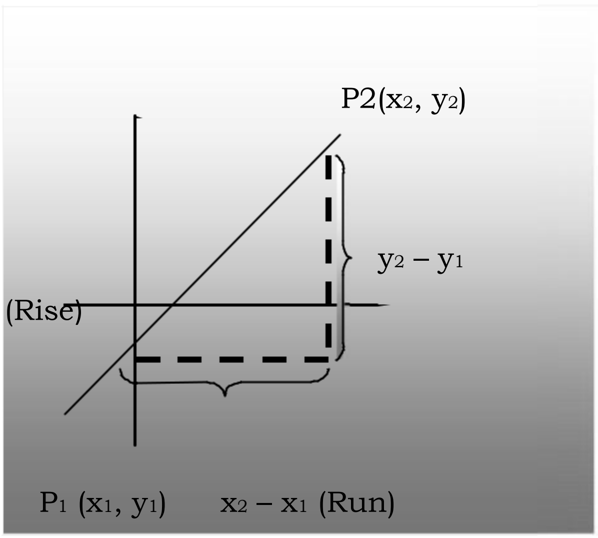
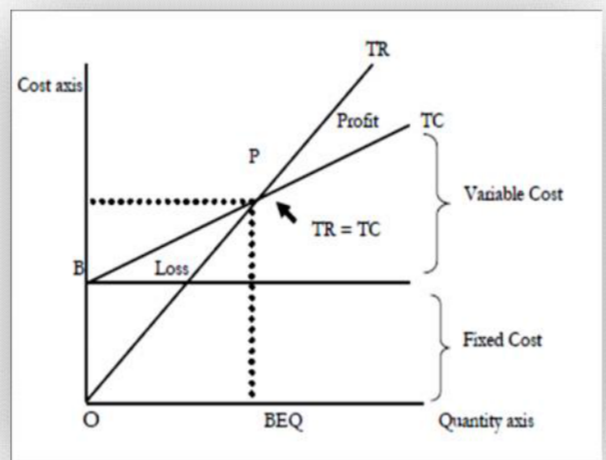
MATHEMATICS FOR MANAGEMENT

(MGMT. 2111)





ARBAMINCH UNIVERSITY

COLLEGE OF BUSINESS & ECONOMICS

DEPARTEMENT OF MANAGEMENT

MODULE

Written By: Gemechu Nemera (PhD)

Merga Abera (MBA)

Revised By: Daniel Tekle (MBA)

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COURSE DESCRIPTION

Managerial decision making is a process by which management when faced with a problem chooses a specific course of action from a set of possible options. A business manager, in making choice attempts to choose the most effective course of action under the given circumstance for attaining the goal of the organization. Traditionally, decision making has been considered purely as an art; however, common sense may be misleading and snap judgments may have painful implications. As such, the present day management need not rely on trial and error approach only, rather various types of mathematical models need to be employed in modern decision making.

The essence of the course mathematics for management is then, to equip students of management or decision making schools with quantitative decision making skills, which may never be a substitute for intuition, but assist same in the process of making choice. The course is designed to include few refreshment topics on mathematical theories and much application based topics so as to achieve the aforementioned targets.

Dear trainees, in this particular module, we would like to introduce linear equations and functions with greater emphasis on their application, in such cases as linear cost, linear revenue, and linear profit functions to you. The breakeven analysis for both manufacturing and merchandising firms are also the focus of this module. In this section, there is possibility for you to appreciate how demand and supply analysis is handled. The use of systems of linear equations and inequalities would also be addressed. The other important part of this fist module is "Matrix Algebra and its application in business problems". The linear programming models, mathematics of finance and introduction to calculus are among the integral part of mathematics for management that is going to make up "Part II" of the module.

Mathematics for management is a prerequisite to many quantitative courses such as statistics for management, and operations research. So it is here, where you can build a good ground for your future success in all those courses.

Finally, we would like to remind you that better understanding and application of mathematics in management rests on the number of exercises you solve. That is why we are providing you with the maximum possible number of exercises possible.. All the given exercise are worth attempting. For that matter you need not limit your self to the exercises on the module, rather you have to try any way out to solve as much number of business problems as possible.

