



Formulation of a Transportation Problem



Determine basic feasible solution using various methods



Understand the MODI, Stepping Stone Methods for cost minimization



Understand how to solve unbalanced Transportation Problem



Assignment Problem Formulation



How to solve the Assignment Problem



How to solve the unbalanced problem using appropriate method



Modify the problem when the objective is to maximize the objective function

3.3 Transportation Problems

Transportation problem deals with the distribution of goods from several points of supplies (sources) to a number of points of demands (destinations). Consider a corporation engaged in the

manufacture of products. Most of such big corporations are of “**multiple-product**” and “**multi-unit**” organizations having production units situated at different places. Items are produced for sales. Sales take place at different markets which are, again located at different places. It is not feasible to co-locate production and market. Markets are located away from the manufacturing places. Hence products are sent to factory warehouses set up near market outlets. **Cost of product** consists of **production cost** and **distribution cost**.

$$\text{Cost of product} = \text{production cost} + \text{distribution cost}$$

Distribution cost consists of mainly the transportation cost of items from its production (manufacturing) center to the warehouses. Transportation techniques are designed to minimize the distribution costs. In order to identify products, it is necessary to work out per unit distribution cost of each product. We also know the production capacity of each product in each factory is not fixed. The holding capacity of a warehouse or potential sales in each marketing center is again a fixed quality which cannot be exceeded.

Basic transportation problem was stated by F.L. Hitchcock in 1941, and later expanded by T.C. Koopmans and W.W. Cooper. Several extensions of transportation model and methods have been subsequently developed. The transportation problem is a particular case of linear programming problem.

The structure of transportation problem involves a large number of shipping routes from several supply origins to several demand destinations. The objective is to determine the number of units which should be shipped from an origin a destination in order to satisfy the required quantity of goods or services at each demand destination, within the limited quantity of goods or services available at each supply origin, at the minimum transportation cost and/or time.

The transportation algorithm discussed in this chapter is applied to minimize the total cost of transporting a homogeneous commodity (product) from supply origin to demand destination. However, it can also be applied to the maximization of some total value or utility, for example, financial resources are distributed in such a way that the profitable return is maximized. Note that

the assumption of homogeneous commodity implies no difference in commodity characteristics among origins.

The characteristics of transportation problem are as follows:

1. A limited supply of one commodity is available at certain sources or origins.
2. There is a demand for the commodity at several destinations
3. The quantities of supply at each source and the demand at each destination are constant.
4. The shipping or transportation costs per unit from each source to each destination are assumed to be constant.
5. No shipments are allowed between sources or between destinations. All supply and demand quantities are given in whole number or integers.
6. The problem is to determine how many units shipped from each source to each destination so that all demands are satisfied at the minimum total shipping costs.

Uses of transportation techniques:

1. Reduce distribution or transportation cost
2. Improve competitiveness of product
3. Assist proper location of warehouses
3. Assist proper location of new factories or plants being planned.
4. Close down warehouses which are found costly and uneconomical.

The objective of transportation problem is:

1. To identify the optimal shipping routes-**minimum cost route**
2. To identify the maximum amount that can be shipped over the optimum route
3. To determine the total transformation cost or the profit of transportation

3.3.1 The Transportation Method

In order to develop a model of transportation problem, it is necessary to know the following information:

- ✓ The sources of supply (origin) which may include production facilities, warehouses, or supply points, characterized by available capacities.
- ✓ The destinations which may include consumption facilities, warehouses or demand points, characterized by required levels of demand.

- ✓ Supply quantity (capacity) of each origin which is stated in rows.
- ✓ Demand quantity of each destination, which is stated in column
- ✓ Unit transportation cost (in birr) for each route.
- ✓ The total quantity available for shipment is equal to the total quantity demanded

The solution algorithm to a transportation problem may be summarized into the following steps:

Step 1: Formulate the problem and set up in the matrix form

The formulation of the problem is similar to the linear programming. Here the objective function is the total transportation cost and the constraints are the supply and demand available at each source and destination respectively.

Step 2: Obtain an initial basic feasible solution

There are **3 methods** to find the initial feasible solution.

- 1. North-West Corner Method (NWCN)**
- 2. Least Cost Method (LCM)**
- 3. Vogel's Approximation Method (VAM)**

The initial solution obtained by any of the three methods must satisfy the following condition:

i. The solution must be feasible

i.e.: It must satisfy all the supply and demand constraints

ii. The number of positive allocations must equal to $m+n-1$, where **m**=the number of rows (or origins or supply centers) and **n**= the number of columns (or destination centers or demand centers)

Example: $m=3$ origins and $n=4$ destinations $\Rightarrow m+n-1=3+4-1=6$ (i.e. the transportation model should have 6 occupied cells).

Note: The number of occupied cells $< m+n-1 \Rightarrow$ degenerate solution

Step 3: Test the initial solution for optimality

If the current solution is optimal, then stop. Otherwise, determine the new improved solution.

Step 4: Repeat step 3 until an optimal solution is reached

3.3.2 Linear programming formulation of the transportation problem

Example1:

Suppose that a firm has three factories /sources of supply/ & four warehouses /point of demand/. The firm's production capacity at the three factories, the demand for the four distribution centers located at various regions & the cost of shipping each unit from the factories to the warehouses through each route is given as follows:

Destinations (dd) =j

Origin (Supply)	W ₁	W ₂	W ₃	W ₄	Factory Capacity =i
F ₁	Br.3	2	7	6	5000
F ₂	7	5	2	3	6000
F ₃	2	5	4	5	2500
Requirements of the Warehouses (Units of demand)	6000	4000	2000	1500	13500

Using all the above information

- I. Express the following transportation problem as an LPP

Solution

Let x_{ij} = The amount of commodity to be transported form source i ($i=1,2,3$) to destination j ($j=1,2,3,4$).

Then the objective function of the problem (minimization of the total transportation cost) can be formulated as:

$$\begin{aligned} \text{Min} Z = & 3x_{11} + 2x_{12} + 7x_{13} + 6x_{14} + \\ & 7x_{21} + 5x_{22} + 2x_{23} + 3x_{24} + \\ & 2x_{31} + 5x_{32} + 4x_{33} + 5x_{34} \end{aligned}$$

Subject to the constraints

a. **Supply constraints:**

$$\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} &= 5000 && F_1 \text{ supply constraint} \\ x_{21} + x_{22} + x_{23} + x_{24} &= 6000 && F_2 \text{ supply constraint} \\ x_{31} + x_{32} + x_{33} + x_{34} &= 2500 && F_3 \text{ supply constraint} \end{aligned}$$

b. Demand constraints:

$$x_{11} + x_{21} + x_{31} = 6000 \quad W_1 \text{ demand constraint}$$

$$x_{12} + x_{22} + x_{32} = 4000 \quad W_2 \text{ demand constraint}$$

$$x_{13} + x_{23} + x_{33} = 2000 \quad W_3 \text{ demand constraint}$$

$$x_{14} + x_{24} + x_{34} = 1500 \quad W_4 \text{ demand constraint}$$

$$x_{ij} \geq 0 \text{ for all } i \& j$$

In the above LPP, there are $m \times n = 3 \times 4 = 12$ decision variables & $m + n = 3 + 4 = 7$ constraints. Thus, if this problem is solved by the simplex method, then it may take considerable computational time.

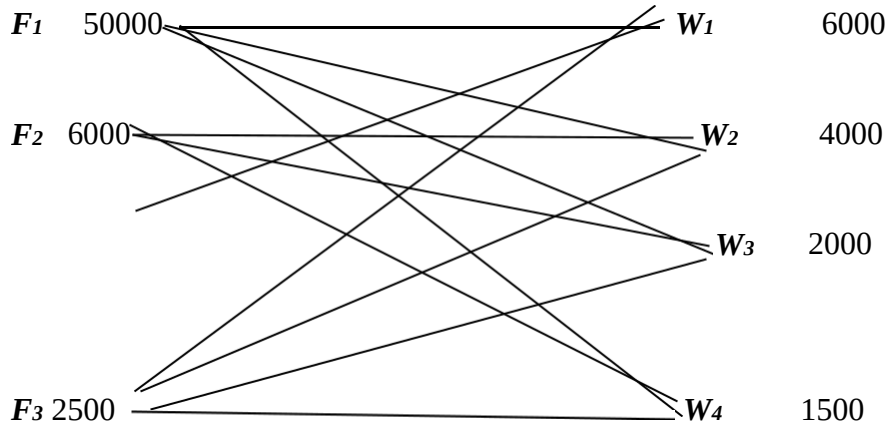
II. The network representation of the transportation LPP is called **Net work flow**

Origin

(Sources of Supply)

Destination centers

(Point of demand centers)



This LPP has 12 shipping routes. The objective is to identify the minimum cost route (Least cost route).

3.4. Methods of Finding Initial Feasible Solution

There are several methods available to obtain an initial feasible solution. Here we shall discuss only three different methods to obtain the initial feasible solution:

3.4.1 North- West Corner Method (NWCM)

The north –west corner method is a systematic approach for developing an initial basic feasible solution. Its *chief advantages* are that it is simple to use and easy to understand. Its *chief drawback* is that it does not take transportation cost in to account. Consequently, such solution may require much additional effort to obtain the optimal solution.

The north-west corner method gets its name because the starting point for the allocation process is the upper left hand (North- West) corner of transportation table.

This method can be summarized as follows:

Step1. Start with the sell at the upper left (north-west) corner of the transportation table and allocate as many units as possible to that cell. This will be the smaller of the row supply and the column demand. Adjust the row and column quantities to reflect the allocation.

Step2. - If allocation made in step 1 is equal to the supply available in the first row, then move vertically down to the cell in the second row and first column.

If allocation made in step1 is equal to the demand of the first destination (in first column), then move horizontally to the cell in the first brow and second column and apply step 1 again for next allocation.

Step3. Continue the procedure step by step till an allocation is made in the south east corner cell of the transportation table.

Example2; Consider the following transportation table and see how to find the initial feasible solution using NWCM.

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	34 34

Required: consider the above table and then find the initial basic feasible solution (total transportation cost) using NWCM.

To find the initial basic feasible solution using NWCM, track the following steps.

1. Beginning in cell (S1, D1) which is the north-west corner cell in the given transportation table. The value for row S1 (i.e.7) and column D1 (i.e.5) are compared. The smaller of the two, i.e. 5 is assigned as the first allocation; otherwise it will violate the feasibility condition. This means that 5 unit of commodities are to be transported from production facility 1 (S1) to warehouse 1 (D1). However, this allocation leaves a supply of 2 ($7-5=2$) units of commodity at S1.
2. Staying in row S1, move horizontally to cell (S1, D2) where supply is now 2 and demand is 8. Allocate 2 units to this cell, exhausting the supply of row S1 and leave 6 units of demand in D2 column.
3. Staying in column D2, move down to cell S2 D2, where supply is now 9 and demand is 6. Allocate 6 units to this cell, exhausting the demand in column D2 and leaving 3 units of supply in row S2.
4. Staying in row S2, move to cell (S2, D3) and allocate 3 units to this cell, exhausting that row's supply and leaving 4 units in column D3.
5. Staying in column D3, move down to cell (S3, D3) and allocate 4 units to this cell, exhausting D3 columns and leaving 14 units of supply.
6. Staying in column S3 move to cell (S3, D4) and allocate 14 units to this cell, exhausting both the rows and columns quantities.

	D1	D2	D3	D4	Supply
S1	19 5	30 2	50	10	7
S2	70	30 6	40 3	60	9
S3	40	8	70 4	20 14	18
Demand	5	8	7	14	34 34

NB: cells with bold quantity indicates the occupied cell)

No of occupied cell= $m + n - 1 = 3+4 -1 =6$, so the table is not degenerate solution.

The total transportation cost of the initial solution derived by the NWCM is obtained by multiplying the quantity in the occupied cells with the corresponding unit costs and adding.

Thus, the total transportation cost of the initial feasible solution derived by the NWCM is:

<u>Route</u>		<u>Unit</u>	<u>Per unit</u>	<u>Total</u>
<u>From</u>	<u>To</u>	<u>Shipped</u>	<u>Xcost (\$)</u>	= <u>Cost (\$)</u>
Plant 1	Store 1	5	19	95
plant 1	Store 2	2	30	60
Plant 2	Store 3	6	30	180
Plant 2	Store 4	3	40	120
Plant 3	Store 4	4	70	280
Plant 3	Store 4	14	20	280
Total Cost= \$ 1015				<u><u>280</u></u>

In terms of minimizing total transportation cost, this solution may or may not be optimal. We shall make that determination very shortly under ‘test for optimality’ section. At this point, our target is simply computing basic initial feasible solution (i.e. total cost =1015).

Note: NWCM **does not** consider the cost factor for allocation.

3.4.2 The Least- Cost Method (Lcm) Or (Largest- Profit) Method

As noted earlier, the main drawback of the north –west corner method is that it does not consider cell (route) costs in making the allocation. Consequently, if this allocation is optimal, that can be attributed to chance rather than the method used. To overcome this drawback, we can use the LCM (which consider the unit transportation cost) to find the initial feasible solution.

LCM is the method used a **minimum cost** in the allocation. It begins a solution by sequentially assigning to the ratios or cells with the **minimum cost as many units as possible**. The first allocation be made to the cell with **the lowest cost** (the **highest profit** in a maximization case).

This approach, also called the minimum –cost method, uses lowest cell as the basis for selecting route. Generally, this method takes in to account the minimum unit cost of transportation for obtaining initial solution and can be summarized as follows:

Step1. Select the cell with the lowest unit cost in the entire transportation table and allocate as much unit as possible to this cell and eliminate (line out) that row or column in which either supply or demand is exhausted.

- ✓ If both a row and column are satisfied simultaneously, only one may be crossed out arbitrarily
- ✓ In case the smallest unit cost cell is not unique, then select the cell where maximum allocation can be made.

Step2. after adjusting the supply and demand for all uncrossed –out rows and columns repeat the procedure with the next lowest unit cost among the remaining rows and columns of the transportation table and allocate a quantity to this cell that is equal to the lower of the available supply of the row and the demand for the column and eliminate (cross out) that row or column in which either demand or supply is exhausted.

Step3. Repeat the procedure until the entire available supply at various sources and demand at various destinations is satisfied. The solution obtained need to be non-degenerate.

Example3: Taking data of the above example, the initial solution using **LCM** and total transportation cost is calculated as follows:

The cell with lowest unit cost is 8 at cell (S3, D2). The maximum units which we can allocate to this cell are selected from (18 and 8) whichever is smallest. Therefore, at this cell we must allocate 8. This meets the complete demand of D2 and leaves 10 units with S3. *Thus, crossed out column D2.*

- ✓ In the reduced table without column D2 (note: D2 cannot be used because it has been crossed out), the next smallest unit transportation cost is 10 in cell (S1, D4). The maximum unit which can be allocated to this cell can be 7 (i.e. compare 14 and 7 and then assign the smallest unit). This exhausts the capacity of S1 and leaves 7 units with D4 as unsatisfied demand. *Thus, eliminate S1 row.*
- ✓ The next smallest cost, without considering S1, is 20 units at cell (S3, D4). The maximum allocation in this cell could be 7 units. This satisfies the entire demand of D4 and leaves 3 units with S3 as remaining supply. *Thus, line out D4.*
- ✓ The next smallest unit cost cell is not unique. That is, there are two cells (S2, D3) and (S3, D1), having the same unit transportation cost of 40 birr. Allocate 7 units in cell (S2, D3) first because it can accommodate more units as compared to cell (S3, D1) and allocate 3 units (only supply left with S3) in cell (S3, D1). The remaining demand of 2 units of D1 is fulfilled from S2. Since supply and demand at each origin in destination is exhausted, the initial solution is arrived at. The following table summarizes the discussion.

	D1	D2	D3	D4	Supply
S1	19	30	50	10 7	7
S2	70 2	30	40 7	60	9
S3	40 3	8 8	70	20 7	18
Demand	5	8	7	14	34 34

Number of occupied cells = $m + n - 1 = (3 + 4 - 1) = 6$ and the solution is not degenerated.

$$\begin{aligned}
 \text{Total transportation cost} = & 7 \times 10 = 70 \\
 & 70 \times 2 = 140 \\
 & 40 \times 7 = 280 \\
 & 40 \times 3 = 120 \\
 & 8 \times 8 = 64 \\
 & 20 \times 7 = 140 \\
 & \text{Birr } 814
 \end{aligned}$$

NB: The **Least- Cost Method** yields not only an initial feasible solution but also one that is close to optimal in small problems.

Example3: Develop an initial feasible solution using least-cost method & compute the total cost.

	W_1	W_2	W_3	W_4	Factory Capacity
F ₁	³ 1000	² 4000	⁷	⁶	5000
F ₂	⁷ 2500	⁵	² 2000	³ 1500	6000
F ₃	² 2500	⁵	⁴	⁵	2500
Demand	6000	4000	2000	1500	13500

<u>Routes</u>		<u>Units</u>	<u>Unit</u>	<u>Total</u>
<u>From</u>	<u>To</u>	<u>Shipped</u>	X <u>Cost</u>	<u>=Cost</u>
F ₁	W ₁	1000	3	\$ 3000
F ₁	W ₂	4000	2	8000
F ₂	W ₁	2500	7	17500
F ₂	W ₃	2000	2	4000
F ₂	W ₄	1500	3	45000
F ₃	W ₁	2500	2	5000
Total transportation cost				<u><u>= \$42,000</u></u>

m= 3, n=4 ==> 3+4-1 =6 occupied calls (Feasible)

Least- Cost method is better than the NWCM because it considers cost factories

3.4.3 Vogel's Approximation Method (VAM) Or Penalty Method

VAM is preferred to the other two methods described above. In this method each allocation is made on the basis of the opportunity (or **penalty** or **extra**) cost that would have incurred if allocation in certain cells with minimum unit transportation cost were missed.

In this method allocation are made so that the penalty cost is minimized. The advantage of this method is that it gives an initial solution which is nearer to an optimal solution or is the optimal solution itself.

VAM determines the penalty for **not using the minimum cost routes**, where the objective is to **avoid large penalties** so that the penalty from not using the routes is minimized.

The steps in VAM are as follows:

1. Calculate penalties for each row (column) by taking **the smallest & the next smallest** unit transportation cost in the same row (column). This difference indicates the penalty or extra cost which has to be paid if one fails to allocate to the cell with the minimum unit transportation cost
2. Select the row or column with the largest penalty & allocate as much unit as possible in the cell having the least cost in the selected row or column satisfying the conditions. If there is a tie in the values of penalties, then it can be broken by selecting the cell where maximum allocation can be made.
3. Adjust the supply & demand & cross out the satisfied row or column. If a row or column is satisfied simultaneously, only one of them is crossed out & the remaining row (column) is assigned a zero supply (demand). Any row or column with zero supply or demand should not be used in computing future penalties.
4. Repeat step 1 to 3 until the entire available supply at various sources & demand at various destinations are satisfied.

Example 1: Using the following transportation table to determine the initial feasible solution using Vogel's approximation method (VAM)

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	34

Solution

- ✓ As shown in the following table, differences (penalty costs) for each row and column have been calculated. In the first round, the maximum penalty, 22 occur in column D2. Thus, the cell (S3, D2) having the least transportation cost 8 is chosen for allocation. The maximum possible allocation in this cell is 8 (compare demand i.e. 8 and supply i.e.18 and assign the smallest unit) and it satisfies demand in column D2. Adjust the supply of S3 from 18 to 10 ($18-8=10$) and cross out column D2.
- ✓ Calculate the new row and column penalties except for column D2 because its demand has been satisfied. The penalties are:
 - D1=21 (i.e. the smallest 19 subtracted from the next smallest 40 units)
 - D3=10 (i.e. the smallest 40 subtracted from the next smallest 50 units)
 - D4=10 (i.e. the smallest 10 subtracted from the next smallest 20 units)
 - S1 = 9 (i.e. the smallest 10 subtracted from the next smallest 19 units)
 - S2 =20 (i.e. the smallest 40 subtracted from the next smallest 60 units)
 - S3 = 20(the smallest 20 subtracted from the next smallest 40 units)

NB: Column D2 should not be considered as it is eliminated (crossed out).

- ✓ The second-round allocation is made in column D1 with target penalty 21 in the same way as in the first round as shown in cell (S1, D1) of the table.
- ✓ In the third round, the maximum penalty 50 occurs at S3. The maximum possible allocation of 10 units is made in cell (S3, D4) having least transportation cost 20 as shown in the table.
- ✓ The process is continued with new allocations till a complete solution is obtained. The initial solution using VAM is in the table.

	D1	D2	D3	D4	Supply	Row difference
S1	19 5	30	50	10 2	7	9 9 40 40
S2	70	30	40 7	60 2	9	10 20 20 20
S3	40	8 8	70	20 10	18	12 20 50 --
Demand	5	8	7	14	34 34	
column difference	21	22	10	10		
	21	--	10	10		
	--	--	10	10		
	--	--	10	50		

Number of occupied cells = $m + n - 1 = 3 + 4 - 1 = 6$ and solution is not degenerated

Total transportation cost = $19 \times 5 = 95$

$10 \times 2 = 20$

$40 \times 7 = 280$

$60 \times 2 = 120$

$8 \times 8 = 64$

$20 \times 10 = \underline{200}$

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Example2: Determine an initial basic feasible solution to the following transportation problem using VAM.

	A	B	C	D	Supply
<i>Factory</i> F ₁	2	2	0	4	25
F ₂	5	9	8	3	25
F ₃	6	4	3	2	10
Demand	20	15	20	5	60

Solution

		Warehouse				
Factory		A	B	C	D	Supply
	F ₁	2 5	2	0 20	4	25
	F ₂	5 15	9 5	8	3 5	25
	F ₃	6 10	4	3	2	10
	Demand	20	15	20	5	60
Row difference or Row penalty or opportunity cost						
		2	2	-	-	-
		2	2	2	2	5 0
		1	2	2	-	-
Column difference or Column penalty or opportunity cost		3	2	3	1	
		3	2	-	1	
		1	5	-	1	
		0	0	-	0	
		0	0	-	-	

$m=3, n=4 \Rightarrow 3+4-1=6$ Occupied cells (feasible)

The transportation cost associated with this solution is:

Total cost = $5 \times 2 + 20 \times 0 + 15 \times 5 + 9 \times 3 + 10 \times 4 = \$185$



Self-check exercise 3.2

1. Determine an initial feasible solution to the following transportation problem using
 - a. North west corner method (NWCN)
 - b. Least cost method (LCM)
 - c. Vogel's approximation method (VAM)

	A	B	C	D	Supply
S ₁	1	5	3	3	34
S ₂	3	3	1	2	15
S ₃	0	2	2	4	12
S ₄	2	7	2	4	19
demand	21	25	17	17	

3.5. Optimality Tests

Once an initial solution is available, the next step is to check its optimality. An optimal solution is one in which there is no opportunity cost. That is, there is no other set of transportation routes (allocations) that will reduce the total opportunity cost. Thus, we have to evaluate each unoccupied cell (represents unused route) in the transportation table in terms of opportunity cost.

The purpose of the optimality test is to see if the proposed solution just generated can be improved or not. The solution to be checked for optimality must be non-degenerate i.e ***the n^0 of occupied cells must be $m+n-1$.***

The Procedure for testing optimality is analogous to that of the simplex method. A distinction is made between ***basic variables***, those associated with ***occupied cells*** & ***non-basic variables***, those associated with the ***empty cells***.

For each empty cell, the effect of changing it to an occupied cell is examined. If any of these changes are favorable, the solution is not optimal & a new solution must be designed. A favorable change means an increase in the value of the objective function in maximization problems or a decrease in minimization problems. Optimum solution to a TP can be obtained by following two methods. These methods are much simpler compared to simplex method of an LPP.

- ✓ Stepping stone Method
- ✓ Modified Distribution method (MODI Method)

3.5.1 Stepping-stone method

The Stepping-stone method is an iterative technique for moving from an initial feasible solution to an optimal solution in transportation problems.

For the **stopping- stone method** to be applied to a transportation problem, one rule about the n^0 of shipping routes being used must be observed.

The rule is:

“The N^0 of occupied routes (or squares) must always be equal to one less than the sum of the n^0 of rows plus the n^0 of columns.”

i.e Occupied shipping routes (squares) = N^0 of rows + N^0 of columns - Non degenerate solution.

Rules for drawing each closed loop:

1. Select an unused square (cell) to be evaluates.
2. Beginning at this cell, trace a closed loop going clockwise draw an arrow to an occupied cell in the same row (or column).
3. Move vertically or horizontally (but never diagonally) to another occupied cell “stepping –over” unoccupied or occupied cells (if necessary) without changing them. Follow the same procedure to other occupied cells until returning to the original empty cell.
4. Begin with a plus (+) sign at the unused cell, place alternative (-) signs and plus signs on each corner square of the closed path just traced. i.e at each turn of the loop (the loop may cross over itself at times), plus and minus signs are alternately placed in the cells, starting with a + sign in an empty cell.
5. There must be exactly one cell with a + sign and exactly one cell with a - sign in any row or column in which the loop turns.
6. An even n^0 of at least four cells must participate in a loop and the occupied cells can be visited once and only once.
7. Repeat steps 1 to 4 until an improvement index has been calculated for all unused squares (cells). If all indices computed are greater than or equal to zero, an optimal solution has been reached. If not, it is possible to improve the current solution and decrease total shipping costs.

Note: In a non-degenerate problem, there is only one possible way of drawing the loop for each empty cell.

How to find the value of a cell evaluator

The value of a cell evaluator is the sum of the per unit shipping costs in the gaining cells less the sum of the per unit shipping costs in the losing cells of the closed loop. This evaluation process must be extended to all unoccupied cells.

1. The test of optimality for a minimization (cost) problem:

If one or more of the cell evaluators is negative, the existing solution is not optimal. i.e: For minimization (cost) problems, all the cell evaluators must be positive for optimality.

- **Analysis of test:** Check all the empty cells and select for improvement the one with the largest improvement potential.
- If the solution is not optimal, the next step in the transportation method is to find a better solution. The operations in this step are:
 - a. Identify the “incoming” cell (the empty cell to be occupied). In a minimization case, the incoming cell is located by identifying the most negative cell evaluator.
 - b. Design an improved solution
 - By shifting units from cell to cell

2. The test of optimality for a maximization (profit) case:

: Is the reversed of minimization case

-If one or more of the cell evaluators is positive, the existing solution is not optimal. i.e: for a maximization (profit) case, all the cell evaluators must be negative for optimality. If any cell evaluation is positive, the solution is not optimal.

Note:

- ✓ A cell evaluator of **0** indicates the existence of another solution just as good as the current solution. Thus, in the final solution, if cell evaluators of **0** exist, this indicates the existence of multiple optimal solutions.
- ✓ If two or more cells have the same value, then either may be selected.
- ✓ If two or more of the "losing" cells contain the same n^0 of units, both will become empty simultaneously and a "degenerate" solution will result.
- ✓ For the minimization case; when one or more cell evaluators are negatives, the cell with the largest negative should be brought into solution because that route has the largest potential for improvement per unit.
- ✓ The loop starts and ends at the selected unoccupied cell. Every corner element of the loop must be an occupied cell.

Example1: Use *NWCM* to find initial feasible solution and test the solution for optimality.

		Project A	Project B	Project C	Ss
<i>F_{or} m</i>	F₁	4	2	8	100
	F₂	5	1	9	200
	F₃	7	6	3	200
	Dd	50	150	300	500

Solution

Initial feasible solution

		Project A	Project B	Project C	SS
F₁		4	2	8	100
		50	50		
F₂		5	1	9	200
			100	100	
F₃		7	6	3	200
				200	
DD		50	150	300	500

$m=3, n=3 \Rightarrow 3+3-1=5$ (Non-degenerate solution)

Total transportation cost = $\$[50 \times 4 + 50 \times 2 + 100 \times 1 + 100 \times 9 + 200 \times 3] = \$1900$

Evaluation path for the unoccupied cells

Table: Test of optimality

<i>Unoccupied cells</i>	<i>Cell evaluators</i>
(F_2, A)	$+5-4+2-1=+2$
(F_1, C)	$+8-9+1-2=-2$
(F_3, A)	$+7-4+2-1+9-3=+10$
(F_3, B)	$+6-1+9-3=+11$

The negative value for cell (F_1, C) indicates an improved solution is possible. For each unit we can shift into that cell, the total cost will decrease by \$2. The next question is how many units can be reallocated into that cell while retaining the balance of supply and demand for that table?

The Stepping- stone path for cell (F_1, C) is:

The + Signs in the path indicate units to be added, the - signs indicate units to be subtracted. The limit on subtraction is the **smallest quantity in a negative position** along the cell path. There are two quantities in negative positions, 50 and 100. Because 50 is the smaller quantity, that amount will be shifted in the following manner: Subtract 50 units from each cell on the path with a - sign and add 50 units to the quantity of each cell with a + sign in it.

With each iteration (new solution), it is necessary to evaluate the empty cells to see if further improvements are possible.

The distribution plan after reallocation of 50 units is:

	A	B	C	Ss
F_1	4 50	2	8 50	100
F_2	5	1 150	9 50	200
F_3	7	6	3 200	200
Dd	50	150	300	500

Table: Test of optimality

Unoccupied cells	Cell evaluators
(F_1, B)	$+2 -8 +9 -1 = +2$
(F_2, A)	$+5 -4 +8 -9 = 0$
(F_3, A)	$+7 -4 +8 -3 = +8$
(F_3, B)	$+6 -1 +9 -3 = +11$

Because none of these numbers is negative, this is an optimal solution. Therefore, the total cost for the distribution plan is:

The total transportation cost = \$ $(50 \times 4 + 50 \times 8 + 150 \times 1 + 50 \times 9 + 200 \times 3) = \underline{\underline{\$1,800}}$



F1 1.

F2

F3

Dd

Project A	Project B	Project C	Ss	
4	2	8	100	
50	50	+		
5	Store 1	Store 2	9	200
100	100	100	3	Supply
A	12	20	15	50
B	9	11	4	15
50	150	14	8	55
C	20	300	500	120
Demand	25	50	45	

- Develop an initial feasible solution using the NWCM. And compute the total cost for this solution.
- Evaluate the solution using the stepping-stone method. Is the solution optimal? Explain.
- What is the total cost for the optimal solution?

3.5.2 Modified Distribution Method (MODI Method)

The steps to evaluate unoccupied cells are as follows:

Step 1: for an initial feasible solution with $m+n-1$ occupied cells, calculate u_i and v_j for rows and columns. The initial solution can be obtained by any of the three methods discussed earlier.

To start with, any one of u_i 's and v_j is assigned the value zero. It is better to assign zero for a particular u_i or where there are maximum number of allocations in a row or column respectively, as it will reduce arithmetic work considerably. Then complete the calculation of u_i and v_j 's for other rows and columns by using the relation

$$c_{ij} = u_i + v_j \text{ for all occupied cells } (i, j).$$

Step 2: for unoccupied cells; calculate opportunity cost by using the relationship

$$O_{ij} = c_{ij} - (u_i + v_j) \text{ for all } i \text{ and } j.$$

Step 3: Examine sign of each

- I. If $O_{ij} > 0$, then current basic feasible solution is optimal
- II. If $O_{ij} = 0$, then current basic feasible solution will remain unaffected but an alternative solution exists
- III. If one or more $O_{ij} < 0$, then an improved solution can be obtained by entering unoccupied cell (i, j) in the basis. An unoccupied cell having the largest negative value of O_{ij} is chosen for entering in to the solution mix (new transportation schedule)

Steps 4: construct a closed path (or loop) for the unoccupied cell with largest negative opportunity cost. Start the closed path with the selected unoccupied cell and mark a plus sign(+) in this cell, trace a path along the rows(or columns) to an unoccupied cell, mark the corner with minus sign(-) and continue down the column (or row) to an occupied cell and mark the corner with plus sign(+) and minus(-) alternatively. Close the path back to the selected unoccupied cell.

Step 5: select the smallest quantity amongst the cells marked with minus sign on the corners of closed loop. Allocate this value to the selected unoccupied cell and add it to other occupied cells marked with plus signs and subtract it from the occupied cells marked with minus signs.

Step 6: obtain a new improved solution by allocating units to the unoccupied cell according to step 5 and calculate the new total transportation cost.

Step 7: test the revised solution further for optimality. The procedure terminates when all ≥ 0 , for unoccupied cells.

Remarks: the loop starts and ends at the selected unoccupied cell. It consists of successive horizontal and vertical (connected) lines whose end points must be occupied cells, except for a end point associated with entering unoccupied cell. This means that every corner element of the loop must be an occupied cell. It is immaterial whether the loop is traced in a clockwise or anti clock wise direction. However, for a given solution only one loop can be constructed for each unoccupied cell.

Example 1: Obtain an optimal solution to the transportation problem by *MODI method* given below:

	Project N ^o 1	Project N ^o 2	Project N ^o 3	Supply
Farm 1	4	2	8	100
Farm 2	5	1	9	200
Farm 3	7	6	3	200
Demand	50	150	300	500

Solution

Note: Both the MODI and the stepping - stone method will yield the same values.

Remark:

Conventionally, we **begin** by assigning a value of **zero** as the index for row 1 ($U_1=0$). Once row index has been established, it will enable us to compute column index numbers for all occupied cells in that row. Similarly, once a column index number has been determined, index numbers for all rows corresponding to occupied cells in that column can be determined.

Consider the initial feasible solution of the given example by NWCM as shown below.

Initial solution, NWCM

	Project Nº 1	Project Nº 2	Project Nº 3	Supply	U _i
Farm 1	4 50	2 50	8	100	U ₁ =0
Farm 2	5	1 100	9 100	200	U ₂ =-1
Farm 3	7	6	3 200	200	U ₃ =-7
Demand	50	150	300	500	
V _j	V ₁ =4	V ₂ =2	V ₃ =10		

To determine C_{ij} , use the occupied cells.

For instance, $C_{11}=4$, $C_{12}=2$, $C_{22}=1$, $C_{23}=9$, and $C_{33}=3$

$$C_{ij} = U_i + V_j$$

$$\Rightarrow C_{11} = U_1 + V_1 \Rightarrow 4 = 0 + V_1 \Rightarrow \underline{V_1 = 4}, U_1 = 0 \text{ by convention}$$

$$\Rightarrow C_{12} = U_1 + V_2 \Rightarrow 2 = 0 + V_2 \Rightarrow \underline{V_2 = 2} \Rightarrow C_{22} = U_2$$

$$+ V_2 \Rightarrow 1 = U_2 + 2 \Rightarrow \underline{U_2 = -1}$$

$$\Rightarrow C_{23} = U_2 + V_3 \Rightarrow 9 = -1 + V_3 \Rightarrow \underline{V_3 = 10}$$

$$\Rightarrow C_{33} = U_3 + V_3 \Rightarrow 3 = U_3 + 10 \Rightarrow \underline{U_3 = -7}$$

Note:

$$C_{ij} \neq U_i + V_j \text{ (For unoccupied cells)}$$

For instance, from the above information, $C_{32} \neq U_3 + V_2 \Rightarrow 6 \neq -7 + 2$

Table: Test of optimality

Unoccupied cells	Cell evaluators $D_{ij} = C_{ij} - (U_i + V_j)$
(1,3)	$C_{13} - (U_1 + V_3) = 8 - (0 + 10) = -2$
(2,1)	$C_{21} - (U_2 + V_1) = 5 - (-1 + 4) = +2$
(3,1)	$C_{31} - (U_3 + V_1) = 7 - (-7 + 4) = 10$
(3,2)	$C_{32} - (U_3 + V_2) = 6 - (-7 + 2) = +11$

In this case, we found that cell (1, 3) had an evaluation of -2, which represented an improvement potential of and **\$ 2 per unit**. Hence, an improved solution is possible. The stepping-stone path for cell (1, 3) is:

	Project № 1	Project № 2	Project № 3	Supply
Farm 1	4 50	2 - 50	8 +	100
Farm 2	5	1 100	9 100	200
Farm 3	7	6	3 200	200
Demand	50	150	300	500

The distribution plan after reallocation of 50 units is:

	Project № 1	Project № 2	Project № 3	Supply	U_i
Farm 1	4 50	2	8 50	100	$U_1=8$
Farm 2	5	1 150	9 50	200	$U_2=9$
Farm 3	7	6	3 200	200	$U_3=3$
Demand	50	150	300	500	
V_j	$V_1=-4$	$V_2=-8$	$V_3=0$		

Table: Test of optimality

Unoccupied cells	Cell evaluators $D_{ij} = C_{ij} - (U_i + V_j)$
(1,2)	$C_{12} - (U_1 + V_2) = 2 - (-8 + 8) = 2$
(2,1)	$C_{21} - (U_2 + V_1) = 5 - (-4 + 9) = 0$
(3,1)	$C_{31} - (U_3 + V_1) = 7 - (-4 + 3) = 8$
(3,2)	$C_{32} - (U_3 + V_2) = 6 - (-8 + 3) = +11$

Because none of the cell evaluators is negative, this is an optimal solution.

Thus, the total cost for the distribution plan = **\$1650**



Self-check exercise 3.3

1. Consider the following transportation problem

	A	B	C	D	Supply
S ₁	1	5	3	3	34
S ₂	3	3	1	2	15
S ₃	0	2	2	4	12
S ₄	2	7	2	4	19
demand	21	25	17	17	

- a. Develop an initial feasible solution using the VAM. And compute the total cost for this solution.
- b. Evaluate the solution using modified distribution method. Is the solution optimal? Explain.
- c. What is the total cost for the optimal solution?

3.6. Special Cases

3.5.1. Degeneracy

A condition that occurs when the N^0 of occupied cells in any solutions less than the N^0 of rows plus the N^0 of columns minus 1 in a transportation table.

i.e. $N_o \text{ of occupied calls} < m + n - 1$ Degeneracy

If the $N_o \text{ of occupied cells} = m + n - 1$, then the solution is non- degenerate.

The degeneracy in the transportation problems may occur at two stages:

1. *when obtaining an initial solution*
2. *During improvement (or at any stage while moving towards optimal solution.*

To resolve degeneracy, we processed by allocating a **very small quantity close to zero** to one or more **unoccupied cell** so as to get **$m+n-1$** number of occupied cells. This amount is denoted by a Greek letter (epsilon) or Δ (delta). This quantity **would not affect the total cost** as well as **supply** and **demand** values.

= Almost zero

In a minimization transportation problem, allocate the smallest transportation cost to .. In a maximization transportation problem, it should be allocated to a cell that has a high pay off value.

Insert Δ when it is able to create a closed loop for each occupied cell.

The purpose of epsilon/delta is to enable evaluation of the remaining empty cells. The choice of location for the epsilon/delta can be somewhat tricky: some empty cells may be unsuitable if they do not enable evaluations of remaining empty cells. Not all choices would be acceptable.

Actually, the N^0 of epsilon/deltas needed will equal the difference between the N^0 of completed cells and $m+n-1$. However; you will only be exposed to the most common case in which one more completed cell is needed.

The epsilon/delta cannot be placed in a cell which later turns out to be in a negative position of a cell path involved in reallocation because epsilon/delta will be the “smallest quantity a negative position “and shifting that minute quantity around the cell path will leave the solution virtually unchanged. Consequently, a certain amount of trial and error may be necessary before a satisfactory location can be identified for epsilon/delta.

Example 1: Solve the following transportation problem.

	1	2	Supply
1	3	3	50
2	4	6	30
Demand	50	30	80

Solution:

Using NWCM and MODI, the initial solution is:

	1	2	Supply	U_i
1	3 50	3	50	$U_1=0$
2	4	6 30	30	$U_2=3$
Demand	50	30	80	
V_j	$V_1=3$	$V_2=3$		

$$C_{ij} = U_i + V_j$$

$$\Rightarrow C_{11} = U_1 + V_1 \Rightarrow 3 = 0 + V_1 \Rightarrow \underline{V_1=3}, U_1=0 \text{ by}$$

$$\text{convention} \Rightarrow C_{12} = U_1 + V_2 \Rightarrow 3 = 0 + V_2 \Rightarrow \underline{V_2=3} \Rightarrow C_{22} =$$

$$U_2 + V_2 \Rightarrow 6 = U_2 + 3 \Rightarrow \underline{U_2=3}$$

$$\Rightarrow C_{33} = U_3 + V_3 \Rightarrow 3 = U_3 + 8 \Rightarrow \underline{U_3=-5}$$

Note: $m=2$ and $n=2 \Rightarrow 2+2-1=3 \Rightarrow$ Occupied cells $= 2 < 3$ (Degeneracy)

Table: Test of optimality

Unoccupied cell	Cell evaluator $K_{ij} = C_{ij} - (U_i + V_j)$
(2,1)	$C_{21} - (U_2 + V_1) = 4 - (3+3) = -2$

The optimal solution is:

	1	2	Supply	U_i
1	3 20	3 30	50	$U_1=0$
2	4 30	6	30	$U_2=1$
Demand	50	30	80	
V_j	$V_1=3$	$V_2=3$		

$$C_{ij} = U_i + V_j$$

$$\Rightarrow C_{11} = U_1 + V_1 \Rightarrow 3 = 0 + V_1 \Rightarrow \underline{V_1=3}, U_1=0 \text{ by}$$

$$\text{convention} \Rightarrow C_{21} = U_2 + V_1 \Rightarrow 4 = U_2 + 3 \Rightarrow \underline{U_2=1}$$

$$\Rightarrow C_{12} = U_1 + V_2 \Rightarrow 3 = 0 + V_2 \Rightarrow \underline{V_2 = 3}$$

Table: Test of optimality

<i>Unoccupied cells</i>	<i>Cell evaluators</i> $K_{ij} = C_{ij} - (U_i + V_j)$
(2,2)	$C_{22} - (U_2 + V_2) = 6 - (1 + 3) = +2$

The total cost = $\$(20 \times 3 + 30 \times 3 + 30 \times 4) = \underline{\underline{\$270}}$

3.6.2 Unbalanced supply and demand

For a feasible solution to exist, it is necessary that the total supply must equal total demand.

Unbalanced case may be of two types: supply > demand or demand > supply.

Case1. Demand more than Supply

If total demand exceeds total supply, a dummy row (called a dummy supply center) can be added to the transportation table to account for the excess demand. The unit transportation cost for the cells in the dummy row is set equal to zero.

Example

	A	B	C	Supply
X	2	4	4	20
Y	8	12	8	50
Z	2	8	12	30
Demand	40	25	85	

In this case the total demand is greater than the total supply ($150 > 100$). Therefore, it needs dummy row as follows.

	A	B	C	Supply
X	2	4	4	20
Y	8	12	8	50
Z	2	8	12	30
Dummy	0	0	0	50
Demand	40	25	85	

After balancing the unbalanced transportation problem, we can obtain the initial solution using one of the three methods and test its optimality using two methods as we have discussed in the above.

Case 2 Supply more than demand

If total supply exceeds total demand, an additional column (called dummy demand center) can be added to the transportation table to absorb the excess supply. The unit transportation cost for these cells in this column is set equal to zero because these represent product items that are not being made and not being sent.

Example: Consider the following transportation problem

	A	B	C	Supply
X	2	4	4	70
Y	8	12	8	50
Z	2	8	12	30
Demand	40	25	45	

Since total supply is greater than total demand by 50 units, add a dummy column with 0-unit cost and 50 units of demand. See the following table.

	A	B	C	Dummy	Supply
X	2	4	4	0	70
Y	8	12	8	0	50
Z	2	8	12	0	30
Demand	40	25	45	50	

Now proceed as above to till optimum solution is obtained using either *stepping stone* method or *MODI* method.

3.6.3. Alternative Optimal solutions

The existence of alternative optimal solution can be determined by an inspection of the opportunity costs, D_{ij} for the unoccupied cells. If an unoccupied cell in an optimal solution has opportunity cost of **zero**, then an alternative optimal solution can be formed with another set of allocations without increasing the total transportation cost.



Self-check exercise 3.4

1. **XYZ Tobacco Company** purchases tobacco and stores in warehouses located in the following four cities.

	City A	City B	City C	City D	Supply
Company L	7	12	10	5	120
Company P	10	9	3	5	100
Company Q	5	4	11	7	110
Demand	90	50	80	60	

- a. Find the optimum distribution for **XYZ Tobacco Company**.
- b. Are there multiple optimum solutions? If there are alternative optimum solutions, identify them.

3.7 Assignment Problems

The Assignment Problem (AP) refers to the class of LPPs that involves determining the most efficient assignment of people to projects, salespeople to territories, contracts to bidders, jobs to machines, and so on. The objective is to assign a number of resources to an equal number of activities so as to minimize total costs or total time or maximize total profit of allocation.

The problem of assignment arises because available resources such as men, machines, etc have varying degrees of efficiency for performing different activities such as job. Therefore, cost, profit or time of performing the different activities is different.