For self- check exercises do not try to look at the answers before you do your level best to find the solution by yourself. Dear trainees, do not hesitate to forward any comment on the weaknesses and flaws of the module for if has great deal to do with shaping the next module on linear programming, mathematics of finance, and calculus.

ICONS USED IN THE MODULE

Brainstorming Questions

Self Check Exercises

Key for Self Check Exercises

Module Objectives

CHAPTER ONE

LINEAR EQUATIONS AND THEIR INTERPRETATIVE APPLICATIONS

1.1 INTRODUCTION

The Equation concept is one of the most important ideas in mathematics. The study of mathematics beyond the elementary level requires a firm understanding of a basic list of elementary equations and functions, their properties and their graphs.

And its application is useful in areas of business and management.

Dear students, in this particular unit we are going to emphasize major features of linear equations and inequalities, such as the slope of a straight line and its interpretation, intercepts, and equations of a line. Besides this, solutions of systems of linear equations and inequalities are discussed. Business applications of linear equations such as linear cost functions, linear revenue functions, linear demand and supply functions and linear profit functions are dealt with in our attempt to relate your knowledge of linear equations and functions to the field of your study. Business problems do not necessarily involve linear equations rather it is not unusual to see problems having inequality features as well. Due to this fact, we need to have a look at some features of linear inequalities and its application in business sense.

1.2 OBJECTIVES OF THE CHAPTER

After completion of this chapter, trainees have to be able to

- Understand the different components of linear equations and functions
- Determine the equation of a straight line, given slope and Y – intercept, slope and one pint on the line, given two points on a given line.
- Apply the concept of linear equation in such business areas as; Cost, revenue, profit, demand and supply functions
- Carryout breakeven analysis for merchandisers and manufactures.

1.3 LINEAR EQUATIONS

BRAIN STORMING QUESTION

What is linear equation?

Equation – Is a mathematical expression that indicates two quantities or algebraic expressions are equal.

Example: y = 2x + 3

3x – 2y = 10

y= 5X – 20

Algebraic expressions- Are a mathematical statement indicating that numerical quantities are linked by mathematical operations.

Example: x + 2

Linear equations: Are equations whose terms (parts is separated by plus, minus, and equal signs) are a constant or a constant times one variable to the power of one.

Example:

3x – 5y = 20

y= 2X

y = 3

All the above are ***linear equations.***

The equation: 3x – 5y = 20 is linear equation, because;

The term 3x is a constant 3 times one variable (x) to the first power (degree one).

The term -5y is a constant 5 times one variable (y) to the first power.

The constant number is 20.

The equation: $y = 2x$ is a ***linear equation*** because;

The term $2x$ is a constant 2 times one variable (x) to the first power

The equation: $y = 3$ is a ***linear equation*** because;

The term 3 is a constant.

However,

$3xy – 2y$,

$y = 2x^2$,

$Y = x^3 + 3$ Are ***not linear equations***.

The equation: $3xy – 2y = 7$ is ***not linear equation,*** because;

The term $3xy$ is a constant 3 times one variable

The equation: $y = 2x^2$ is ***not linear equation,*** because;

The term $2x^2$ is a constant times one variable with degree two.

Dear students, If you answered the first question that “Linear equations are equations in which all variables are raised to the fist power and no cross product of variables occurs”, you are correct.

In Linear equation, no x^2 terms, no x/y and no xy terms are allowed and the slope of linear equations is constant throughout the line.

1.4 Linear Function

RAIN STORMING QUESTION

What is linear function?

____

____

A function: Is a method that produces a correspondence between two sets of elements such that to each element in the first set, there corresponds one and only one element in the second set. The first set is called domain (independent variable) and the set of corresponding elements in the second set is called range (dependent variable). A ***linear function*** is always represented by a straight line as shown in the following figure. Whenever we come across situations in which both the variables increase or decrease proportionally we use linear functions. This implies that the slope (gradient) of the linear function is one and the same at all points.

Generally;

Linear functions are the relationship between dependent and independent variables.

A ***linear function*** is always represented by a straight line

For each value of x there is one and only one corresponding value of y.

Y is written in terms of x.

Y is a function of x.

The general notion of a linear function is expressed in a form;

$$\mathbf{y = mx + b}$$

Where;

m = slope

b = The Y- intercept

y = Dependent variable and

x = Independent variable

BRAIN STORMING QUESTION

Why do you study linear equations?