Assumptions:

The AP is a special case of TP under the condition that the number of origins is equal to the number of destinations. Viz. $m=n$. Hence assignment is made on the basis of 1:1.

Following are the assumptions:

- ✓ Number of jobs is equal to the number of machines or persons
- ✓ Each man or machine is loaded with one and only one job.
- ✓ Each man or machine is independently capable of handling any of the job being presented.
- ✓ Loading criteria must be clearly specified such as “minimizing operating time” or “maximizing profit” or “minimizing production cost” or “minimizing throughout (production cycle) time” etc.

Remark:

- ✓ The AP is considered as a special TP in which the supply at each source and the demand at each destination are always one unit.
- ✓ Since the supply and demand are always equal to one unit in each row and column, there is no need to write them in the assignment table.

Example:

Service costs of different team assignment (\$ in thousands)

Table: The assignment

Zone Service Team	Z ₁	Z ₂	Z ₃	Zone Service Team
S ₁	20	15	31	
S ₂	17	16	33	
S ₃	18	19	27	

	Z ₁	Z ₂	Z ₃	Supply
S ₁	20	15	31	1
S ₂	17	16	33	1
S ₃	18	19	27	1
Demand	1	1	1	

The above problem can be presented as a LPP as follows:

$$\text{Min}Z = 20x_{11} + 15x_{12} + 31x_{13} + 17x_{21} + 16x_{22} + 33x_{23} + 18x_{31} + 19x_{32} + 27x_{33}$$

Subject to the constraints

a. Supply constraints:

$$x_{11} + x_{12} + x_{13} = 1 \quad S_1 \text{ constraint}$$

$$x_{21} + x_{22} + x_{23} = 1 \quad S_2 \text{ constraint}$$

$$x_{31} + x_{32} + x_{33} = 1 \quad S_3 \text{ supply constraint}$$

b. Demand constraints

$$x_{11} + x_{21} + x_{31} = 1 \quad Z_1 \text{ constraint}$$

$$x_{12} + x_{22} + x_{32} = 1 \quad Z_2 \text{ constraint}$$

$$x_{13} + x_{23} + x_{33} = 1 \quad Z_3 \text{ constraint}$$

$$x_{ij} \text{ either } 0 \text{ or } 1 \text{ for all } i, j$$

Since all x_{ij} can be either 0 or 1, there will be one assignment in each supply constraint and one assignment in each demand constraint.

As in the transportation problem, assignment problems can be balanced or not. In a balanced case, the number of objects to be assigned equals the number of objects to which they are assigned. Unbalanced problem can be balanced by adding a dummy (dummies) with zero cost coefficients.

3.7.1 Methods of Solving Assignment Problems

An assignment problem can be solved by using the following methods

1. Enumeration method
2. Simplex algorithm
3. Transportation method
4. Hungarian method

3.7.1.1. Hungarian Method/Flood's Technique/

The Hungarian Method (developed by Hungarian mathematician D.Konig) of assignment provides us with efficient method of finding the optimal solution without having to make a direct comparison of every solution. It works on the principle of reducing the given cost matrix (the principle of matrix reduction) to a matrix of opportunity costs, which means that by subtracting and adding appropriate numbers in the cost table or matrix, we can reduce the problem to a matrix of opportunity costs.

Opportunity costs show the relative penalties associated with assigning resource to an activity as opposed to making the best or least-cost assignment. If we can reduce the cost matrix to the extent of having at least one zero in each row and column, then it will be possible to make optimal assignments.

Steps in solving assignment problems:

Step.1. Develop the cost table from the given problem

If the number of rows does not equal the number of columns and vice versa, then a dummy row or dummy column must be added. The assignment costs for dummy cells are always zero.

Step 2. Find the opportunity cost table

i.e. The transformation of the cost matrix to what is termed as a total-opportunity cost matrix.

It involves two operations:

a. Perform row operation

i.e. Locate the smallest element in each row of the given cost table and then subtract that the given cost table and then subtract that from each element of that row

b. Perform column operation

I.e. In the reduced matrix obtained from 2(a), locate the smallest element in each column and then subtract that from each element of that column. Notice that each row and column now have at least one zero value.

Step 3: Make assignment in the opportunity cost matrix

- a) Examine rows successively until a row with exactly one unmarked zero is obtained. Make an assignment to this single zero by making a square around it
- b) For each zero value that becomes assigned, eliminate (strike off) all other zeros in the same row and/or column
- c) Repeat steps 3(a) and 3(b) for each column also with exactly single zero value cell that has not been assigned
- d) If a row and/or column has two or more unmarked zeros and one cannot be chosen by inspection, then chose the assigned zero cell arbitrarily
- e) Continue this process until all zeros in rows/columns are either enclosed (assigned) or strike off (x)

Step 4: Optimality criteria

If the number of assigned cells is equal to the number of rows/columns, then it is optimal solution. The total cost associated with this solution is obtained by adding original cost figures in the occupied cells.

Steps 5: Revise the opportunity cost table

Draw a set of horizontal and vertical lines to cover all the zeros in the revised cost table obtained from step 3, by using the following procedure

- a) For each row in which no assignment was made, mark a tick
- b) Examine the marked rows, if any zero cell occurs in those rows, mark a tick to the respective columns that contain those zeros

- c) Examine market columns. If any assigned zero occurs in those columns, tick the respective rows that contain those assigned zeros
- d) Repeat this process until no more rows or columns can be marked
- e) Draw a straight line through each marked column and each unmarked row

If the number of lines drawn (or total assignment) is equal to the number of rows (or columns), the current solution is the optimal solution, otherwise go to step 6

Step 6 develop the new revised opportunity cost table

- a) From among the cells are not covered by any line, choose the smallest element. Call this value k
- b) Subtract k from every element in the cell not covered by a line
- c) Add k to every in the cell covered by the two lines, i.e. intersection of two lines
- d) Elements in cells not covered by one line remain unchanged

Step 7: Repeat steps

Repeat steps 3 to 6 until optimal is obtained

Example1: A computer center has three programmers. The center wants three application programs to be developed. The head of the computer center, after studying carefully the programs to be developed, estimate the computer time in minutes required by the experts for the application programs as follows:

	Programs		
1	120	100	80
2	80	90	110
3	110	140	120

Assign the programmers to the programs in such a way that the total computer time is minimum.

Solution:

Steps 1 and 2:

A. Perform row reduction

The minimum time element in row 1, 2, and 3 is 80, 80 and 110 respectively. Subtract those elements from all elements in their respective row. The reduced time matrix is:

Table: After row reduction

1	40	20	0
2	0	10	30
3	0	30	10

B. Column reduction

Since column B has no one '0', perform also column reduction. The minimum time element in columns A, B and C is 0, 10 and 0 respectively. Subtract these elements from all elements in their respective column to get the reduced time matrix.

Table: After column reduction

40	10	0
0	0	30
0	20	10

Step 3: make assignment in the opportunity cost matrix

- a. In 1 assignment, start with row/column having one zero and cancel the alternative zeros(x) Table: Test of optimal assignment

40	10	0
0	0	30
0	20	10

- b. Count the number of occupied cells

If the number occupied cells are equal to the number of rows/columns, the optimal solution is obtained.

The pattern of assignment among programmers and programs with their respective time (in minute) is given below:

<u>Programmer</u>	<u>Program</u>	<u>Time (in minutes)</u>
1	C	80
2	B	90
3	A	110
		Total time= <u>280 minutes</u>

Example 2. A department has five employees with five jobs to be performed. The time (in hours) each man will take to perform each job is given in the effectiveness matrix.

		Employees				
Jobs	A	10	5	13	15	16
	B	3	9	18	13	6
	C	10	7	2	2	2
	D	7	11	9	7	12
	E	7	9	10	4	12

How should the jobs be allocated, one per employees, so as to minimize the total man-hours?

Solution:

Table: After row reduction

A	5	0	8	10	11
B	0	6	15	10	3
C	8	5	0	0	0
D	0	4	2	0	5
E	3	5	6	0	8

Since the number of lines less than the number of rows/columns, an improvement is possible.

Step 4. Improve the present opportunity cost table

This is done by the following operations;

- Select the smallest entry (element) among all uncovered elements by the lines and subtract it from all entries in the uncovered cells. this cell is 2
- Add the same smallest entry to those cells in which lines intersect (cells with two lines them).
- Cells with one line through them are unchanged to the improved table.

A	7	0	8	12	11
B	0	4	13	10	1
C	10	5	0	2	0
D	0	2	0	0	3
E	3	3	4	0	6

Since the number of assigned cells equals to the number of rows/columns, the solution is optimal.

Job	Employees	Time (in minutes)
A	II	5
B	I	3
C	V	2
D	III	9
E	IV	4
		Total time= <u>23 Hours</u>



Self-check exercise 3.5

1. A manager has prepared the following table, which shows the costs for various combinations of job-machine assignments:

		Machine (Cost in '000s))		
		A	B	C
Job	1	20	15	31
	2	17	16	33
	3	18	19	27

- a. What is the optimal (minimum-cost) assignment for this problem?
- b. What is the total cost for the optimum assignment?

3.8. Special Cases in The Assignment Problems

Certain situations can arise in which the model deviates slightly from that previously described. Among those situations are the following:

3.8.1 Multiple Optimal Solutions

While making an assignment in the reduced assignment matrix, it is possible to have two or more ways to strike off a number of zeros. Such situation indicates multiple optimal solutions with the same optimal value of objective function. In such cases the more suitable solution may be considered by the decision-maker.

In multiple optimal solutions, no unique 0 will exist at some point, resulting in more than one choice for assignment and hence, more than one optimal solution. It should be noted that all optimal solutions will yield the same value of the objective function.

Example:1. Given this final assignment table, identify two optimal solutions.

		Machine		
Job	A	4	0	0
	B	0	3	2
	C	1	0	0

Solution

The first assignment must be B-1, because B-1 is the only 0 that appears in a single row or column. Having made that assignment, there are two choices for the remaining two rows, and two choices for the remaining two columns. This results in two possible solutions, as shown:

		Machine		
A	4	0	~~0~~	
B	0	~~3~~	2	
C	1	~~0~~	0	

	Machine			
A	4		~~0~~	0
B	0	3	0	2
C	1		0	~~0~~

Example 2. The foreman of a machine shop wants to determine a minimum cost matching for operators and machines. The foreman has determined hourly cost for of four operators for the four machines, as shown in the following cost table.

	Operator	Machine			
		1	2	3	4
		70	80	75	64
		55	52	58	54
		58	56	64	68
		62	60	67	70

Required:

- Determine the minimum-cost assignment for this problem
- What is the total cost for the optimal assignment?
- Is there an alternative optimal assignment? What is it? Calculate the total cost for the alternate optimal assignment.

Solution:

Table: After row reduction

	1	2	3	4
1	6	16	11	0
2	3	0	6	2
3	2	0	8	12
4	2	0	7	10

Table: After column reduction

	1	2	3	4
1	4	16	5	0
2	1	0	0	2
3	0	0	2	12
4	0	0	1	10

Table: Optimal Assignments

	1	2	3	4
1	4	16	5	0
2	1	✗	0	2
3	0	✗	2	12
4	✗	0	1	10

Optimal Assignment

Operator	Machine	Cost (in \$)
4	A	62
3	B	56
2	C	58
1	D	<u>64</u>
		Total cost = <u>\$240</u>

c. Yes!

Alternative optimal assignment

Operator	Machine	Cost (in \$)
1	D	64
2	C	58
3	A	58
4	B	<u>60</u>
		Total cost= <u>\$240</u>

3.8.2 Maximization case in assignment problems

There may arise situations when the assignment problem calls for maximization of profit, revenue, etc as the objective function. Such problem may be solved by converting the given maximization problem into a minimization problem by the following procedure:

- i. Find the largest profit coefficient in the entire.
- ii. Subtract each entry in the original table from the largest profit coefficient.

The transformed assignment problem so obtained can be solved by using the Hungarian method.

Example 1. A company has four territories open, and four salesmen available for an assignment. The territories are not equally rich in their sales potential. Based on the past performance, the following table shows the annual sales (in \$) that can be generated by each salesman in each territory. Find the optimal assignment and the maximum expected total sales.

		Territory			
Salesman		I	II	III	IV
	A	42	35	28	21
	B	30	25	20	15
	C	30	25	20	15
	D	24	20	16	12

Solution:

Convert maximization problem into minimization problem by subtracting all elements from the highest element (i.e 42) Thus, the equivalent cost table is:

	I	II	III	IV
A	0	7	14	21
B	12	17	22	27
C	12	17	22	27
D	18	22	26	30

	I	II	III	IV
A	0	3	6	9
B	0	1	2	3
C	0	1	2	3
D	0	0	0	0

Thus, after improvement of the table, the optimal assignment is:

	I	II	III	IV
A	0	2	4	7
B	0	0	0	1
C	0	0	0	1
D	2	1	0	0

The pattern of two alternative optimal assignments among territories and salesmen with respective sale is given below:

Assignment set I			Assignment set II		
Salesman	Territory	Sales (\$)	Salesman	Territory	Sales (\$)
A	I	42	A	I	42
B	III	20	B	II	25
C	II	25	C	III	20
D	IV	<u>12</u>	D	IV	<u>12</u>
Total= \$ <u>99</u>			Total= \$ <u>99</u>		



Self-check exercise 3.6

- Five salesmen are to be assigned to five territories. Based on the past performance, the following table shows the annual sales man in each territory. Find the optional assignment.

	T ₁	T ₂	T ₃	T ₄	T ₅
S ₁	26	14	10	12	9
S ₂	31	27	30	14	16
S ₃	15	18	16	25	30
S ₄	17	12	21	30	25
S ₅	20	19	25	16	10

3.8.3 Unbalanced Assignment problem

The Hungarian method of assignment requires that the number of columns and rows in the assignment matrix be equal. However, when the given cost matrix is not a square matrix, the assignment problem is called an unbalanced problem. In such cases a dummy row(s) or column(s) are added in the matrix (with zeros as the cost elements) to make it a square matrix. After making the given cost matrix a square matrix, the Hungarian method may be used to solve the problem.

Example 1. MEGA printing press, a publisher headquartered in Addis Ababa, wants to assign three recently hired college graduates, Marta, Bakcha and Hirut to regional sales districts in Mekelle, Bahir Dare, and DireDawa. But the firm also has an opening in Gambela and would send one of the three there if it were more economical than a move to Mekelle, Bahir Dar and Dire Dawa. It will cost Br. 1,000 to relocate Marta to Gambela, Br. 800 to relocate Baklcha there, and Br. 1,500 to move Hirut. What is the optimal assignment of personnel to offices?

Office Hire	Mekelle	Bahir Dare	Dire Dawa
Marta	Br.800	Br 1,100	Br 1,200
Bekcha	Br. 500	Br 1,600	Br 1,300
Hirut	Br. 500	Br 1,000	Br 2,300

Solution

To balance the problem, we add a dummy row (person) with a zero-relocation cost to each city.

	City			
	C ₁	C ₂	C ₃	C ₄ (Gambela)
P ₁	800	1,100	1,200	1,000
P ₂	500	1,600	1,300	800
P ₃	500	1,000	2,300	1,500
Dummy	0	0	0	0

Table: After row reduction

	C ₁	C ₂	C ₃	C ₄
P ₁	0	300	400	200
P ₂	0	1,100	800	300
P ₃	0	500	1800	1000
Dummy	0	0	0	0

Table: Optimal Assignment

	C ₁	C ₂	C ₃	C ₄
P ₁	~~100~~	0	100	0
P ₂	0	~~700~~	400	0
P ₃	0	100	1400	700
Dummy	~~400~~	0	0	100

Thus, an optimal assignment can be made at zero cells (squares).

Person	City
Dummy (No person)	Dire Dawa
Hirut	Mekelle
Bekcha	Gambela
Marta	Bahir Dare

$$\text{Cost} = \text{Br. } (0 + 500 + 800 + 1,100) = \underline{\underline{\text{Br. 2,400}}}$$

3.8.4 Restrictions on Assignments

In certain instances, it may happen that a particular match or pairing may be either undesirable or otherwise unacceptable. For example, an employee may not have the skills necessary to perform a particular job or a machine may not be equipped to handle a particular operation. In such cases, the cost of performing that particular activity by a particular resource is considered to be very large (written as M or) so as to prohibit the entry of this pair of employee-job into the final solution. When such a restriction is present, a letter (M) is often placed in the

table in the position that would represent a pairing. Analysis is performed as usual except the M is ignored throughout the analysis. That is, M is not used in any reductions, nor is any value added to it or subtracted from it during the course of the analysis.

Example 1. In the modification of a plant layout of a factory four new machines M₁, M₂, M₃ and M₄ are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine M₂ cannot be placed at C and M₃ cannot be placed at A. the cost of placing of machine at place i (in \$) is shown below.

		<i>Location</i>				
Machine		A	B	C	D	E
	M ₁	9	11	15	10	11
	M ₂	12	9	-	10	9
	M ₃	-	11	14	11	7
	M ₄	14	8	12	7	8

Find the optimal assignment schedule.

Solution:

As the cost matrix is not balanced, add one dummy row (machine) with a zero-cost element in that row. Also assign a high cost, denoted by M, to the pair (M₂, C) and (M₃, A).

Apply the Hungarian method to solve the problem

	A	B	C	D	E
M ₁	9	11	15	10	11
M ₂	12	9	M	10	9
M ₃	M	11	14	11	7
M ₄	14	8	12	7	8
M ₅	0	0	0	0	0

Table: Optimal assignment

	A	B	C	D	E
M ₁	<u>0</u>	2	6	1	2
M ₂	3	<u>0</u>	M	1	~~8~~
M ₃	M	4	7	4	<u>0</u>
M ₄	7	1	5	<u>0</u>	1
M ₅	~~0~~	~~0~~	<u>0</u>	~~0~~	~~0~~

The total minimum cost (\$) and optimal assignments made are as follows:

Machines	Location	Costs (\$)
M ₁	A	9
M ₂	B	9
M ₃	E	7
M ₄	D	7
M ₅ (Dummy)	C	<u>0</u>
		Total = \$ <u>32</u>

CHAPTER SUMMERY

The transportation problem is a particular case of linear programming problem. The structure of transportation problem involves a large number of shipping routes from several supply origins to several demand destinations.

In order to find initial solution of a transportation methods the following methods used:

- ✓ North- west corner method
- ✓ Least cost method, and
- ✓ Vogel's approximation (or penalty) method.

The north –west corner method is a systematic approach for developing an initial basic feasible solution. Its *chief advantages* are that it is simple to use and easy to understand.

List cost approach, also called the minimum –cost method, uses lowest cell as the basis for selecting route.

Vogel's approximation (penalty or regret) method is a heuristic method and is preferred to the other two methods described above.

The assignment problem is a special case of Transportation problem in which the number of sources and destinations are the same, and the objective is to assign the given job (tasks) to most appropriate machine (person) so as to optimize the objective function like minimizing cost or maximizing profit.

Solution of assignment problem is calculated based on the principles of reducing the given cost matrix to a matrix of opportunity costs. Opportunity cost shows the relative penalties associated with assigning resources to an activity as opposed to making the best (or least assignment).

CHAPTER FOUR

DECISION THEORY

4.1 Introduction

The success or failure that an individual or organizations experiences, depends to a large extent on the ability of making appropriate decisions. Making of a decision requires an enumeration of feasible and viable alternatives (courses of action or strategies), the projection of consequences associated with different alternatives and a measure of effectiveness (or an objective) by which the most proffered alternative is identified. Decision theory provides an analytical and systematic approach to the study of the decision making in other words, decision theory provide a method of natural decision making where in data concerning the occurrence of different outcomes (consequences') may be evaluated to enable the decision maker to identify alternative.

Decision models useful in helping decision makers make the best possible decisions are classified according to the degree of certainty. The scale of certainty can range from complete certainty to complete uncertainty. The region which falls between these two extreme points corresponds to the decision making under risk.

4.2 Objective of The Chapter

After completing this chapter, you should be able to:

- ☞ Describe and give example of decision making under the three conditions (certainty, risk and complete uncertainty)
- ☞ Construct and understand the payoff table
- ☞ Make conclusion by using maximin, maximax, minimax regret, insufficient reason and expected value criteria.
- ☞ Find out the expected value of perfect information

4.3 Terms in Decision Theory

Irrespective of the type of decision model, there are certain essential characters which are common to all as listed below.

1. **Decision alternatives:** these are a finite number of decision alternatives available with the decision maker at each point at a time when a decision is made. The number and type of such alternatives may depend on the previous decision made and on what has happened

subsequent to those decisions. These alternatives are also called course of action (actions, act or strategies) and are under control and known to the decision maker.

2. **State of nature:** a possible future condition (consequence or event) resulting from the choice of a decision alternative depends up on certain factors beyond the control of a decision maker. These factors are called state of nature (future). For example, if the decision is to carry an umbrella or not, the consequence (get wet or do not) depends on what action nature takes.

The states of nature are mutually exclusive and collectively exhaustive with respect to any decision problem. The state of nature may be described numerically such as, demand of 100 units of an item or non-numerically such as, employee strike, etc.

3. **Payoff:** a numerical value resulting from each possible combination of alternatives and state off nature is called payoff. The payoff values are always conditional values because of unknown state of nature. A tabular arrangement of these conditional outcome (pay off) values as pay off matrix

Steps in decision theory approach

The decision-making process involves the following steps:

1. Identify and define the problem
2. Listing of all possible future events called state of nature, which can occur in the context of the decision problem. Such events are not under the control of decision maker because these erratic in nature
3. Identification of all the course of action (alternatives or decision choices) which are available to the decision maker. The decision maker has control over this course of action.
4. Expressing the payoffs resulting from each pair of course of action and state of nature. These pay offs are normally expressed in a monetary value
5. Apply an appropriate mathematical decision theory model to select best course of action from the given list on the basis of some criterion (measure of effectiveness) that results in the optimal (desired) pay off.

Example 1: a firm manufactures three types of products. The fixed and variable costs are given below.

	Fixed cost in birr	Variable cost per unit
Product A	25,000	12
Product B	35,000	9
Product C	53,000	7

The likely demand (units) of the products is given below:

Poor demand	3000
Moderate demand	7000
High demand	11,000

If the sale price of each type of product is birr 25, then prepare the payoff matrix

Solution: let D1, D2 and D3 be the poor, moderate and high demand, respectively. Then payoff is given by

Payoff= sales revenue-cost

The calculation for payoff (in '000's) for each pair of alternative demand (course of action) and the types of product (state of nature) are shown below:

$$\text{Payoff} = (p \cdot q) - [(v \cdot q) + FC]$$

$$D1, A = (3000 \times 25) - [(12 \times 3000) + 25000] = 14000$$

$$D1, B = (3000 \times 25) - [(9 \times 3000) + 35000] = 13000$$

$$D1, C = (3000 \times 25) - [(7 \times 3000) + 53000] = 1000$$

$$2. = (7000 \times 25) - [(12 \times 7000) + 25000] = 66000$$

$$2. = (7000 \times 25) - [(9 \times 7000) + 35000] = 77000$$

$$2. = (7000 \times 25) - [(7 \times 7000) + 53000] = 73000$$

$$3. = (11000 \times 25) - [(12 \times 11000) + 25000] = 118000$$

$$3. = (11000 \times 25) - [(9 \times 11000) + 35000] = 141000$$

$$3. = (11000 \times 25) - [(7 \times 11000) + 53000] = 145000$$

The payoff values are shown below:

Product type	D1	D2	D3
A	14	66	118
B	13	77	141
C	1	73	145

4.4 Types of decision-making environments

Decisions are made based up on the data available about the occurrence of events as well as the decision situation (or environment). There are three types of decision-making environment: certainty, uncertainty and risk.

Type1: decision making under certainty

In this case the decision maker has the complete knowledge (perfect information) of consequence of every decision choice (course of action) with certainty. Obviously, he/she will select an alternative that yields the largest return (payoff) for the known future (state of nature)

Type 2: decision making under risk

In this case the decision maker has less than completes knowledge with certainty of the consequence of every decision choice (course of action). This means there is more than one state of nature (future) and for which he makes an assumption of the probability with which each state of nature will occur. For example, probability of getting head in the toss of a coin is 0.5

Type 3 decision making under uncertainty

In this case the decision maker is unable to specify the probabilities with which the various state of nature (futures) will occur. Thus, decision under uncertainty are taken with even less information than decision under risk. For example, the probability that Mr. X will be the prime minister of the country 15 years from now is not known.

4.4.1 Decision Making Under Uncertainty

In the absence of knowledge about the probability of any state of nature (future) occurring the decision maker must arrive at a decision only on the actual conditional pay off values, together with a policy (attitude). There are several different criteria of decision making in this situation and these are:

- a. Maximax or Minimin
- b. Maximin or Minimax
- c. Equally likely

- d. Criterion of realism
- e. Criterion of regret

A. Criterion of Optimism (Maximax Or Minimin)

In this criterion the decision maker ensures that he should not miss the opportunity to achieve the largest possible profit (Maximax) or lowest possible cost (Minimin). Thus, he/she selects the alternative that represents the maximum of maximum (or minimum of minimum) pay off. The working method is summarized as follows:

- a) Locate the maximum (or minimum) pay off values corresponding to each alternative
- b) Select an alternative with the best anticipated pay off value (maximum for profit and minimum for cost)

Since in this criterion the decision maker selects an alternative with largest (or lowest) possible pay off value it is also called an optimistic decision criterion

Example 2: To illustrate the maximaxi criteria, let us refer the example given under *Example 1*.

Required: based on Example 1, which strategy (decision alternative) should the manufacturing company adopt?

Solution

In order to determine the strategy (alternative action) first determine the maximum value for each column. Then, the column with the largest value is considered as the best strategy.

<i>Product type</i>	<i>demand (alternative courses of action)</i>		
<i>(State of nature)</i>	<i>D1</i>	<i>D2</i>	<i>D3</i>
Product A	14	66	118
Product B	13	77	141
Product C	1	73	145
<i>Column maximax</i>	<i>14</i>	<i>77</i>	<i>145</i> ↑ Maximax

The maximum of column maxima is 145. Hence the manufacturing company should adopt alternative product D3 (sale in the high demand).

B. Criterion of Pessimism (Maximin Or Minimax)

In this criterion the decision maker ensures that he would earn no less (or pay no more) than some specified amount. Thus, he selects the alternative that represents the maximum of the minima (or minimum of the maxima in case of loss) payoff in case of profits. The working method is summarized as follows:

- a) Locate the minimum (or maximum in case of profit) payoff value in case of loss (or cost) data corresponding to each alternative, then
- b) Select an alternative with the best anticipated payoff value (maximum for profit and minimum for loss or cost)

Since in this criterion the decision maker is conservative about the future and always anticipates worst possible outcome (minimum for profit and maximum for cost or loss), it is called a pessimistic decision criterion. This criterion is also known as Walds criterion.

Example 3: Select the best strategy according to criteria of pessimism (Maxmin or Minimax)

Product type	D1	D2	D3
A	14	66	118
B	13	77	141
C	1	73	145

Solution

<i>Product type</i> <i>(State of nature)</i>	<i>demand (alternative courses of action)</i>		
	<i>D1</i>	<i>D2</i>	<i>D3</i>
Product A	14	66	118
Product B	13	77	141
Product C	1	73	145
<i>Column maxmini</i>	<i>1</i>	<i>66</i>	<i>118</i>
			↓ Maxmin

The minimax of column maximan is 118. Hence the manufacturing company should adopt alternative D3 (sale in the high demand).

C. Equally Likely Decision (Laplace) Criterion

Since the probabilities of state of nature are not known, it is assumed that all state of nature will occur with equal probability, i.e. each state of nature is assigned an equal probability. As

stats of nature are mutually exclusive and collectively exhaustive, so the probability of each of these must be $1/(\text{number of states of nature})$. The working method is summarized as follows:

- a) Assign equal probability value to each state of nature by using the formula: $1/(\text{number of states of nature})$
- b) Compute the expected (or average) payoff for each alternative (course of action) by adding all the payoffs and dividing by the number of possible states of nature or by applying the formula:
 $(\text{Probability of state of nature } j) * (\text{payoff value for the combination of alternative, } i \text{ and state of nature } j.)$
- c) Select the best expected payoff value (maximum for profit and minimum for cost)

This criterion is also known as criterion of insufficient reason because, except in few cases, some information of the likelihood of occurrence of state of nature is available

D. Criterion of Realism (Hurwicz Criterion)

This criterion suggests that a rational decision maker should be neither completely optimistic nor pessimistic and therefore, must display a mixture of both. Hurwicz, who suggested this criterion, introduced the idea of a sufficient of optimism (denoted by α) to measure the decision maker's degree of optimism. This coefficient lies between 0 and 1, where 0 represents a complete pessimistic attitude about the future and 1 a complete optimistic attitude about the future. Thus, if α is the coefficient of optimism, then $(1-\alpha)$ will represent the coefficient of pessimism.

The hurwicz approach that the decision maker must select an alternative that maximizes

$H (\text{criterion of realism}) = \alpha (\text{maximum in column}) + (1-\alpha) (\text{minimum in column})$

The working method is summarized as follows:

- a) Decide the coefficient of optimism α and then coefficient of pessimism $(1-\alpha)$
- b) For each alternative select the largest and lowest payoff value and multiply these with α and $(1-\alpha)$ values, respectively. Then calculate the weighted average by using the formula
- c) Select an alternative with best anticipated weighted average payoff value

E. Criterion of Regret (Savage Criterion)

This criterion is also known as opportunity loss decision criterion or minimax regret decision criterion because the decision maker feels regret after adopting a wrong course of action (or alternative) resulting in an opportunity loss of payoff. Thus, he/ she always intend to minimize this regret. The working method is summarized as follows:

- a) From the given payoff matrix, develop an opportunity loss (or regret) matrix as follows I. Find the best payoff corresponding to each state of nature, and
II. Subtract all other entries (payoff) in that row from this value
- b) For each course of action (strategy or alternative) identify the worst or maximum regret value. Record this number in a new row
- c) Select the course of action (alternative) with the smallest anticipated opportunity –loss value

Example 4: Given the following payoff tables, determine which alternatives would be chosen using each of these decision criteria.

- i. maximaxi, ii) maximin iii) minimax regret iv) laplace criterion

State of nature	alternative			
	a	b	c	d
1	12	17	22	14
2	18	10	16	14
3	15	14	10	14

Solution

i. Maximaxi approach

The maximaxi approach to select the best alternative using maximaxi approach; first select the maximum values from each column. Then, the best strategy is the minimum column row with the highest values. See the following computation.

State of nature	Strategy (alternative)			
	a	b	c	d
1	12	17	22	14
2	18	10	16	14
3	15	14	10	14
Column	18	17	22	14
Maximum			↑	

mximaxi

The maximum value from the maximum column row is 22. Thus, select strategy C

ii. Maximin approach.

To select the best alternative using maximini approach, first select the minimum values from each column. Then, the best strategy is the column with the highest values. See the following computation.

State of nature	Strategy (alternative)			
	a	b	c	d
1	12	17	22	14
2	18	10	16	14
3	15	14	10	14
Column minimum	12	10	10	14

↑
maximin column row
Maximum

The maximum value from the maximum column row is 14. Thus, select strategy d.

iii. Minimax regret

In order to the maximin regret decision, we must first obtain the opportunity loss or regret table. To get this table, first identify the best payoff in each row (i.e. 22, 18, and 15 for row 1, 2, and 3 respectively). Then subtract every payoff in each row from the best pay off rows. See the following.

State of nature	(alternative) Strategy nature			
	A	B	c	d
1	(22-12=10)	(22-17=5)	(22-22=0)	(22- 14=8)
2	(18-18=0)	(18-10=8)	(18-16=2)	(18-14=4)
3	(15-15=0)	(15-14=1)	(15-10=5)	(15-14) =1)

Thus, the regret (opportunity loss) table is:

State of nature	Strategy (alternative)			
	A	B	C	D
1	10	5	0	8
2	0	8	2	4
3	0	1	5	1
Worst regret (Column Maximum)	10	8	5	8
	Minimum regret			

The minimum value from the worst regret row is 4. Thus, select strategy C.

iv. Laplace criterion

To select the best alternative using Laplace approach, first we must determine the average payoff for each alternative. Then select the minimum values from each column. Then, the best

strategy is the column with the highest values. To illustrate the laplace criterion application, let us refer the above Example once again.

State of nature	Strategy(alternative)			
	a	b	c	d
1	12	17	22	14
2	18	10	16	14
3	15	14	10	14
Average payoff $(12+18+15)/3$	$(17+10+14)/3$	$(22+16+10)/3$	$(14+14+14)/3$	
= 15	=13.67	=16	=14	
Best payoff				

Since the largest payoff is obtained from alternative C, select this strategy.

4.4.2 Decision Making Under Risk

Decision making under risk is a probabilistic decision situation, in which more than one state of nature exists and the decision maker has sufficient information to assign probability values to the likely occurrence of each of these states. Knowing the probability distribution of the state of nature, the best decision is to select that course of action which has the largest expected payoff value. The most widely criterion for evaluating various course of action under risk is the expected monetary value (EMV) or expected utility

1. Expected Monetary Value

The expected monetary value (EMV) for a given course of action is the weighted average payoff which is the sum of the payoffs for each course of action multiplied by the probabilities associated with each state of nature.

Steps for calculating EMV

- Construct a payoff matrix listing all possible courses of action and state of nature. Enter the conditional payoff values associated with each possible combination of course of action and state of nature along with the probabilities of the occurrence of each state of nature
- Calculate the EMV for each course of action by multiplying the conditional payoffs by the associated probabilities and add these weighted values for each course of action
- Select the course of action that yields the optimal EMV

2. Expected Monetary Loss (EML)

An alternative approach to minimizing expected monetary value (EMV) is to minimize the expected opportunity loss (EOL), also called expected value of regret. The EOL is defined as the difference between the highest profit (or payoff) for a state of nature and the actual profit obtained for the particular course of action taken. In other words, EOL is the amount of payoff that is lost by not selecting the course of action that has the greatest payoff for the state of nature that actually occurs. The course of action due to which EOL is minimums recommended. Since EOL is an alternative decision criterion for reaching decision making under risk, therefore, the result will always be the same as those obtained by EMV criterion discussed earlier. Thus, only of the two methods should be applied to reach a decision.

Steps for calculating EOL

- a) Prepare a conditional profit table for each course of action and state of nature combination along with the associated probabilities
- b) For each state of nature calculate the conditional opportunity loss (COL), values by subtracting each payoff from the maximum payoff for that outcome.
- c) Calculate EOL for each course of action by multiplying the probability of each state of nature with the COL value and then adding the values
- d) Select a course of action for which the EOL value is minimum

3. Expected Value of Perfect Information (EVPI)

In decision making under risk each state of nature is associated with the probability of its occurrence. However, if the decision maker can acquire perfect (complete and accurate) information about the occurrence of various state of nature, then he/she will be able to select a course of action that yields the desired payoff for whatever state of nature that actually occurs.

We have seen that the EMV or EOL criterion helps the decision maker select a particular course of action that optimizes the expected payoff without any additional information. Expected value of perfect information (EVPI) represents the maximum amount of money the decision maker has to pay to get this additional informational about the occurrence of various state of nature before a decision has to made.

Examples

1. A company manufactures goods for a market in which the technology of the product is changed very rapidly. The R&D department has produced a new product which appears to

have potential for commercial exploitation. A further birr 60,000 is required for development testing. The company has 100 customers and each customer might purchase at the most one unit of the product. Market research suggest that a selling price of birr 6000 for each unit with total variable costs and selling estimates are birr 2000 for each unit. From the previous experience, it has been possible to drive a probability distribution relating to the proportion of customers who will buy the product as follows:

Proportion of customers	0.04	0.08	0.12	0.16	0.20
Probability	0.10	0.10	0.20	0.40	0.20

Required: given the above data,

- show the expected opportunity loss value using the opportunity loss table
- state whether or not the company should develop the product

Solution

Given: selling price= birr 6000, variable cost = birr 2000,

Number of units sold=100p, cost of testing 60,000

- Let 'p' be the proportion of customers who purchase the product,

S1 be *develop the new product* alternative

S2 be *do not develop the new product*

N state of nature, then

First determine the conditional profit payoff as follows

The conditional profit is= (contribution to profit X number of units sold) -development test

$$= (6000-2000) \times (100p) -$$

$$60000 = 400000p - 60,000$$

The conditional profit values (payoffs) for each pair of state of nature (proportion of customers who will buy the product) and alternative (to develop or not to develop the new product) is shown in the following table.

(Proportion of customers, p)	<u>courses of action</u>	
	S1 (Develop)	S2 (Do not develop)
0.04	-44,000	0
0.08	-28000	0
0.12	-12000	0
0.16	4000	0
0.20	20,000	0

		State of nature (Proportion of customers, p)			
		<u>Conditional profit</u>		<u>opportunity loss</u>	
State of nature	prob.	S1	S2	S1	S2
0.04	0.1	-44,000	0	44,000	0
0.08	0.1	-28000	0	28000	0
0.12	0.2	-12000	0	12000	0
0.16	0.4	4000	0	0	4000
0.20	0.2	20000	0	0	20,000

Using the given estimate of probabilities of associated with each state of nature, the expected opportunity loss (EOL) for each alternative is given below:

**EOL of developing the product (S1) is:*

$$=0.1(44000) + 0.1(28000) + 0.2(12000) + 0.4(0) + 0.2(0) = \text{birr } 9600$$

**EOL of do not develop the product (S2) is:*

$$=0.1(0) + 0.1(0) + 0.2(0) + 0.4(4000) + 0.2(20,000) = \text{birr } 5600.$$

Since the company wants to minimize the expected opportunity loss, it should select courses of action S2 (do not develop the product) with minimum EOL.

2. A company needs to increase its production beyond its existing capacity. It has narrowed the alternatives to two approaches to increase the production capacity: expansion, at a cost of birr 8 million or modernization at a cost of birr5 million. Both approaches would require the same amount of time for implementation. Management believe that over the required payback period, demand will either be high or moderate. Since high demand is considered to be somewhat less likely than moderate demand, the probability of high demand has been set at 0.35. If the demand is high, expansion would gross an estimated additional birr 12 million but modernization only an additional birr 6 million, due to lower maximum production capability. On the other hand, if the demand is moderate, the comparable figure would be birr 7 million for expansion and birr 5 million for modernization.

Required: given the above data:

- a) calculate conditional profit

- b) If the company wishes to maximize EMV, should it modernize or expand?
- c) Calculate EVPI
- d) Construct the conditional opportunity loss table and also calculate EOL.

Solution

Given: - state of nature: high demand (N1) and moderate demand (N2)
 -alternative action (courses of action): expand (S1) and modernize (S2)

- a. The calculation for conditional profit values are shown below:

NB: since the probability that the demand is high estimated at 0.35, the probability of moderate demand must be $(1-0.35) = 0.6$

State of nature (Demand)	<u>conditional profit (in '000000's</u>	
	expand (S1)	modernize (S2)
High demand (N1)	$12-8=4$	$6-5=1$
Moderate demand (N2)	$7-8=-1$	$5-5=0$

- a. The calculation of EMV for each courses of action S1 and S2 is given below:

$$\text{EMV (S1)} = 0.35(4) + 0.65(-1) = 0.75\text{million}$$

$$\text{EMV (S2)} = 0.35(1) + 0.65(0) = 0.35\text{milliun}$$

To maximize EMV, the company must choose courses of action S1 (expand). Thus, the EMV=0.75million

- b. To calculate EVPI, we shall first calculate expected payoff with perfect information (EPPI). To calculate EPPI, we chose optimal courses of action for each state of nature, multiply its conditional profit by the given probability to get weighted profit, and then sum these weights as shown below:

State of nature (Demand)	probability	optimal	conditional	
		course of action	profit	weighted profit
High demand	0.35	S1	4	$4 \times 0.35 = 1.40$
Moderate DD	0.65	S2	0	$0 \times 0.65 = 0$
				EPPI = 1.40

The optimal EMV is 0.75million corresponding to the courses of action

S1. Then, $\text{EVPI} = \text{EPPI} - \text{EMV (S1)}$

$$= 1.40 - 0.75$$

$$= \text{birr } 0.65 \text{ million}$$

c. The opportunity loss values are shown below:

State of nature (Demand)	probability	<u>conditional profit</u>		<u>conditional loss</u>	
		S1	S2	S1	S2
High demand	0.35	4	1	0	3
Moderate DD	0.65	-1	0	1	0

$$EOL(S1) = 0.35(0) + 0.65(1) = 0.65 \text{ million}$$

$$EOL(S2) = 0.35(3) + 0.65(0) = 1.05 \text{ million}$$

Since decision makers seek to minimize the EOL, he must select alternative S1.



Self-check exercise 4.1

1. Discuss the importance of decision model.
2. Discuss the different types of decision-making environments.

4.5 Decision Tree Analysis

Decision making problems discussed so far are referred to as single stage decision problems, because the payoffs, state of nature, course of action and probabilities associated with the occurrence of state of nature are not subject to change. However, situations may arise when a decision maker needs to revise his previous decision on getting new information and make sequences of other decisions. Thus, the problem becomes a multi stage decision problem because the consequence of one decision affects the future decisions. For example, in the process of marketing a new product, the first decision is often testing marketing and the alternative course of action might be either intensive testing or gradual testing. Given various possible consequences- good, fair or poor, the decision maker may be required to decide between redesigning the product, an aggressive advertising campaign or complete withdrawal of product, etc. given that decision, there will be an outcome which leads to another decision and so on.

Decision tree analysis involves construction of a diagram showing all the possible course of action, state of nature and the probabilities associated with the state of nature. The decision diagram looks very like a drawing of a tree, therefore also called decision tree.

A decision tree consists of nodes, branches, probability estimates and payoffs. There are two types of nodes: decision nodes and chance nodes. A decision node is usually represented by a square and indicates where a decision maker must make a decision. Each branch leading away from a decision node represents one of the several possible courses of action available to the decision maker. The chance node is represented by a circle and indicates a point, at which the decision maker will discover the response to his decision, i.e. different possible course occurred from a chosen course of action.

Branches emanate from and connect various nodes and are either decisions or state of nature. These are two types of branches: decision branches and chance branches. Each branch leading away from a decision node represents a course of action or strategy that can be chosen at a decision point, whereas a branch leading away from a chance node represents the state of nature of a set of chance factors. Associated probabilities are indicated alongside of respective chance branch. These probabilities are the likelihood that the end of decision tree, i.e., it is not followed by either a decision or chance node, is called a terminal branch. A terminal branch can represent either a course of action or chance outcome. The terminal points of a decision tree are said to be mutually exclusive points so that exactly one course of action will be chosen.

The payoff can be positive (i.e. revenue or sales) or negative (expenditure or cost) and they can be associated either with decision or chance branches. The optimal sequence of decisions in tree is found by starting at the right-hand side and rolling backward. The aim of this operation is to maximize the return from the decision situation. At each node, an expected return is calculated (called position value). If the node is chance node, then the position value is calculated as the sum of products of the probabilities or the branches emanating from the chance node and their respective position values. If the node is a decision node, then the expected return is calculated for each of its branches and the highest return is selected. The procedure continues until the initial node is reached. The position value for this node corresponds to the maximum expected return obtained from the decision sequence.

Remark: decision tree versus probability trees decision tree are basically an extension of probability tree. However, there are several basic differences:

1. The decision tree utilizes the concept of ‘rollback’ to solve a problem. This means starting at the right-hand terminus with the highest expected value of the tree and working back to the current or beginning decision point to determine the decision or decisions that should

be done. Most decisions require trees with numerous branches and more than one decision point. It is the multiplicity of decision points that make the rollback process necessary.

2. The probability tree is primarily concerned with calculating the correct probabilities, whereas the decision tree utilizes probability factors as a means of arriving at a final answer
3. The most important feature of the decision tree, not found in probability tree, is that it takes time differences of future earnings in to account. At any stage of the decision tree, it may be necessary to weight differences in immediate cost or revenue against differences in value.

Example 1: An oil company has recently acquired rights in a certain area to conduct surveys and test drillings to lead to lifting oil if it is found in commercially exploitive quantities. The area is considered to have good potential for finding oil in commercial quantities. At the outset the company has the choice to conduct further geological tests or to Carry out a drilling program immediately. On the known conditions, the company estimates that there is a 70:30 chance of future tests showing a success.

Whether a test show the possibility of ultimate success or not or even if no test is undertaken at all, the company could still pursue its drilling program or alternatively consider selling its rights to drill in the area. Therefore, However, if it carries out the drilling program, the likely hood of final success or failure is considered depends on the forgoing stages. Thus;

- If successful tests have been carried out the expectation of success in drilling is given as 80:20.
- If the test indicates failure, then the expectation of success in drilling is given as 20:80.
- If no tests have been carried out at all the expectation of success in drilling is given as 55:45.