Here are some reasons for studying linear equations. Would you mention some other reasons apart from what are mentioned below?

We can represent /approximate many of the real world problems in linear equation.

The analysis of linear mathematics is easier than that of nonlinear relations

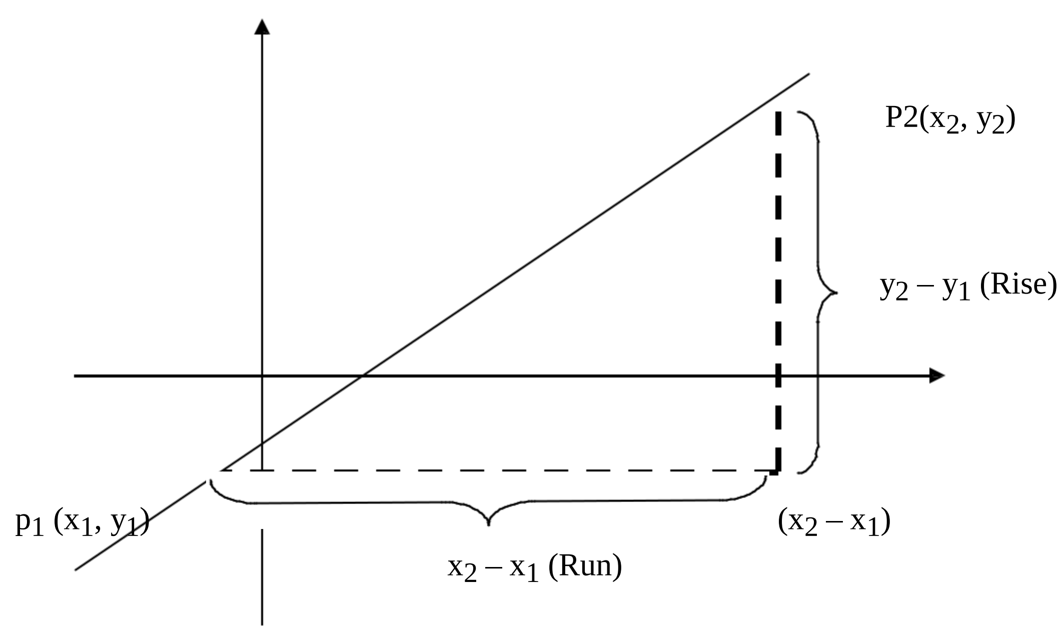
The techniques we use in the analysis of nonlinear relationship are the extension of those used in linear relationship.

1.5COMPONENTS OF LINEAR EQUATIONS

1.5.1 Slopes (Gradient) – ***is a measure of the inclination of the line to the horizontal axis. It refers the steepness of a line segment*****.** The slope of the line measures the change in the dependent variable, Y, due to change in the independent variable, X. It indicates both steepness and direction of change.

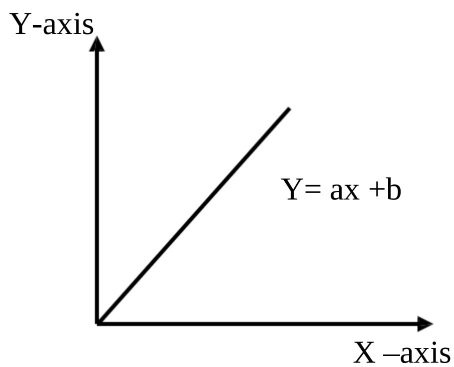
The slope of m of a non vertical line which passes through the points p1 (x1, y1) and p2(x2, y2) and $x_1 \neq x_2$ is defined as the ratio of vertical increase (increase in y) to horizontal increase (increase in x).

$$Slope = \frac{\Delta y}{\Delta x} = \frac{y1 - y2}{x1 - x2} = \frac{y2 - y1}{x2 - x1} = \frac{Rise}{Run}$$



Generally, the slope of a line can be negative, positive, zero or undefined. The following graphical presentations and detail explanation describe this phenomenon.

Positively sloped lines – it shows that there is a direct relationship between the dependent and the independent variables. Therefore, as X increases, Y also increases and as X decreases, Y also decreases.