Costs and revenues have been estimated for all possible outcomes and the net present value of each is as follows:

Outcomes	Net present value (birr million)
Success	
With prior list	100
Without prior list	120
Failure	
With prior list	-50
Without prior list	-40

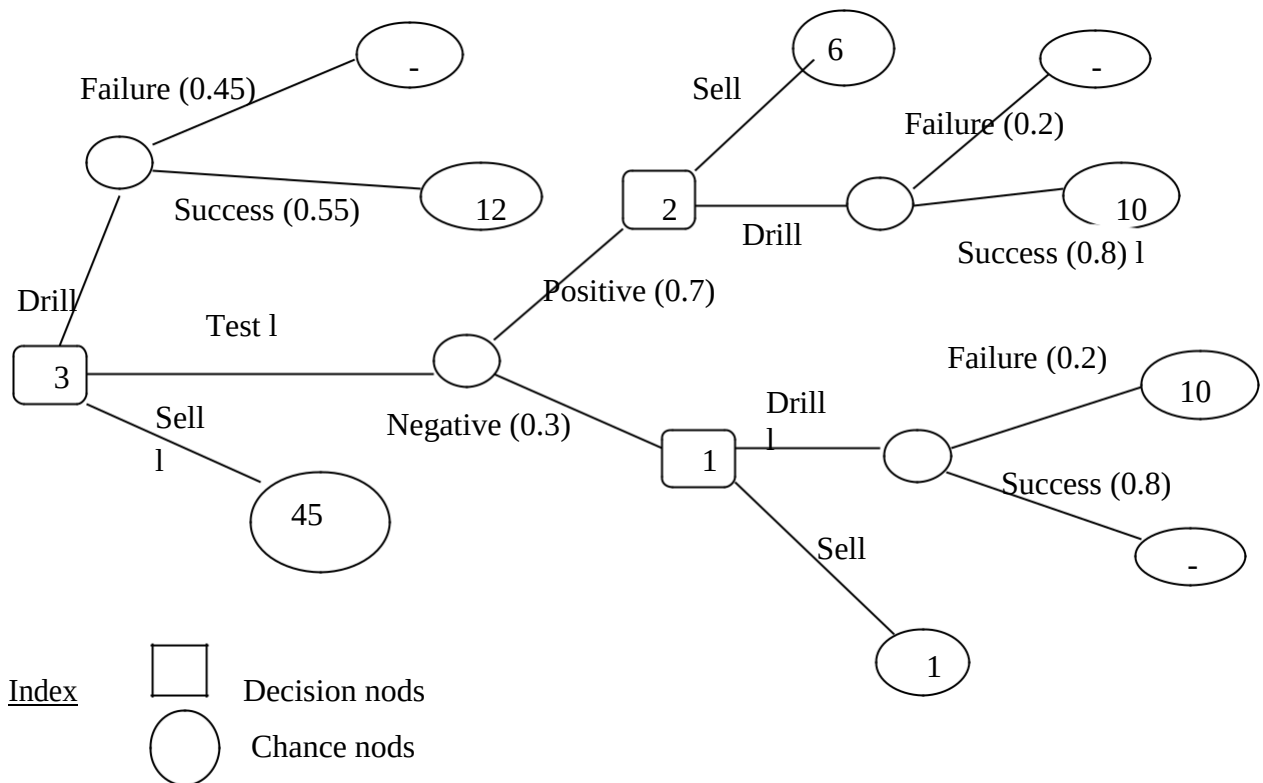
Sale of exploitation right:

Prior test show success

Prior test show failure

Without prior test

- A. Draw up a decision (probability) tree diagram to represent the above information: and
- B. Evaluate the tree in order to advise the management of the company on its best course of action.



As you observe that the tree has several branches which originate from squares or from circles. A square represents a decision node or decision fork at which the decision maker has to take a decision, while a circle represents a chance node or chance fork at which events (i.e. the state of nature) are branched out.

The method of solution: as mentioned earlier the decisions have to be evaluated in a backward manner by evaluating the best course at the later stages so as to decide on the best course of actions on the early stage. Thus, the solution to the problem is obtained by working backwards, from right to left, through the tree.

Evaluation of decision node 1

Alternatives	Out come	Probability	Conditional value	Expected value
1. Drill	Success	0.2	100	20
	Failure	0.8	(-50)	(40)
	Total			(20)
2. Sell		1.0	15	15

The expected value of the drilling option is a loss of Birr 20 million while if the site is sold, we can get birr 15 million. On the basis of the criterion of maximization of the expected profit, our decision would be to sell the site.

Evaluation of decision node 2

Alternatives	Out come	Probability	Conditional value	Expected value
1. Drill	Success	0.8	100	80
	Failure	0.2	(-50)	(10)
	Total			70
2. Sell		1.0	65	15

The two-alternative course of action, selling and drilling have expected value equal to 65 million and birr 70 million, respectively. Obviously, therefore, provided that a positive result is indicated by the test, the best course would be to go in for oil drilling.

Now at this stage we have two conditional decisions, sell the site if negative is obtained on the test and drill in case the test indicates a positive. At each node the branch at which we have not to move. Next, we move to the decision node where a decision has to be taken whether to drill at the outset.

Evaluation of decision node

Alternatives	Out come	Probability	Conditional value	Expected value
1. Drill	Success	0.55	120	66
	Failure	0.45	(40)	(18)
	Total			48
2. Test	Positive	0.7	70	49
	Negative	0.3	15	4.5
	Total			53.5
3. Sell		1.0	45	45

The expected value of the alternative of carrying out a test is birr 53.5 million, which is the highest of the three. Therefore, it is better to test before carried out. If it proves negative, the right should be sold to give a return of birr 15 million. To proceed with relining, if that happens, would expectedly lead to a loss. However, if the test proves positive the drilling should be under taken.



Self-check exercise 4.2

1. An investor is given the following investment alternatives and percentage rate of return.

	State of nature (market condition)		
	Low (%)	Medium (%)	High (%)
Regular sharing	2	5	8
Risky sharing	-5	7	15
Property	-10	10	20

Over the past 300 days, 150 days have been medium market conditions and 60 days have had high market increases. On the basis of these data, state the optimal investment strategy for the investor. According to the f given information the probabilities of low medium and high market conditions would be $90/300$ Or 0.30 , $150/300$ or 0.50 and $60/300$ or 0.20 , respectively.

- a. Calculate the expected payoffs for each of the alternatives
- b. Calculate the expected return
- c. In which alternative the investor invests his money?

CHAPTER SUMMARY

Knowledge of the likely hood of each of the state of nature can play an important role in selecting a course of action. There are three decision making environments: certainty, uncertainty and risk. Decision under certainty is the case where the decision maker has a perfect knowledge (information) about the state of nature

Decision making under uncertainty is a situation in which decision maker has neither previous knowledge of the outcomes of his decisions nor is in a position to attribute a probability of occurrence of such outcomes.

The most widely used criterion for evaluating various courses of action (alternatives) under risk includes: expected monetary value (EMV), expected opportunity loss (EOL), expected value of perfect information (EVPI), Expected monetary value is the sum of the payoffs for each course of action multiplied by the probabilities associated with each state of nature.

Expected value of perfect information (EVPI) represents the maximum amount of money the decision maker has to pay to get information about the occurrence of various state of nature before a decision has to be made.

A decision tree is a graphic representation of the sequence of action event combination available the decision maker. It depicts in a systematic manner all possible sequences of decisions and consequences.

CHAPTER FIVE

NETWORK MODEL

5.1 Introduction

Dear learners! One of the most challenging jobs that any managers take is the management of projects that require coordinating numerous activities throughout the organization. A myriad of details must be considered in planning how to coordinate all these activities, in developing a realistic schedule, and then in monitoring the progress of the project. Now a days, two closely related operations research techniques, PERT (program evaluation and review technique) and CPM (critical path method), are available to support the project manager in carrying out these responsibilities. These techniques make heavy use of to help plan and display the coordination of all the activities. They also normally use a software package to deal with all the data needed to develop schedule information and then to monitor the progress of the project. In this chapter we will be discuss critical path and program evaluation and review techniques of managing projects effectively and efficiently.

5.2 Objectives of The Chapter

After going through this chapter, you should be able to:

- ☞ Define network, project management and network diagram
- ☞ State and explain the characteristics and phases of project management
- ☞ Describe the major difference between PERT and CPM
- ☞ Explain PERT/CPM Network Components and precedence Relationship
- ☞ Demonstrate project time cost trade-off

5.3 The Concept of Network Model

Network model is a technique used for planning, and scheduling large projects in the field of construction, maintenance, fabrication, purchasing computer system, distribution, transportation, facility allocation, and many others. There are several reasons for the wide range of applications. First many problems that can be formulated as LP models can also be visualized with the context of **networks**. **Second** network models of problems are often easier to formulate than alternative LP models of the same problem. Third the pictorial representation of network models makes them relatively easy to understand, particularly by the non-technical

decision makers. Finally, the advances in network optimization procedure have resulted in some amazing results in solving network model.

Network diagram is a practical representation of the various events and activities.

Network modeling: Techniques that enable complex projects to be scheduled, taking into account the precedence of each activity.

The different systems satisfying the definition of network in the physical world

Physical situation	Nodes	Branches	Flow
Highway systems	Intersections	Roads	Vehicles
Communication systems	Switching points	Wires	Messages
Fluid supply systems	Pumping stations	Pipes	Fluid
Production systems	Work centers	Handling routs	Jobs
Project Management	Decision points	Activities	Time
Airway systems	Air port	Air lines	Aircraft

5.3.1 Projects

Definition: A **project** is a set of proposals for the investment of resources in to a clearly identified set of actions that are expected to produce future benefits that exceeds costs.

A project is a combination of interrelated activities all of which must be executed in a certain order for its completion.

Examples of a project:

- *New business establishment*
- *Expansion of existing business*
- *Launching new product*
- *Cost reduction project*
- *Total Quality of Management (TQM), etc*

Characteristics of a project:

1. *The duration of a project lasts long*
2. *A project is complex in nature involving many interrelated activities and participants*
3. *Delay in completion time may be very costly*
4. *Project activities are sequential*

5.3.2 Phases of Project Management:

Project management consists of three phases;

1. project planning phase

In order to visualize the sequencing or preceding requirements of the activities in a project, it is helpful to draw a network diagram. For this the following steps are adopted:

- i) Identify various activities (tasks or work elements) to be performed in the project, that is, develop a breakdown structure (WBS).
- ii) Determine requirement of resources such as man, material, machines, money, e.t.c. for carrying out activities listed above.
- iii) Assign responsibility for each work package. The work package corresponds to the smallest work effort defined in a project and from the set of elements which are the base for planning, scheduling and controlling the project.
- iv) Allocate resources to work package.
- v) Estimate cost and time at various level of project completion.
- vi) Develop work performance criteria.
- vii) Establish control channels for project personnel.

2. Project scheduling phase

Once all packages have been identified and given unique names or identification, scheduling of the project, i.e. when each of the activities required to be performed, is taken up. The various steps are listed below:

- i) Identify all people who will be responsible for each task.
- ii) Estimate the expected duration of each activity, taking in to consideration the resource required for their execution in most economic manner.
- iii) Specify the interrelationship (i.e. precedence relationship) among various activities.
- iv) Develop a network diagram showing the sequential interrelationship between various activities.
- v) Based on the time estimate, calculate the total project duration, identify critical path; calculate floats; carry out resource smoothing (leveling) exercise for critical (or scarce) resources taking in to account resource constraints if any.

3. project control phase

It refers to evaluating actual progress (status) against the plan. If significant difference is observed then the scheduling and resource allocation decision are changed to update and revise the uncompleted part of the project.

There are two basic planning and control technique that utilize network to complete predetermined Project or schedule. These are Program Evaluation and Review Technique (PERT) and Critical Path Method.

PERT is an acronym for "Program Evaluation and Review Technique". This was created as a means to plan and accelerate the development of the Polaris Ballistic Missile. In USA the defense department developed a nuclear missile to be launched from beneath the ocean's surface by a mobile submarine, which would be an effective deterrent against aggression by an enemy. This paved way to plan how to design, develop and plan the different stages in the production of a missile and how quickly this task could be completed. A planning and scheduling technique named PERT gave the answer to these questions.

In any new venture, uncertainties are bound to creep in. PERT incorporated these uncertainties into a model, which provides a reasonable answer to these uncertainties. There are certain statistical aspects scheduling large projects consisting of numerous activities whose completion times are uncertain and are independent of one another. PERT is an event-oriented technique. By 'event' we mean reaching a certain stage of completion of the project.

Another technique, Critical Path Method, abbreviated as CPM, has emerged simultaneously. It is also a network technique but it is concerned with obtaining the trade-off, between cost and completion date for large projects. In any project consisting of several activities each activity can be completed in a normal duration with normal cost. If we employ more persons or skilled people or given overtime to the workers, the activity could be completed in a reduced duration known as crash duration. But this involves an increased cost in the form of additional resources. With CPM the amount needed to complete the various activities is assumed to be known with certainty. So, the direct costs for the activities increase and hence the cost of the project also increases. By reducing the activity duration of some or all possible completed ahead of the schedule. This will naturally reduce the overhead cost for the entire project. On one hand the direct expenses increase, if we shorten the activity duration, but, the indirect expenses for the project are reduced. We have to strike a balance or an optimum time schedule, or at least cost

schedule is to be obtained. This is the purpose of the Critical Path Method. Thus, CPM is not concerned with uncertain job times as in PERT. PERT is useful in research and developmental projects, whereas CPM is mostly used in construction projects, or in situations already handled, so that the details like the normal completion time, crash duration and cost of crashing are already known.

The following are the suggested applications when PERT or CPM is found useful.

- ➡ The construction of a building or of a highway.
- ➡ Planning and launching a new product.
- ➡ Scheduling maintenance for a project.
- ➡ The manufacture and assembly of a large machine tool.
- ➡ To conduct a music or drama festival.
- ➡ Preparation of budget for a company.



Self-check exercise 5.1

1. Discuss the importance of networking model in the area of business.
2. Define projects.
3. How networking model is important to manage projects? Discuss in detail.

5.4 Networking Algorithms

An algorithm is a procedure or formula for solving a problem, based on conducting a sequence of specified actions. In networking there are different techniques to be followed to solve a given problem. The most widely used techniques are Gantt chart, critical path method and project evaluation and review technique. In this chapter the latter two technique will be discussed. Critical path method (CPM) is a method that can be useful for deterministic time estimate and project evaluation and review technique (PERT) is useful when the time estimates are not clearly known or probabilistic. In networking algorism there is sequence of activities to be carried out. These sequences represented by activity on node and on arrow. The most common one is the latter one which activity on arrow. We first define the preceding relationship between activities and the time each activity required to perform. Thereafter, we are going to calculate the earliest start, earlies finish, latest start, latest finish and float time.

5.4.1 Basic Difference between PERT and CPM

PERT

1. It assumes a probability distribution for the duration of each activity. Thus, completion time estimates for all of the activities are needed
2. To perform PERT analysis on a project, the emphasis is given on the completion of a task rather than the activities required to perform to reach a particular event or task. Thus, it is also called *event-oriented technique*.
3. It is used for one-time project involving activities of none-repetitive nature (i.e. activities which may never have been performed before) in which time estimates are uncertain, such as redesigning an assembly line or installing a new information system
4. It helps in identifying critical areas in a project so that necessary adjustment can be made to meet the schedule completion date of the project.

CPM

1. This technique was developed in connection with a construction and maintenance project in which duration of each activity was known with certainty.
2. It is suitable for establishing a trade-off for optimum balancing between schedule time and cost of the project.
3. It is used for completion of project involving activities of repetitive nature.

5.4.2 PERT/CPM Network Components and precedence Relationship

EVENTS

Events in the network diagram represent project milestones, such as the start or the completion of an activity (task) or activities, and occur at a particular instant of time at which some specific part of the project has been or is to be achieved. Events are commonly represented by circles (nodes) in the network diagram. The events can be further in to following two categories. One or more activities can start and end simultaneously at an event.

- I. **MERGE EVENT:** an event which represents the joint completion of more than one activity is known as merge event

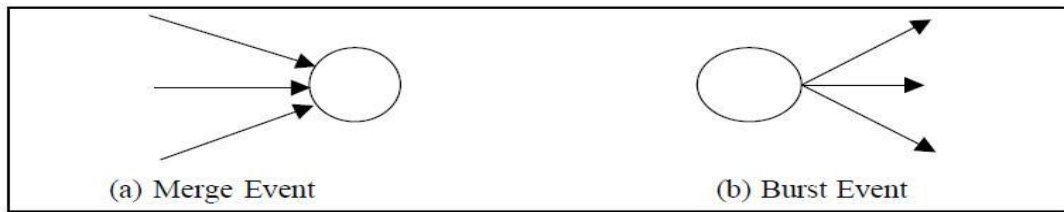


Fig 1 types of events

- II. **BURST EVENT:** an event which represents the initiation (beginning) of more than one activity is known as burst event

Events in the network diagram are identified by numbers. Each event should be identified by a number higher than that allotted to its immediately preceding event to indicate progress of the work. The numbering of events in the network diagram must start from left (start of the project) to the right (completion of the project) and top to bottom. Care should be taken that there is no duplication in the numbering.

ACTIVITIES

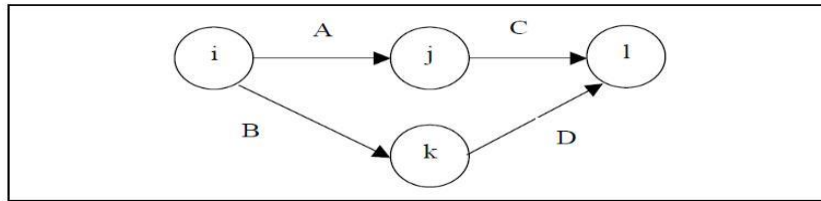
Activities in network diagram represent project operation or tasks to be conducted. As such each activity except dummy consumes time and resource and incurs costs. An arrow is commonly used to represent an activity with its head indicating the direction of progress in the project. Activities are identified by the numbers of their starting (tail or initial) and ending (head or terminal) event. An arrow (i,j) between two events; the tail even I represent the start of the activity and the head event j represents the completion of the activity as shown in fig 2



Fig 2 activity –node relationship in network diagram

The activities can be further classified in to the following three categories:

- Predecessor activity: an activity which must be completed before one or more activities start is known as predecessor activity
- Successor activity: an activity which started immediately after one or more of other activities are completed is known as successor activity

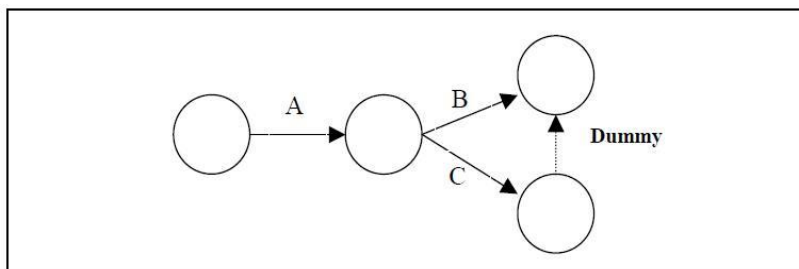


Activities A and B precede activities C and D respectively

- c) **Dummy activity:** an activity which does not consume either may resource and/ or time is known as dummy activity

A dummy activity in the network is added only to establish the given precedence relationship among activities of the project and is needed when

- a) Two or more parallel activities in a project have same head and tail events or
- b) Two or more activities have some (but not at all) of their immediate predecessor activities in common. A dummy activity is depicted by dotted line in the network diagram



Network models use following two types of precedence network to show precedence requirements of the activities in the project

Activity on node (AON) network: in this type of precedence network each node (or circle) represents a specific task while the arcs represent the ordering between tasks. AON network diagrams place the activities within the nodes, and the arrows are used to indicate sequencing requirements. Generally, these diagrams have no particular starting and ending nodes for the whole project. The lack of dummy activities in these diagrams always makes them easier to draw and to interpret

Activity on arrow (AOA) network: in this type of precedence network at each end of the activity arrow is node (or circle). These nodes represent points in time or instants, when an activity is starting or ending. The arrow itself represents the passage of time required for that activity to be performed. These diagrams have a single beginning node from which all activities with no followers come together. Three important advantage of using AOA are as follows:

- i. Many computer programs are based on AOA network
- ii. AON diagrams can be superimposed on a time scale with the arrows drawn, the correct length to indicate the time requirement
- iii. AOA diagrams give a better sense of the flow of time throughout a project

In this chapter, only AOA network diagram will be used fig 3 illustrates AON and AO diagrammatically

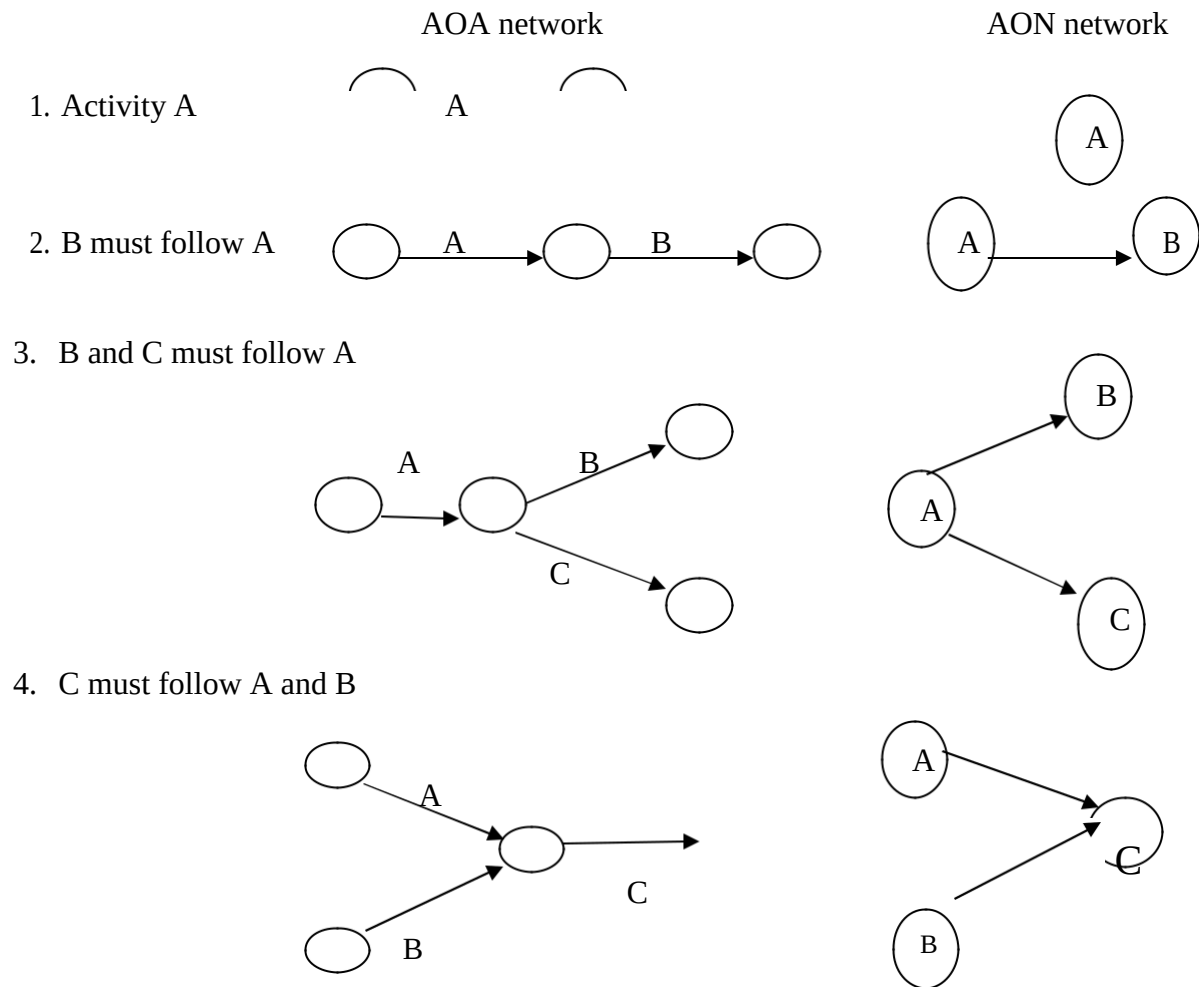


Fig 3 AOA and AON diagramming

Rules of AOA network construction

Following are some of the rules which have to be followed while constructing a network

1. In network diagram, arrows represent activities and circles the events. The length of an arrow is of no significance

2. Each activity should be represented by only on arrow and must start and end in a circle called event. The tail of an activity represents the start and head the completion of the work
3. The event numbered 1 denotes start of the project and is called initial event. All activities emerging (or taking off) from event 1 should not be preceded by any other activity or activities. Event carrying the highest number denotes the completion event. A network should have only one initial event and only one terminal event
4. The general rule for numbering the event is that the head event should always be numbered larger than that at its tail. That is, events should be numbered such that for each activity (I,j), $i < j$
5. An activity must be uniquely identified by its starting and completion event which implies that
 - a) An event number should not get repeated or duplicated
 - b) Two activities should not be identified by the same completion event
 - c) Activities must be represented either by their symbols or by the corresponding ordered pair of starting-completion events
6. The logical sequence (or inter- relationship) between activities must follow these rules:
 - a) An event cannot occur until all the incoming activities into have been completed
 - b) An activity cannot start unless all the preceding activities on which it depends have been completed
 - c) Though a dummy activity does not consume either any resource of time, even then it has to follow the rules of dummies

Errors and dummies in network

1. **Looping and dangling:** lopping (cycling) and dangling are considered to as fault in a network. Therefore, must be avoided

(a) A case of endless loop in a network diagram which is also known as looping is shown below where activities A, B, and C form a cycle

Due to precedence relationship, it appears from fig 4 that every activity in looping (or cycle) is a predecessor of itself. In this case it is difficult to number three events associated with activity A, B and C so as to satisfy rule 6 of constructing a network

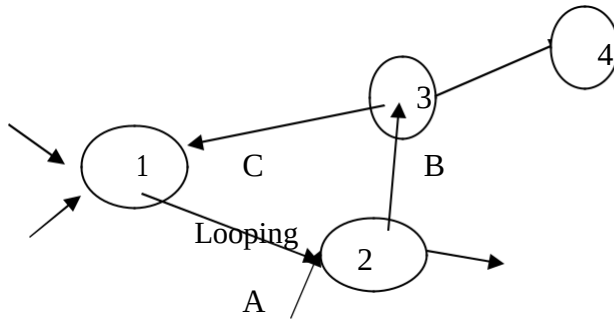


Fig 4.1 looping in network diagram

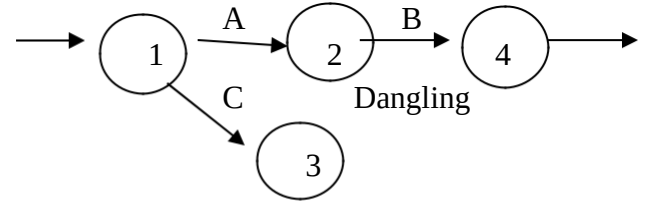


fig 4.2 dangling in network diagram

(b) A case of disconnect activity before the completion of all activities which is also known as dangling is shown on fig 4.2. In this case, activity C does not give any result as per rules of the network. The dangling is avoided by adopting rule 5 of constructing the network

Dummy (or redundant) activity: the following are the two cases in which use of dummy activity may help in drawing the network correctly as per the various rules.

- When two or more parallel activities in a project have the same head and tail events, i.e. two events are connected with more than one arrow (A), activities B and C have a common predecessor activity A. at the same time, they have activity D as common successor. To get correct network, a dummy activity for the ending event B to show that D may not start before B and C are completed is shown in fig (B).

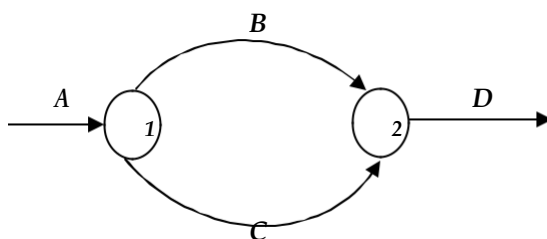


Fig: a) Parallel Activities

Activity

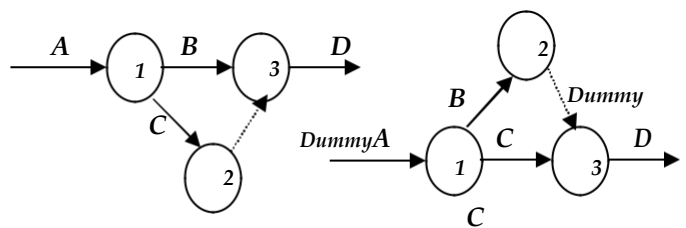


Fig: b) Dummy

5.5 Critical path Analysis

The objective of critical path analysis is to estimate the total project duration & to assign starting & finishing times to all activities involved in the project. This helps in checking actual progress against the scheduled duration of the project.

The duration of individual activities may be uniquely determined (in case of CPM) or may involve the three-time estimates (in case of PERT) out of which the expected duration of an activity is computing. Having done this, the following factors should be known to prepare project scheduling.

- i) Total completion time of the project
- ii) Earliest & latest start time of each activity
- iii) Float for each activity i.e. the amount of time by which the completion of an activity can be delayed without delaying the total project completion time.
- iv) Critical activities & critical path.

Consider the following notation for the purpose of calculating various times of event & activities.

- A. =Earliest occurrence time of an event, i. It is the earliest time at which an event can occur without affecting the total project time.
- B. =Latest occurrence time of event i. It is the latest time at which an event can occur without affecting the total project time
- C. =Earliest start time for activity (i, j). It is the time at which the activity can start without affecting the total project time
- D. =Latest start time for activity (i, j). It is the latest possible time by which an activity must start without affecting the total project time
- E. =Earliest finish time for activity (i, j). It is the earliest possible time at which an activity can finish without affecting the total project time.
- F. =Latest finish time for activity (i, j). It is the latest time by which an activity must get completed without delaying the project completion.
- G. Duration of activity (i, j).

Expected time estimate

In a network diagram, there should only be one initial event & one final event, while other events are numbered consecutively with integers 1, 2... n such that $i < j$, for any two event i & j connected by an activity which starts at i & finish at j

For calculating the above-mentioned times, there are two methods; namely;

- 1) Forward pass method &
- 2) Backward pass method

1. Forward Pass Method (For Earliest Event Time)

In this method calculation begins from the initial event 1, proceeds through the network visiting event in an increasing order of event number & ends at the final event, say N. At each event we calculate earliest occurrence event time (E) & earliest start & finish time for each activity that begins at that event. When calculations end at the final event N, its earliest occurrence time gives the earliest possible completion time of the entire project.

The method may be summarized as follows:

- 1) Set the earliest occurrence time of initial event 1 to zero, i.e. $E_1 = 0$, $E_1 = 0$
- 2) Calculate earliest start time for each activity that begins at event i ($i=1$). This is equal to earliest occurrence time of event, i (tail event). That is E_i , for all activities (i, j) starting at event i .
- 3) Calculate the earliest finish time of each activity that begins at event i . This is equal to the earliest start time of the activity plus the duration of the activity. That is, $E_i + t_{ij}$ for all activities (i, j) beginning at event i .
- 4) Proceed to the next event, say j ; $j > i$
- 5) Calculate the earliest occurrence time for the event j . This is the maximum of the earliest finish times of all activities ending into that event, i.e.
 $E_j = \max \{ E_i + t_{ij} \}$, for all immediate predecessor activities.
- 6) If $j = N$ (final event), then earliest finish time for the project, i.e. the earliest occurrence time for the final event is given by

$$E_N = \max \{ E_i + t_{ij} \}, \text{ for all terminal activities.}$$

$E_j = \max \{ E_i + t_{ij} \}$ & stop the method.

2. Backward Pass Method (For Latest Allowable Event Time)

In this method calculations begin from final event N, proceed through the network visiting events in the decreasing order of event number & end at the initial event 1. At each event, we calculate the *latest occurrence event time (L)* for the corresponding event, latest finish & start

time for each activity that is terminating at the event, such that the earliest finish time for the project remains the same. The method may be summarized as follows.

1. Set the latest occurrence of last event N equal to its earliest occurrence time (known from forward pass method), that is

$$L_N = E_N$$

2. Calculate latest finish time of each activity which ends at event j. This is equal to latest occurrence time of final event N. That is,

$$L_j = L_N \text{ for all activities (i, j) ending at event j.}$$

3. Calculate the latest start times of all activities ending at j. It is obtained by subtracting the duration of the activity from the latest finish time of the activity, that is,

$$L_i = L_j - t_{ij} \text{ for each activity (i, j) ending at Event j,}$$

4. Proceed backward to the event in the sequence that decreases j by 1.
5. Calculate the latest occurrence time of event i (i < j). This is the minimum of the latest start times of all activities starting from that event. That is,

$$L_i = \min \{ L_j - t_{ij} \}$$

6. If j = 1 (initial event), then the latest finish time for project, i.e. latest occurrence time L_1 for the initial event is given by.

$$L_1 = \min \{ L_j - t_{1j} \}$$
 For all immediate successor activities

$$L_1 = \min \{ L_j - t_{1j} \}$$
 & stop the method

3. Float (or slack) of an Activity and Event

The *float (slack)* or *free* time is the length of time to which a non-critical activity and/or an event can be delayed the total project completion time. The *float* of an activity is the amount of time by which it is possible to delay its completion time without affecting the total project completion time.

1. Event Float

The **float** (also sometimes called '**slack**') of an event is the difference between its latest time (L_i) and its earliest time (E_i). That is

$$F_i = L_i - E_i$$

It is a measure of how much later than expected a particular event could occur without delaying the completion of the entire project.

2. Activity Float

It is the float (or slack) in the activity time estimates.

Types of activity floats:

a. Total Float

It is the difference between the time available to perform the activity (**measured from the earliest start time to the latest finish time**) and the expected completion time of the activity.

That is, for activity (i, j), the total float is given by

$$TF_{ij} = (L_j - E_i) - (E_j - E_i)$$

b. Free Float

For calculating the total float, only a particular activity was considered with respect to its tail and head event times or by considering latest start and finish time of an activity with respect to its earliest start and finish time. However, it may be needed to know that how much an activity's scheduling time can be increased or decreased without changing its immediate successor activities.

The time by which the completion of an activity can be delayed beyond its **earliest finish time** without affecting the earliest start time of a succeeding activity is called **free float**.

Independent float () = (-) - = { - } - The negative value of independent float is considered zero.

1. Since latest event time is always greater than or equal to the earliest event time (i.e.

This implies the value of free float may range from zero to total float and can never exceed total float value.

2. The calculation of various floats can help the decision-maker in identifying the underutilized resources, flexibility in the total schedule and possibilities of redeployment of resources.
3. Once the float of an activity is disturbed, float of all other activities of the project is changed and should be recalculated.
4. If total float value is

→ *resources are not adequate* and activity may not finish in time

b. Zero (i.e $L-E=0$)

c. Positive (i.e $L-E>0$)



Resources are surplus



Surplus resources can be deployed elsewhere or execution of the activity can be delayed.

Critical Path

Path: A path is unbroken chain of activities that connects from the start node to the end node (terminal node).

Certain activities in a network diagram of a project are called **critical activities** because delay in their execution will cause further delay in the project completion time. Thus, all activities having zero total float value are identified as **critical activities**.

The critical path is the continuous chain of critical activities in a network diagram. It is the **longest path** starting from first to the last event & is shown by the thick line or double lines in the network diagram most of the time.

The length of the critical path is the sum of the individual times of the critical activities living on it & defines the minimum time required to complete the project.

The critical path on a network diagram can be identified as:

- i) For all activities (i, j) lying on the critical path the E-values & L-values for tail & head events are equal That is, $E_i = L_j$
- ii) On critical path $E_i = L_i = E_j = L_j$

Finding the critical path is important for directing decision-makers attention & effort to critical activities where improvement will pay the largest dividend.

Example

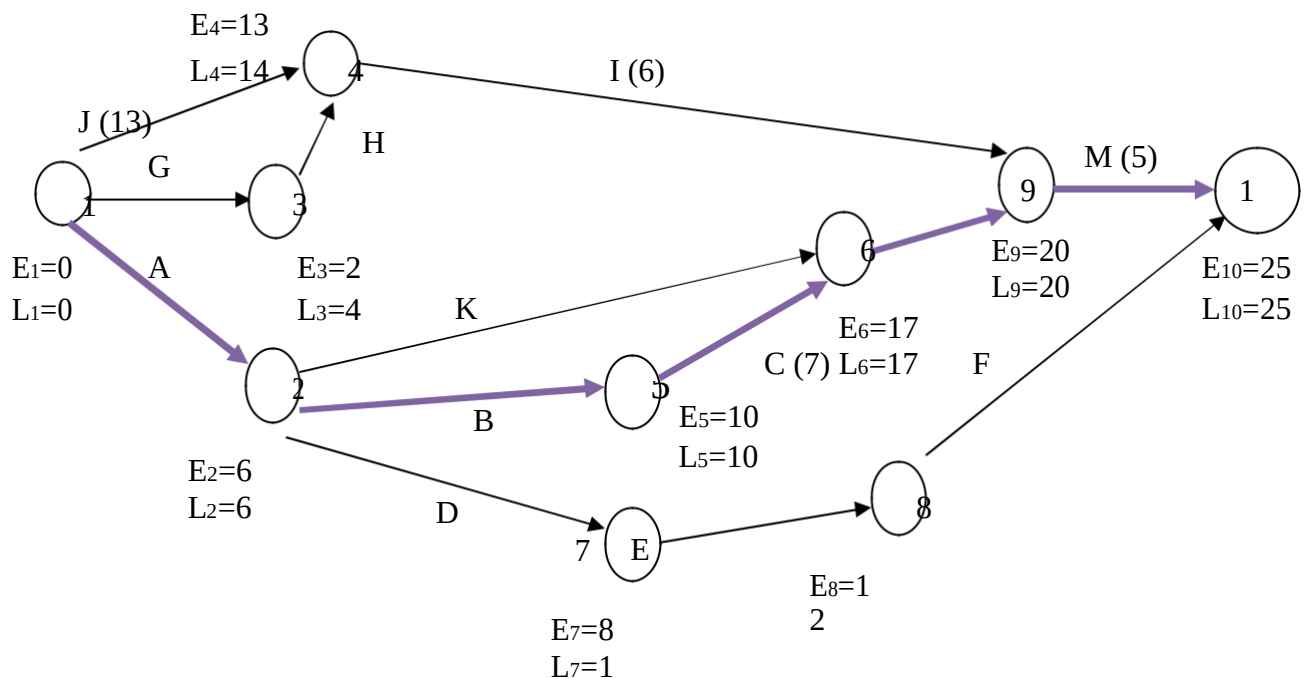
1. An established company has decided to add a new product to its line. It will buy the product from manufacturing concern; package it & sell it to a number of distributors selected on a geographical basis. Market research has identified the volume expected & the size of sales force required. The steps are given below;

<u>Activity</u>	<u>Description</u>	<u>Time (Week)</u>
A	Organize sales office	6
B	Hire Salesman	4
C	Train salesman	7
D	Select advertising agency	2
E	Plan advertising campaign	4
F	Conduct	10
G	Design package	2
H	Setup packaging facilities	10
I	Package initial Stocks	6
J	Order Stock from manufacture	13
K	Select distributors	9
L	Sell to distributors	3
M	Ship stocks	5

Required:

- Draw an arrow diagram
- Indicate the critical path; the critical activities & the total project time
- For each non-critical activity, find the total float

Solution: (a). the arrow diagram for the given project along with E-values and L-values is shown in the following diagram.



Forward pass method

Determine the earliest start time E_i and latest finish L_j for each event by processing as follows.

$$E_1=0$$

$$E_2= E_1+t_{1,2}=0+6$$

$$E_3= E_1+t_{1,3}=0+2=2$$

$$E_4=\text{Max} \{E_1+t_{1,4}; E_3+t_{3,4}\} \quad i=1, 3$$

$$E_5= E_2+t_{2,5}=6+4=10$$

$$E_6=\text{Max} \{E_i + t_{i,6}\} =\text{Max} \{E_2+t_{2,6}; E_5+t_{5,6}\}$$

$$\text{Max} \{0+13, 2+10\} =13$$

$$= 2,5$$

$$E_8= E_7+t_{7,8}=8+4=12$$

$$\text{Max} \{6+9; 10+7\} =17$$

$$E_{10}=\text{Max} \{E_i+t_{i,10}\}$$

$$E_7= E_2+t_{2,7}=6+2=8$$

$$i=8,9$$

$$E_9=\text{Max} \{E_i+t_{i,9}\} =\text{Max} \{E_4+t_{4,9}; E_6+t_{6,9}\}$$

$$=\text{Max} \{E_8+t_{8,10}; E_9+t_{9,10}\}$$

$$= 4,6$$

$$=\text{Max} \{12+10; 20+5\} =25$$

$$=\text{Max} \{13+6; 17+3\} =20$$

Backward pass method

$$L_{10}=E_{10}=25$$

$$L_9=L_{10} - t_{9,10}=25-5=20$$

$$L_8=L_{10} - t_{8,10}=25-10=15$$

$$L_7=L_8 - t_{7,8}=15-4=11$$

$$L_6- t_{6,9}=20-3=17$$

$$L_5=L_6 - t_{5,6}=17-7=10$$

$$L_4=L_9-t_{4,9}=20-6=14$$

$$L_3=L_4 - t_{3,4}=14-10=4$$

$$L_2=\min \{L_j- t_{2,j}\}$$

$$L_2=\min \{L_j- t_{1,j}\}$$

$$J=5, 6, 7$$

$$j=2, 3, 4$$

$$=\min \{L_5, - t_{2,5}; L_6, t_{2,6}; L_7-t_{2,7}\}$$

$$=\min \{L_2, - t_{1,2}; L_3, t_{1,3}; L_4-t_{1,4}\}$$

$$=\min \{10-4; 17-9; 11-2\} =6$$

$$=\min \{6- 6, 5; 4-2, 14-13\} =0$$

- B. The vertical path in the network diagram above has been shown by thick lines by joining all those events where two values E_i & L_j are equal. The critical path of the project is: 1-2-5-6-9-10 the critical activities are A, B, C, L&M. the total project time is 25 weeks
- C. For each non-critical activity, the total float is:

Activity (i,j)	Duration (t _{ij})	Earliest time		Last time		Float	
		Start (E _i)	Finish (E _i +t _{ij})	Start (L _j -t _{ij})	Finish (L _j)	Total [(L _j -t _{ij})-E _i]	Free (E _j -E _i)-t _{ij}
1-3	2	0	2	2	4	2	0
1-4	13	0	13	1	14	1	0
2-6	9	6	15	8	17	2	2
2-7	2	6	8	9	11	3	0
3-4	10	2	12	4	14	2	1
4-9	6	13	19	14	20	1	1
7-8	4	8	13	11	15	3	0
8-10	10	12	22	15	25	3	3

Remark:

- What does the critical path really mean?



The critical path shows the shortest time in which a project can be completed. Even though the critical path is the longest path, it represents the shortest time it takes to complete a project.



There can be more than one critical path on a project. Project managers should closely monitor performance of activities on the critical path to avoid late project completion. If there is more than one critical path, project managers must keep their eyes on all of them

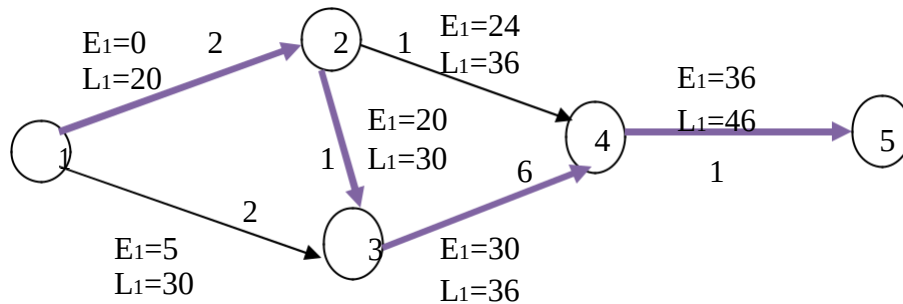
Example2. The following table gives the activities of a construction project and duration.

Activity	1-2	1-3	2-3	2-4	3-4	4-5
Duration (days)	20	25	10	12	6	10

- (a) Draw the network for the project.
- (b) Find the critical path

(c) Find the total, free and independent floats of each activity.

Solution: The first step is to draw the network and fix early start and early finish schedule and then late start-late finish schedule.