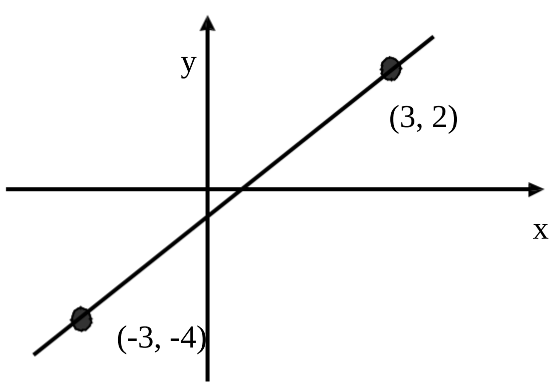


Example 1: Sketch a line through (-3, -4) and (3, 2) pair of points and find the slope of the line.

Let point (x_1, y_1) be equal to (-3, -4) and (x_2, y_2) be equal to (3, 2). Then, using the formula for calculating the slope of a straight line as;

$$Slope = m = \frac{y2 - y1}{\quad}, x1 \neq x2$$

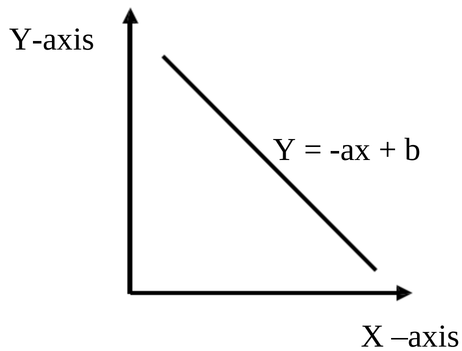
$$Slope = m = \frac{2 - (-4)}{3 - (-3)} = \frac{6}{6} = 1$$



Interpretation

For every one unit change of x is accompanied by 1 unit change in y or the line rise (increase by 1 unit as x increase by 1 unit).

Negatively sloped lines- shows there is inverse relationship between the dependent and independent variables. Hence, as X increases, Y decreases and as X decreases, Y increases. The relationship between the variables can be depicted using graphical method as follows:

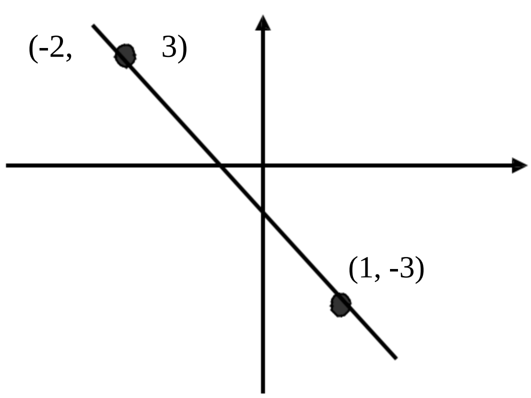


Example 2: Sketch a line through (-2, 3) and (1, -3) pair of points and find the slope of the line.

Let point (x_1, y_1) be equal to (-2, 3) and (x_2, y_2) be equal to (1, -3) Then, using the formula for calculating the slope of a straight line as;

$$Slope = m = \frac{y2 - y1}{x2 - x1}, x1 \neq x2$$

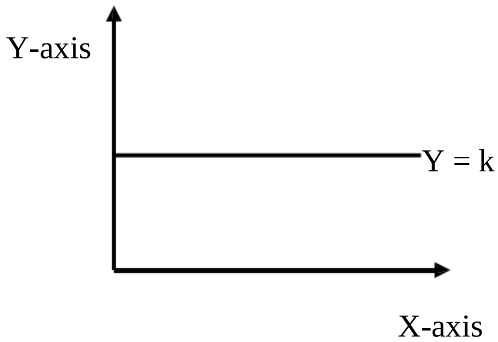
$$Slope = m = \frac{-3 - 3}{1 - (-2)} = \frac{-6}{3} = -2$$



Interpretation

For every one unit change of x is accompanied by 2 unit change in y or the line fall (decrease by 2 units as x increase by 1 unit).