Activity	Total Slack	Slack	Independent Slack
1-2	0	0	0
1-3	5	5	5
2-3	0	0	0
2-4	4	4	4
3-4	0	0	0
4-5	0	0	0

To find the critical path, connect activities with 0 total slack and we get 1-2-3-4-5 as the critical path.

Check with alternate paths.

$$1-2-4-5 = 42$$

$$1-2-3-4-5 = 46 \text{ (critical path)} \quad 1-3-4-5 = 41$$

5.6 PERT Networks (Project Scheduling with Uncertain Activity Times)

Program Evaluation and Review Technique (PERT) - is a network analysis technique used to estimate project duration when there is a high degree of uncertainty about the individual activity duration estimates. PERT applies the critical path method to a weighted average duration estimate.

If the duration of activities in a project is uncertain than activity scheduling calculations are done by using the expected value of the durations. However, such expected duration estimated may not be given an accurate answer. Thus, rather than estimating directly the expected completion time of an activity, three values are considered in PERT.

The three-time estimates are:

1. Optimistic time (t_o or a): This is the shortest possible time to perform an activity, assuming that everything goes well.
2. Pessimistic time (t_p or b): This is the maximum (longest) time that is required to perform an activity, under extremely bad conditions. However, such conditions do not include acts of God like earthquakes, flood, etc.
3. Most likely time (t_m or m): This is the most realistic time to complete the activity. Statistically, it is the modal value of duration of the activity i.e. consider the time estimate under the normal situation.

The β distribution is not necessarily symmetric; the degree of skewness depends on the location of a and b . Thus, the range specified by the optimistic time (t_o) and pessimistic time (t_p) estimates is assumed to enclose every possible estimates of the duration of the activity. The most likely time (t_m) estimate may not coincide with the midpoint ($(t_o + t_p)/2$) and may occur to its left or its right as shown in the following figure.

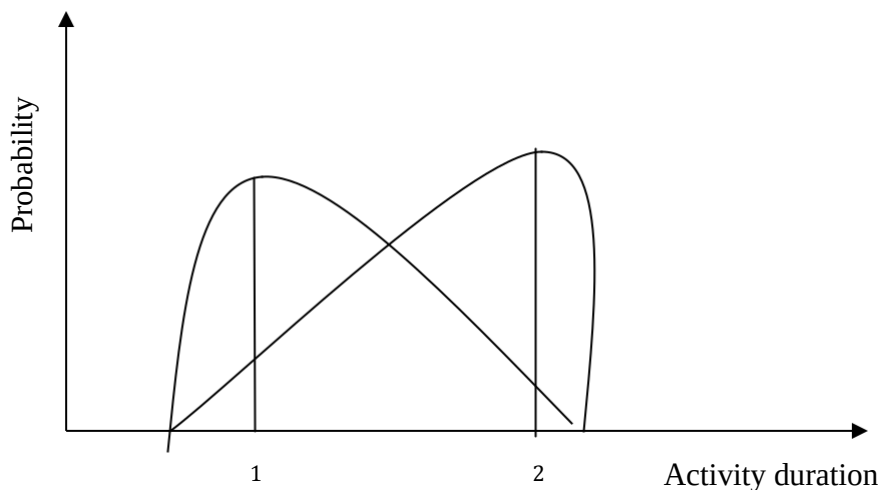


Fig an example of a β distribution

In beta distribution the midpoint $(a + b)/2$ is assumed to weight half as much as the most likely point (m). Thus, the expected or mean (t_e or μ) value of the activity duration can be approximated as the arithmetic mean of $(t_o + t_p)/2$ and $2t_m$. That is

Expected activity time (t_e) = $((t_o + t_p)/2 + 2t_m)/3 = ((t_o + 4t_m) + t_p)/6$

Estimating the variance is apparently based on an analogy to the normal distribution where 99% of the area under normal curve is within $\pm 3\sigma$ from the mean or fall within the range approximately 6 standard deviation in length, therefore t_o , t_p or range – enclose about 6 standard deviation of a symmetric distribution, thus, if σ denotes standard deviation, then

$$\begin{aligned} \text{Variance of activity time, } \sigma^2 &= [1/6 (t_o - t_p)^2] \\ \text{Standard deviation, } \sigma &= \sqrt{\text{Variance}} \end{aligned}$$

Estimation of project completion time

As we are expecting variability in the activity duration, the total project may not be completed exactly in time. Thus, it is necessary to calculate the probability of actually meeting the schedule time to the project as well as activities.

The probability distribution of time for completing an event can be approximated by the normal distribution due to central limit theorem. Thus, the probability of completing the project by schedule time (T) is given by

Prob ($Z = (T - t_e)/\sigma$) Where t_e = expected completion time of the project

Z = number so standard deviations the scheduled time or target date lies away from the mean or expected date.

In order to find out the probability of completing the project in some given time, we shall consider only the expected length of the critical path and its variance. The expected time of the project can be calculated by adding the expected time of each activity lying on the critical path can be known by adding variance of critical activity.

Example1.

1. A project is represented by a network shown below & has the following data

Task:	A	B	C	D	E	F	G	H	I
Optimistic time	5	18	26	16	15	6	7	7	3
Pessimistic time:	10	22	40	20	25	12	12	9	5
Most Likely time:	8	20	33	18	20	9	10	8	4

Determine the following

- Expected task time & their variance
- The earliest & latest expected times to reach each event
- The critical path
- The probability of an event occurring at the expected completion date if the original scheduled time of completing the project is 41.5 weeks.
- The duration of the project that will have 95% chance of being completed.

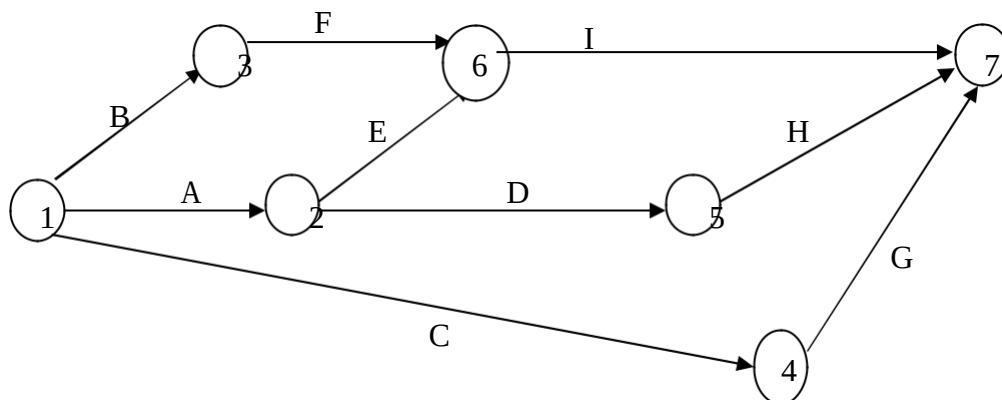
Solution (a) using the following formula, the expected activity time (t_e) or μ and variance (σ^2) is given in the following table.

$$= ((t_o + 4t_m) + t_p)/6 \text{ and } \sigma^2 = [1/6(t_p - t_o)]^2$$

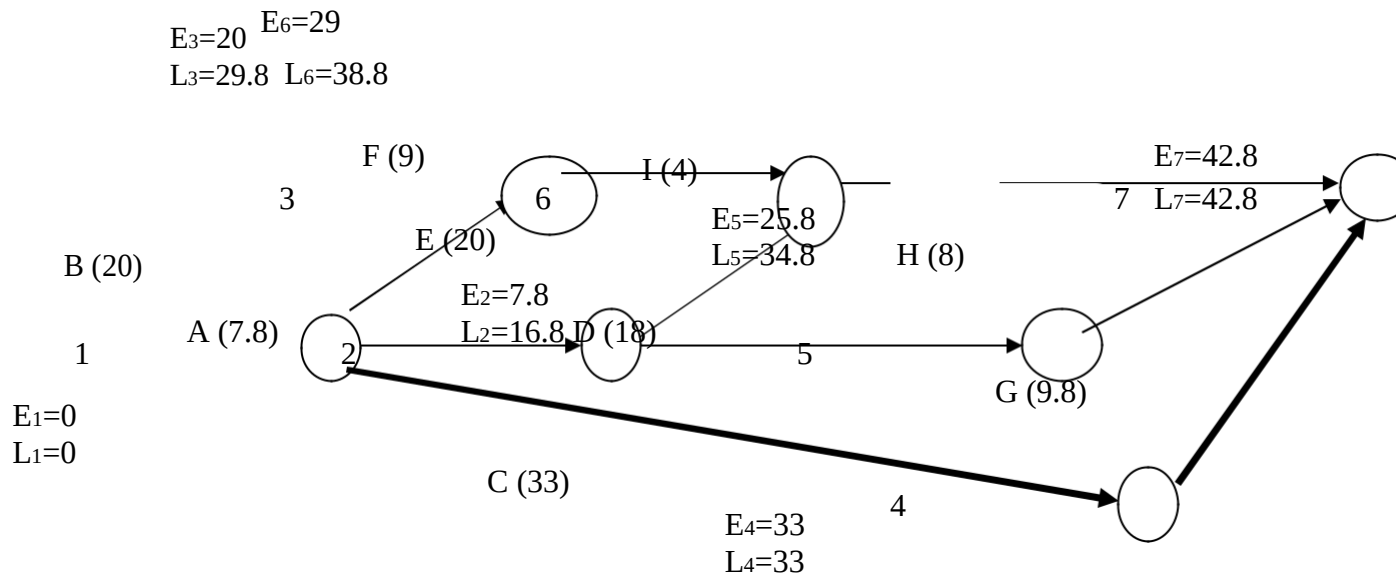
Activity	To	Tp	tm	$t_e = ((t_o + 4t_m) + t_p)/6$	$\sigma^2 = [1/6(t_p - t_o)]^2$
1-2	5	10	8	7.8	0.694
1-3	18	22	20	20.0	0.444
1-4	26	40	33	33.0	5.444
2-5	16	20	18	18.0	0.444
2-6	15	25	20	20.0	2.778
3-6	6	12	9	9.0	1.000
4-7	7	12	10	9.8	0.694
5-7	7	9	8	8.0	0.111
6-7	3	5	4	4.0	0.111

Solution:

(a) Expected task time & their variance



(b) The earliest and latest estimated time presented in the following graph.



(c) The critical path is shown by thick line in the above figure where E values and L values are the same. The critical path is 1-4-7 and the critical completion time for the project is 42.8 week.

(d) The last event 7 will occur only after 42.8 weeks. For this, we require only the duration of critical activity. This will help us in calculating the standard of the duration of the last event.

Expected length of critical path = $33+9.8=42.8$

Variance of critical path length = $5.429+0.694=6.123$

It is given that $T_s=41.5$, $T_e=42.8$ and the scheduled time is given by Prob($\leq$)

$= \sqrt{6.123} = 2.474$. Therefore, probability of meeting
 $\rightarrow \text{prob}(Z \leq -0.5) = 0.3$ from normal distribution

table

Thus, the probability that the project can be completed in less than or equal to 41.5 weeks is 0.30. In other words, the probability that the project will be delayed beyond 41.5 weeks is 70%

(e) Given that Prob ($Z \leq t_s - t_e / \sigma_e$) = 0.95

But $Z_{0.95}=1.64$, from normal distribution table. Thus
 $1.64 = (T_s - 42.8) / 2.474$ or $T_s = 1.64 \times 2.474 + 42.8 = 46.85$ weeks.



Self-check exercise 5.2

1. Discuss the difference between critical path method (CPM) and program evaluation and review techniques (PERT).

5.7 Project Time-Cost Trade-Off

The project completion time can be reduced by reducing (crashing) the normal completion time of critical activities. The reduction in normal time of completion will increase the total budget of the project. However, the decision maker always looks trade-off between total cost of project and total time required to complete it.

5.7.1 Project Crashing and Cost

Crashing is employed to reduce the project completion time by spending extra resource (cost). However, as shown in the figure beyond point A, cost increases more quickly when time is reduced. Similarly, beyond point B, the time increases while the cost decrease. Since for technical reasons time may not be reduced indefinitely, therefore, we call this limit as crash point. There is also a cost-efficient duration called *normal point*. Thus, extending the activity duration beyond normal point may increase costs.

For simplicity the relationship between normal time and cost as well as crash time and cost for an activity is assumed to be linear instead of being concave and/or discrete. Thus, the crash cost per unit of time can be estimated by computing the relative change in cost (cost slope) per unit change in time.

Remark: crashing an activity means performing it in the shortest technically possible time by allocating to it necessary resources.

Time cost trade-off procedure

The method of establishing Time cost trade-off procedure for the completion of a project can be summarized as follows:

Step1: Determine the normal project completion time and associated critical path for the following two cases.

- (a) When all critical activities are completed with their normal time. This provides the starting point for crashing analysis.
- (b) When all critical activities are crashed. This provides the stopping point for crashing analysis.

Step 2: Identify critical activity and compute the cost slope for each of these by using the relationship

$$\text{Cost slope} = (\text{crash cost} - \text{normal cost}) / (\text{crash time} - \text{normal time})$$

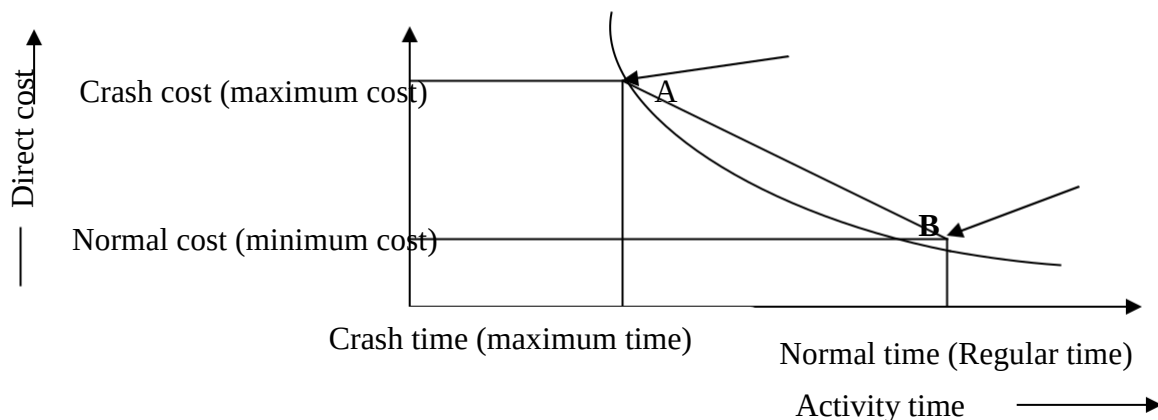
The values of cost slope for critical activities indicate the direct extra cost required to execute an activity per unit of time.

Step 3: For reducing total project completion time, identify and crash an activity time on the critical path with lower cost slope value to the point where

- a) Another path in the network become critical, or
- b) The activity has been crashing to its lowest time.

Step 4: If the critical path under crashing is still critical, return to step three. However, if due to crashing of an activity time in step 3, other path(s) in the network also become critical, then identify and crash the activities(s) on the critical path (s) with the minimum joint cost slope.

Steps 5: Terminate the procedure when each critical activity has been crashed to its lowest possible time. Determine total project cost (indirect cost-plus direct cost) corresponding to different project durations.



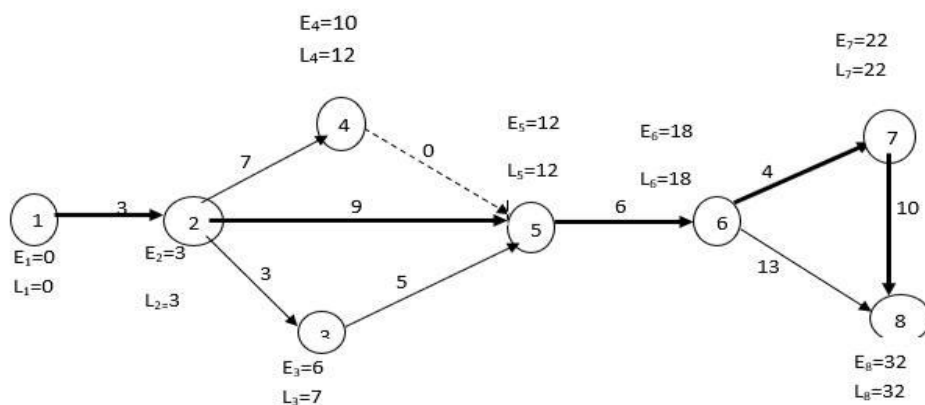
Example 1: The following table gives data on normal time, and cost and crash time and cost for a project

Activities	Normal		Crash	
	Time (weeks)	Cost (Birr)	Time (weeks)	Cost (Birr)
1-2	3	300	2	400
2-3	3	30	3	30
2-4	7	420	5	580
2-5	9	720	7	810
3-5	5	250	4	300
4-5	0	0	0	0
5-6	6	320	4	410
6-7	4	400	3	470
6-8	13	780	10	900
7-8	10	1000	9	1200

Required:

- Draw the network and identify the critical path with a double line.
- What are the normal project duration and associated cost?
- Find out the total float associated with each activity.
- Crash the relevant activities systematically and determine the optimal project completion time and cost.

Solution (a) the network for normal activity times indicates a project completion time of 32 weeks with the critical path: 1- 2- 5- 6 – 7 -8, as shown in the following figure.



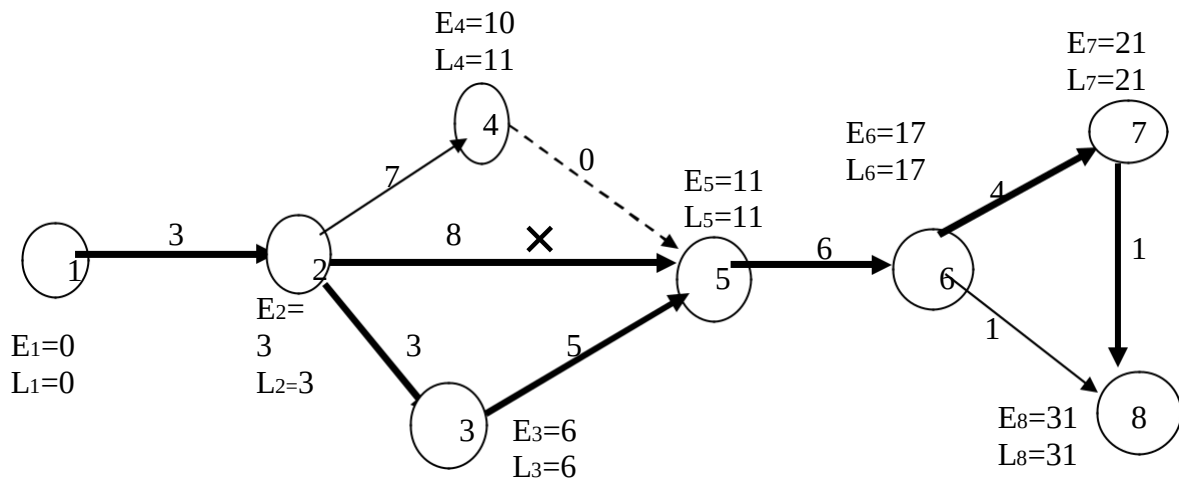
(b). normal project duration is 32 weeks and the associated cost is as follow:

Total cost=Direct normal cost-plus indirect cost for 32 weeks

(c). Calculation for total float associated with each activity are shown in the following table.

Activities	Total float
	(-) -
1-2	$(3-0)-3=0$
2-3	$(7-3)-3=1$
2-4	$(12-3)-7=2$
2-5	$(12-3)-9=0$
3-5	$(12-6)-5=1$
4-5	$(12-10)-0=2$
5-6	$(18-12)-6=0$
6-7	$(22-18)-4=0$
6-8	$(32-18)-13=1$
7-8	$(32-22)-10=0$

(d). for critical activities, crash cost slope is given.



The minimum value of crash cost per week is for activity 2-5 and 5-6. Hence, crash activity 2-5 by 2 days from 9 weeks to 7 weeks. But the time should be reduced by 1 week only otherwise another path 1-2-3-4-6-7-8 becomes a parallel path. Network as shown in the above figure is developed when it is observed that the new project time is 31 weeks and the critical paths are 1-2-5-6-7-8 and 1-2-3-5-6-7-8. The following table shows the crash cost per week (in birr).

Critical activities	Crash cost per week (Birr)
1-2	$\frac{400 - 300}{3 - 2} = 100$
2-5	$\frac{810 - 720}{9 - 7} = 45$
5-6	$\frac{410 - 320}{6 - 4} = 45$
6-7	$\frac{470 - 400}{4 - 3} = 70$
7-8	$\frac{1200 - 1000}{10 - 9} = 200$

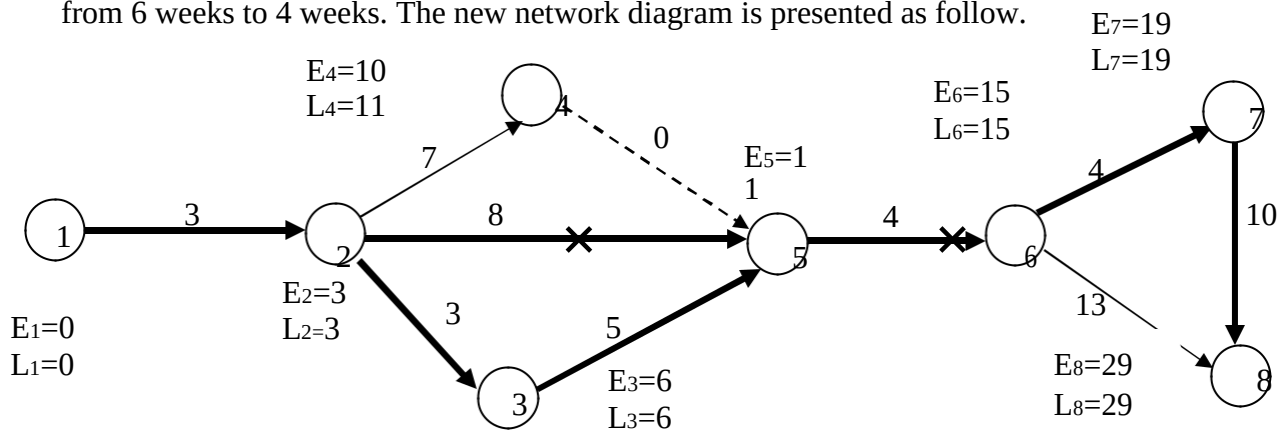
With crashing of activity 2-5 the new total cost involved can be calculated as follows;

$$= \text{Normal cost} + \text{Crash cost} = (4,220 + 1 \times 45) + 31 \times 50 = 4,265 + 1,550 = 5,815$$

Dear learners now with respect to the new network given, the new possibilities for crashing in the critical paths are listed in the following table.