Zero slope - shows that y is constant in spite of the value of X

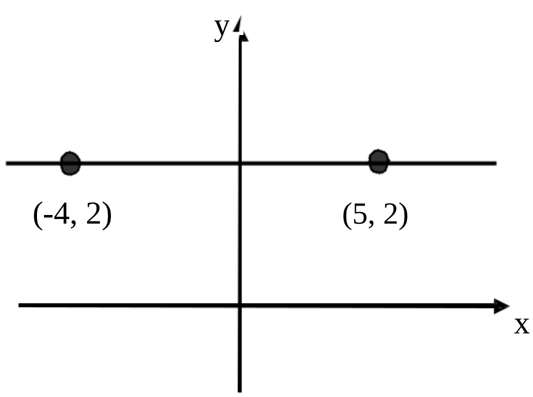


Example 3: Sketch a line through (-4, 2) and (5, 2) pair of points and find the slope of the line.

Let point (x_1, y_1) be equal to (-4, 2) and (x_2, y_2) be equal to (5, 2). Then, using the formula for calculating the slope of a straight line as;

$$Slope = m = \frac{y2 - y1}{}, x1 \neq x2$$

$$Slope = m = \frac{2-2}{5-(-4)} = \frac{0}{9} = 0$$

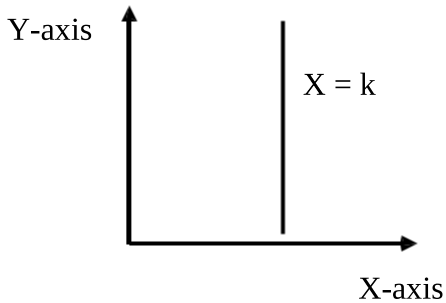


Interpretation

For every one unit change of x is accompanied by 0 unit change in y or y does not change as x changes.

Note: ***The slope of a horizontal line is zero (0).***

Infinite/ undefined slope- shows that the value of x remains constant regardless of the change in the value of Y.

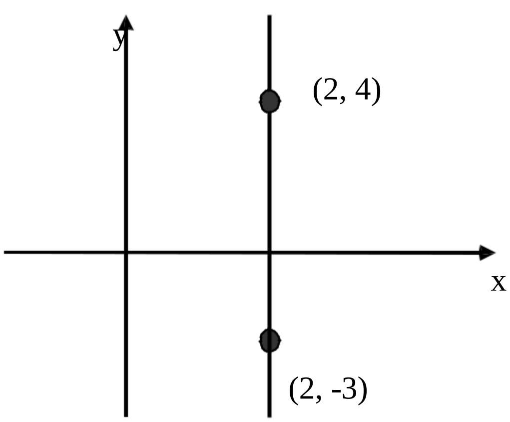


Example 4: Sketch a line through (2, 4) and (2, -3) pair of points and find the slope of the line.

Let point (x_1, y_1) be equal to (2, 4) and (x_2, y_2) be equal to (2, -3). Then, using the formula for calculating the slope of a straight line as;

$$Slope = m = \frac{y2 - y1}{x2 - x1}, \; x1 \neq x2$$

$$Slope = m = \frac{-3-4}{2-2} = \frac{-7}{0}$$ □ Slpe is not defined



Interpretation

X does not change as y changes. Hence x1 = x2 and its slope is not defined.

$$\text{Slope} = \frac{y1 - y2}{x1 - x2} = \frac{y2 - y1}{0} = \text{not defined}$$

SELF CHECK EXERCISE 1.1

1.Find the slope of the line that passes through the following coordinate points and describe the type of relationship that exists between the variables.

A. (2, 4) and (-3, 7)

B. (2, 5) and (4, 9)

C. (4, 8) and (3, 8)

D. (2, 6) and (2, 11)

1.6 Forms of Linear Equations (Developing Equation of a Line)

There are about three different situations under which we need to determine the equation of a line, such as;

1.6.1. Slope Intercept Form

1.6.2. Slope-Point Form and

1.6.3. Two-Point Form

1.6.1. Slope Intercept Form

If a line meets the x-axis at (a, 0) and the y-axis at (0, b), then **"a"** is called the **x-intercept** and **"b"** is called the **y-intercept of the line**.

If the line has slope **m** and a y-intercept **b**, then the ***slope-intercept*** form of its equation is;

$$\mathbf{y = mx + b}$$

Example 1 Find the equation of a line whose slope is 3 and whose y-intercept is -4.