<i>Critical activities</i>	Crash cost per week (Birr)
1-2	100
2-5	× (Crashed)
5-6	0 (no crashing is needed)
6-7	50
7-8	45
1-2	70
2-5	200

The minimum value of crashed cost slope is for activity 5-6. Hence, crashing it by 2 weeks from 6 weeks to 4 weeks. The new network diagram is presented as follow.



As you see in the above figure the critical paths are not changed because activity 5-6 is common for the two critical paths. But with crashing of activity 5-6 by 2 weeks, the new cost involved is:

$$h = 5 - 6 + \frac{29}{5805} = (4,220 + 1 \times 45 + 2 \times 45) + 29 \times 50 = 5805$$

With respect to the new network depicted above the new possibility for crashing in the critical paths are listed below.

<i>Critical activities</i>	<i>Crash cost per week (Birr)</i>
1-2	100
2-3	0 (no crashing is needed)
2-5	× (Crashed)
5-6	× (Crashed)
6-7	70
7-8	200

The further crashing of 6-7 activity time from 4 weeks to 3 weeks will results in increased direct cost than the gain due to reduction in project time. Consequently, here we must stop

further crashing. The optimal project duration is 29 weeks with associated cost of birr 5,805 as shown in the table below.

Project duration (weeks)	Crashing Activity And Weeks	Direct cost (Birr)			Indirect cost (Birr)	Total cost (Birr)
		Normal	Crashing	Total		
32	--	4,220	--	4,220	$32 \times 50 = 1,600$	5,820
31	2-5(1)	4,220	$1 \times 45 = 45$	4,265	$31 \times 50 = 1,550$	5,815
29	5-6(2)	4,220	$45 + 45 \times 2 = 135$	4,355	$29 \times 50 = 1,450$	5,805
28	6-7(1)	4,220	$135 + 1 \times 70 = 205$	4,425	$28 \times 50 = 1,400$	5,825



Self-check exercise 5.3

1. The owner of a chain of fast food restaurants is considering a new computer system for accounting and inventory control. A computer company sent the following information about the computer system installation.

Activity	Description	Predecessors	Expected time (days)		
			Optimistic	Most likely	Pessimistic
A	Select the computer model	-	4	6	8
B	Design input output system	A	5	7	15
C	Deign monitoring system	A	4	8	12
D	Assemble computer hard ware	B	15	20	25
E	Develop the main programs	B	10	18	26
F	Develop input-output routines	C	8	9	16
G	Create database	E	4	8	12
H	Install the system	D, F	1	2	3
I	Test and implement	G, H	6	7	8

Required:

- A. Construct an arrow diagram for this problem
- B. Determine the critical path and compute the expected completion time
- C. Determine the probability of completing the project in 55 days

CHAPTER SUMMERY

Network model is a technique used for planning, and scheduling large projects in the field of construction, maintenance, fabrication, purchasing computer system, distribution, transportation, facility allocation, and many others.

A project is a set of proposals for the investment of resources in to a clearly identified set of actions that are expected to produce future benefits that exceeds costs.

There are two basic planning and control technique that utilize network to complete predetermined Project or schedule. These are Program Evaluation and Review Technique (PERT) and Critical Path Method.

Program Evaluation and Review Technique (PERT) - is a network analysis technique used to estimate project duration when there is a high degree of uncertainty about the individual activity duration estimates. PERT applies the critical path method to a weighted average duration estimate.

If the duration of activities in a project is uncertain than activity scheduling calculations are done by using the expected value of the durations. However, such expected duration estimated may not be given an accurate answer. Thus, rather than estimating directly the expected completion time of an activity, three values are considered in PERT i.e Optimistic time (t_o or a), Pessimistic time (t_p or b) and Most likely time (t_m or m).

Critical path analysis helps to estimate the total project duration & to assign starting & finishing times to all activities involved in the project. This helps in checking actual progress against the scheduled duration of the project.

UNIT SIX

GAME THEORY

6.1 Introduction

The concepts of game theory provide a common language to formulate structure, analyze and eventually understand different strategical scenarios. Generally, game theory investigates conflict situations, the interaction between the agents and their decisions. A game in the sense of game theory is given by a (mostly finite) number of players, who interact according to given rules. Those players might be individuals, groups, companies, associations and so on. Their interactions will have an impact on each of the players and on the whole group of players, i.e. they are interdependent. To be more precise: A game is described by a set of players and their possibilities to play the game according to the rules, i.e. their set of strategies.

The subjects of game theory are situations, where the result for a player does not only depend on his own decisions, but also on the behavior of the other players. Game theory has its historical origin in 1928. By analyzing parlor games, John von Neumann realized very quickly the practicability of his approaches for the analysis of economic problems. Despite the deep insights he gained from game theory's applications to economics, von Neumann was mostly interested in applying his methods to politics and warfare, perhaps descending from his favorite childhood game, Kriegspiel, a chess-like military simulation. He used his methods to model the Cold War interaction between the U.S. and the USSR, picturing them as two players in a zero-sum game.

6.2 Objectives of The Chapter

At the end of this chapter, you should be able to:

-  Understand the concept of game theory
-  Aware the concept of two-person zero-sum game
-  Make a decision under the condition of pure strategy (game with saddle point)
-  Understand the concept of mixed strategy (game without saddle point) differentiate it with pure strategy.
-  Define and understand the rule of dominance

6.3 The Concept of Game Theory

As far as games have defined in mathematical or numerical values, let us describe the ingredients and concepts of the game.

Game: A game is a formal description of a strategic situation.

Game theory: Game theory is the formal study of decision-making where several players must make choices that potentially affect the interests of the other players.

Rules: Mathematical games have strict rules. They specify what is allowed and what isn't. Though many real-world games allow for discovering new moves or ways to act, games that can be analyzed mathematically have a rigid set of possible moves, usually all known in advance.

Outcomes and payoffs: Children (and grown-ups too) play games for hours for fun. Mathematical games may have many possible outcomes, each producing payoffs for the players. The payoffs may be monetary, or they may express satisfaction. You want to win the game.

Uncertainty of the Outcome: A mathematical game is “thrilling” in that its outcome cannot be predicted in advance. Since its rules are fixed, this implies that a game must either contain some random elements or have more than one player.

Decision making: A game with no decisions might be boring, at least for the mind. Running a 100-meter race does not require mathematical skills, only fast legs. However, most sport games also involve decisions, and can therefore at least partly be analyzed by game theory.

No cheating: In real-life games cheating is possible. Cheating means; not playing by the rules. If, when your chess opponent is distracted, you take your queen and put it on a better square; you are cheating, as in poker, when you exchange an 8 in your hand with an ace in your sleeve. Game theory doesn't even acknowledge the existence of cheating.

Common knowledge: A fact is common knowledge if all players know it, and know that they all know it, and so on. The structure of the game is often assumed to be common knowledge among the players.

Dominating strategy: A strategy dominates another strategy of a player if it always gives a better payoff to that player, regardless of what the other players are doing. It weakly dominates the other strategy if it is always at least as good.

Extensive game: An extensive game (or extensive form game) describes with a tree how a game is played. It depicts the order in which players make moves, and the information each player has at each decision point.

Mixed strategy: A mixed strategy is an active randomization, with given probabilities that determine the player's decision. As a special case, a mixed strategy can be the deterministic choice of one of the given pure strategies.

Nash equilibrium: Nash equilibrium, also called strategic equilibrium, is a list of strategies, one for each player, which has the property that no player can unilaterally change his strategy and get a better payoff.

Payoff: A payoff is a number, also called utility that reflects the desirability of an outcome to the player, for whatever reason. When the outcome is random, payoffs are usually weighted with their probabilities. The expected payoff incorporates the player's attitude towards risk.

Perfect information: A game has perfect information when at any point in time only one player makes a move, and knows all the actions that have been made until then.

Player: A player is an agent who makes decisions in a game.

Rationality: A player is said to be rational if he seeks to play in a manner which maximizes his own

Strategic form: A game in strategic form, also called normal form, is a compact representation of a game in which players simultaneously choose their strategies. The resulting payoffs are presented in a table with a cell for each strategy combination.

Strategy: In a game in strategic form, a strategy is one of the given possible actions of a player. In an extensive game, a strategy is a complete plan of choices, one for each decision point of the player.

Zero-sum game: A game is said to be zero-sum if for any outcome, the sum of the payoffs to all players is zero. In a two-player zero-sum game, one player's gain is the other player's loss, so their interests are diametrically opposed.



Self-check exercise 6.1

4. What is the importance of game theory in the area of business?
5. What is dominant strategy?

6.4 Two Person Zero-Sum Game

A game is called zero-sum if the sum of payoffs equals zero for any outcome. That means that the winnings of the winning players are paid by the losses of the losing players. For zero-sum two-player games, the bi-matrix representation of the game can be simplified: the payoff of the second player doesn't have to be displayed, since it is the negative of the payoff of the first player.

Example 1: Assume we are playing ROCK-SCISSORS-PAPER for one dollar. Then the payoff matrix is;

	Rock	Scissor	Paper
Rock	0	1	-1
Scissor	-1	0	1
Paper	1	-1	0

The first cell says "0", which stands for "0; 0" a payoff of 0 for both players. The second cell entry of "1" should be read as "1; -1", a payoff of 1 for Ann which has to be paid by Beth, therefore a payoff of -1 for Beth.

6.5 Pure strategies (Game with saddle point)

In a zero-sum game, the pure strategies of two players constitute a saddle point if the corresponding entry of the payoff matrix is simultaneously a maximum of row minima and a minimum of column maxima. This decision-making is referred to as the **maximin-minimax principle** to obtain the best possible selection of a strategy for the players.

In a pay-off matrix, the minimum value in each row represents the minimum gain for player A. Player A will select the strategy that gives him the maximum gain among the row minimum values. The selection of strategy by player A is based on maximin principle. Similarly, the same pay-off is a loss for player B. The maximum value in each column represents the maximum loss for Player B. Player B will select the strategy that gives him the minimum loss among the column maximum values.

The selection of strategy by player B is based on minimax principle. If the maximin value is equal to minimax value, the game has a saddle point (i.e., equilibrium point). Thus, the strategy selected by player A and player B are optimal.

The game theory for saddle point problem is shown below

Example 1: Consider the example to solve the game whose pay-off matrix is given in the following table as follows:

Game Problem

		Player B	
		1	2
Player A	1	1	3
	2	-1	6

The game is worked out using maximin procedure for player A and minimax for player B. Find the smallest value in each row and select the largest payoff of these values. Next, find the largest value in each column and select the smallest of these numbers. The procedure is shown in the following table.

Maximin-Minimax Procedure

		Player B		
		1	2	Row Min
Player A →	1	1	3	1
	2	-1	6	-1
Col Max →		1	6	

If maximum value in row is equal to the minimum value in column, then saddle (equilibrium) point exists.

Maximin = Minimax (1=1)

Therefore, there is a saddle point and player A plays strategy A1 and player B plays strategy B1 and the value of the game is 1.

Example 2: Solve the game with the pay-off matrix for player A as given in table.

Game Problem

		Player B		
		B ₁	B ₂	B ₃
Player A	A ₁	-4	0	4
	A ₂	1	4	2
	A ₃	-1	5	-3

Solution: Find the smallest element in rows and largest elements in columns.

Maximin-Minimax Procedure

		Player B			
		B ₁	B ₂	B ₃	Row min
Player A	A ₁	-4	0	4	-4
	A ₂	1	4	2	(1)
	A ₃	-1	5	-3	-3
→ → Column Max		(1)	5	4	

Select the largest element in row and smallest element in column. Check for the minimax criterion;

Maximin = Minimax (1=1)

Therefore, there is a saddle point and it is a pure strategy.

Optimum strategy; player A plays strategy A2 and player B plays strategy B1 then, the value of the game is 1.

Example 3: Check whether the following game is given in Table, determinable and fair. **Game Problem**

		Player B	
		1	2
Player A	1	7	0
	2	0	8

Solution: The game is solved using maximin-minimax criteria as shown in Table. **Maximin-minimax Procedure**

		Player B		
		1	2	Row Min
Player A	1	7	0	0
	2	0	8	0
Column Max		7	8	

The game is strictly neither determinable nor fair.

Example 4: Solve the game with the payoff matrix given in table and determine the best strategies for the companies A and B and find the value of the game for them.

Game Problem

		Company B		
		1	2	3
Company A	1	2	4	2
	2	1	-5	-4
	3	2	6	-2

Solution: The matrix is solved using maximin criteria, as shown in table below. **Maximin Procedure**

		Company B			
		1	2	3	Row Min
Company A	1	2	4	2	2
	2	1	-5	-4	-5
	3	2	6	-2	-2
Column Max		2	6	2	

$$\text{Max Min} = \text{Min Max}$$

$$2 = 2$$

Therefore, there is a saddle point and the optimum strategy for company A is A1 and for company B is B1 or B3.

6.6 Mixed Strategies: Game without Saddle Point

In certain cases, no pure strategy solutions exist for the game. In other words, saddle point does not exist. In all such game, both players may adopt an optimal blend of the strategies called Mixed Strategy to find a saddle point. The optimal mix for each player may be determined by assigning each strategy a probability of it being chosen. Thus, these mixed strategies are probabilistic combinations of available better strategies and these games hence called Probabilistic games. The probabilistic mixed strategy games without saddle points are commonly solved by any of the following methods.

No	Methods	Application to
1	Analytical method	2x2 games
2	Graphics method	2x2, mx2 and 2xn
3	Simplex method	2x2, mx2, 2xn and mxn games

(a) Analytical Method: A 2 x 2 payoff matrix where there is no saddle point can be solved by analytical method. Given the matrix;

$$\begin{matrix} & a_{11} & a_{12} \\ a_{21} & a_{21} & a_{22} \end{matrix}$$

$$\begin{matrix} a_{21} & a_{22} \end{matrix}$$

value of the game is;
$$\frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

With the coordinates,

$$x = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$y = \frac{a_{11} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$x = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$y = \frac{a_{11} - a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

Alternative procedure to solve the strategy

- ✓ Find the difference of two numbers in column 1 and enter the resultant under column 2. Neglect the negative sign if it occurs.
- ✓ Find the difference of two numbers in column 2 and enter the resultant under column 1. Neglect the negative sign if it occurs.
- ✓ Repeat the same procedure for the two rows.

Example 5;

It is a 2 x 2 matrix and no saddle point exists. We can solve by analytical method

A	B	
	5	1
	3	4

$$\begin{aligned}
 & \begin{matrix} & & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{matrix} \\
 & \begin{matrix} 3 & 2 \end{matrix} \\
 & = \frac{11 \cdot 22 - 21 \cdot 12}{5(4) - 3(1)} = \frac{20 - 3}{-9 - 4}
 \end{aligned}$$

$$V=17/5 \quad S_A = (x_1, x_2) = (1/5, 4/5) \quad S_B = (y_1, y_2) = (3/5, 2/5).$$

Example 2: Solve the given matrix;

$$\begin{matrix} & B \\ A & \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \end{matrix}$$

Solution

It is a 2 x 2 matrix and no saddle point exists. We can solve by analytical method

$$\begin{array}{c}
 \text{A} \quad \begin{array}{cc} & \text{B} \\ \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 3 \end{array} \\ \begin{array}{c} 1 \\ 3 \end{array} & \end{array} \\
 \\
 V = \frac{a_{11}a_{22} - a_{21}a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{0 - 1}{2 + 2}
 \end{array}$$

$$V = -1/4$$

$$S_A = (x_1, x_2) = (1/4, 3/4)$$

$$S_B = (y_1, y_2) = (1/4, 3/4)$$

(b) Graphical method

The graphical method is used to solve the games whose payoff matrix has

- ✓ Two rows and n columns (2 x n)
- ✓ m rows and two columns (m x 2)



Draw two vertical axes 1 unit apart. The two lines are $x_1 = 0$, $x_1 = 1$



Take the points of the first row in the payoff matrix on the vertical line $x_1 = 1$ and the points of the second row in the payoff matrix on the vertical line $x_1 = 0$.



The point a_{1j} on axis $x_1 = 1$ is then joined to the point a_{2j} on the axis $x_1 = 0$ to give a straight line.



Draw 'n' straight lines for $j=1, 2, \dots, n$ and determine the highest point of the lower envelope obtained. This will be the **maximin point**.



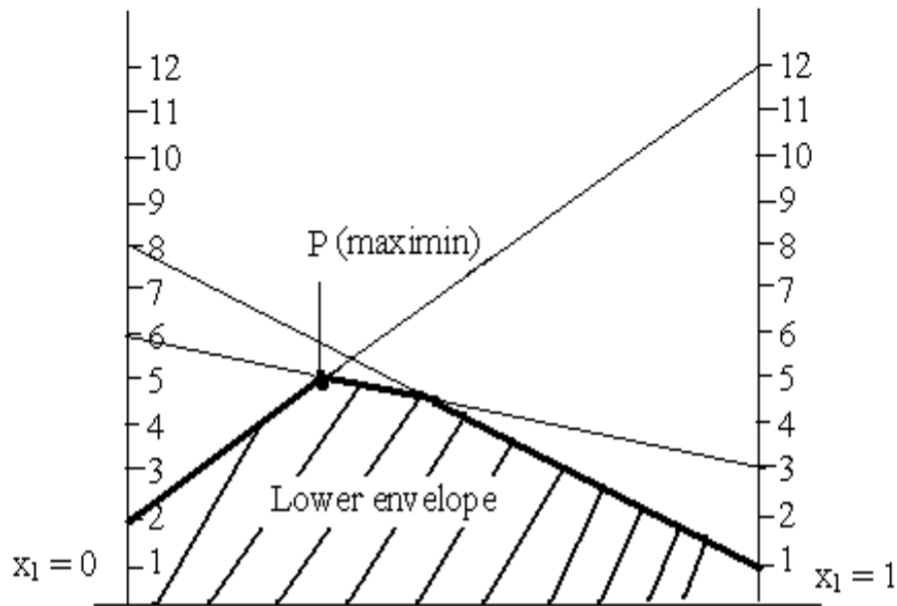
The two or more lines passing through the maximin point determines the required 2 x 2 payoff matrix. This in turn gives the optimum solution by making use of analytical method.

Example 6;

Solve by graphical method

	B1	B2	B3
A1	1	3	12
A2	8	6	2

Solution



	B2	B3	
A1	3	12	4
A2	6	2	9
	10	3	

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{6 - 72}{5 - 18}$$

$$V = 66/13$$

$$SA = (4/13, 9/13)$$

$$SB = (0, 10/13, 3/13)$$

6.7 The Rule of Dominance

The principle of dominance in Game Theory (also known as dominant strategy **or** dominance method) states that if one strategy of a player dominates over the other strategy in all conditions then, the later strategy can be ignored.

A strategy dominates over the other only if it is preferable over other in all conditions. The concept of dominance is especially useful for the evaluation of **two-person zero-sum games** where a saddle point does not exist.

The rules of dominance are used to reduce the size of the payoff matrix. These rules help in deleting certain rows and/or columns of the payoff matrix, which are of lower priority to at least one of the remaining rows, and/or columns in terms of payoffs to both the players. Rows / columns once deleted will never be used for determining the optimal strategy for both the players. This concept of domination is very usefully employed in simplifying the two – person zero sum games without saddle point. In general, the following rules are used to reduce the size of payoff matrix.

Rules (principles) of dominance

-  Rule 1: If all the elements in a row (say i^{th} row) of a payoff matrix are less than or equal to the corresponding elements of the other row (say j^{th} row) then, the player A will never choose the i^{th} strategy and we say i^{th} strategy is dominated by j^{th} strategy and will delete the i^{th} row.
-  Rule 2: If all the elements in a column (say r^{th} column) of a payoff matrix are greater than or equal to the corresponding elements of the other column (say s^{th} column) then, the player B will never choose the r^{th} strategy or in the other words the r^{th} strategy is dominated by the s^{th} strategy and we delete r^{th} column .
-  Rule 3: A pure strategy may be dominated if it is inferior to average of two or more other pure strategies. Now, consider some simple examples.

Example 7: Given the payoff matrix for player A, obtain the optimum strategies for both the players and determine the value of the game.

A	B		
	6	-3	7
	-3	0	4

Solution

A	B		
	B1	B2	B3
A1	6	-3	7
A2	-3	0	4

When A chooses strategy A1 or A2, B will never go to strategy B3. Hence strategy B3 is redundant.

A	B			Raw minima
	B1	B2	B3	-3
A1	6	-3	7	-3
A2	-3	0	4	
Column maxima	6	0		

Minimax (=0), maximin (= -3). Hence, this is not a pure strategy with a saddle point. Let the probability of mixed strategy of A for choosing A1 and A2 strategies are p_1 and $1 - p_1$ respectively. We get $6p_1 - 3(1 - p_1) = -3p_1 + 0(1 - p_1)$ or $p_1 = 1/4$. Again, q_1 and $1 - q_1$ being probabilities of strategy B, we get $6q_1 - 3(1 - q_1) = -3q_1 + 0(1 - q_1)$ or $q_1 = 1/4$.

Hence, optimum strategies for players A and B will be as follows:

$$S_A = (A_1=1/4 \text{ \& } A_2=3/4)$$

$$S_B = (B_1=1/4, B_2=3/4 \text{ \& } B_3=0)$$

$$\text{Expected value of the game} = q_1(6p_1 - 3(1 - p_1)) + (1 - q_1)(-3p_1 + 0(1 - p_1)) = 3/4.$$



Self-check exercise 6.2

- Find the equilibrium point of the following game

		X	Y
Players	A	10, 10	15, 5
	B	5, 16	12, 11

CHAPTER SUMMARY

The subjects of game theory are situations, where the result for a player does not only depend on his own decisions, but also on the behavior of the other players. A game is said to be zero-sum if for any outcome, the sum of the payoffs to all players is zero. In a two-player zero-sum game, one player's gain is the other player's loss, so their interests are diametrically opposed.

In certain cases, no pure strategy solutions exist for the game. In other words, saddle point does not exist. In all such game, both players may adopt an optimal blend of the strategies called Mixed Strategy to find a saddle point. The optimal mix for each player may be determined by assigning each strategy a probability of it being chosen.

The principle of dominance in Game Theory (also known as dominant strategy **or** dominance method) states that if one strategy of a player dominates over the other strategy in all conditions then, the later strategy can be ignored. A strategy dominates over the other only if it is preferable over other in all conditions.