Solution: we know that Y = mx + b

m = 3

b = -4

y = mx + b we have

Hence **<u>y = 3x -4</u>**

Example 2 Given the equation 2y + 4x = 20;

Write the equation in slope-intercept form
State Slope
Find the x-intercept
Find the y-intercept
Plot the equation by intercept

Solution

2y + 4x = 20

2y = -4x + 20

y = -2x + 10

m = -2

y-intercept is obtained at x = 0.

y = -2(0) + 10

y = 10 is the y-intercept at the point (0, 10)

Ā

-intercept is obtained at y = 0.

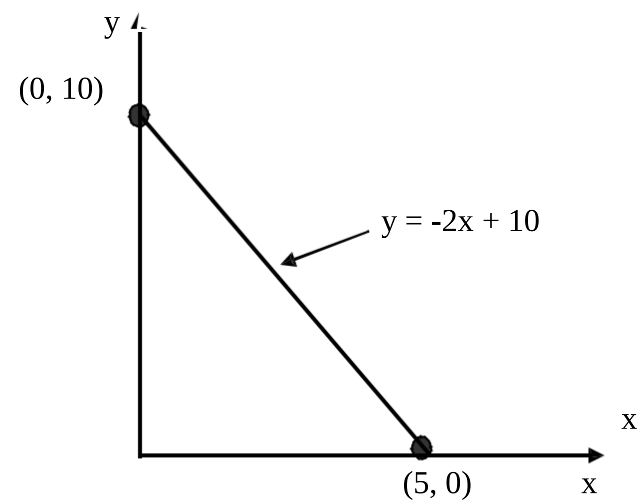
y = -2x + 10

0 = -2x = 10

2x = 10

x =5 is the x-intercept at the point (5, 0)

e)



Dear learners, having seen the mathematical concept of how to develop equation of a straight line given a slope and y-intercept, we will now take one application example and see how to form equation of a cost function.

Example 3: It costs Birr 200 to set up the machinery needed to assemble a television. After set up, it costs Birr 5 per television to assemble. Let x be the number of televisions assembled and y be the total cost of making this number of televisions.

Write the equation expressing total cost (y) is a function of the number of televisions assembled (x).

State the slope of a line and interpret it.

State the y-intercept of the line and interpret it.

What is the cost for every additional unit of a television?

Find the total cost if 30 televisions are assembled.

Solution

m = slope = Marginal cost (MC) = 5

b = y-intercept = fixed cost =

200 y = mx + b

y = 5x + 200, and hence, this is a linear equation in slope-intercept form.

m = MC = 5, means that every additional television assembled adds Birr 5 to the total cost.

The y-intercept which is Birr 200 is total cost when there is no television assembled. That is, at x = 0;

y = 5x = 200 y

= 5(0) + 200 **y**

= Birr 200

Marginal cost is the extra cost for every additional units of television assembled. Hence, MC = slope = Birr 5.

y = 5x + 200

At x = 30 units of television assembled,

y = 5(30) + 200

y = 150 + 200

y (Total Cost) = Birr 350

1.6.2 Slope- Point Form

Linear equation of a straight line the passes through the point P1 (x_1, y_1) with slope m is given by

$y – y_1 = m(x – x_1)$

The point-slope form is extremely useful since it enables us to find an equation of a straight line given its slope and the coordinates of a point on the line.

Example 1: Find the equation of a line that has slope $\frac{3}{5}$ and passes through the point (-3, 1).

Solution

$M = \frac{3}{5}$ and $(x_1, y_1) = (-3, 1)$. Then,

$y - y_1 = m(x - x_1)$

$y - 1 = \frac{3}{5}(x - (-3))$

$y - 1 = \frac{3}{5}(x + 3)$

$y - 1 = \frac{3}{5}x + \frac{9}{15}$ or $\frac{3}{5}$

$\mathbf{y = \frac{3}{5}x + \frac{8}{5}}$

Example 2: The total cost (y) and the number of units made (x) is a linear relationship and if costs increased by Birr 10 for each additional unit made and if total cost of 50 units is Birr 1100.

Find the equation that relates total cost (y) and the number of units made (x).

State the slope of a line and interpret it.

State the y-intercept of the line and interpret it.

Find the total cost if 30 units are made.

What is the cost for every additional units of output?

Solution:

TC = 1100 at x = 50 unit and m = MC = 10

y = mx + b

1100 = 10 (50) + b

b = TFC = 600. Then the equation

is y = mx + b

y = <u>10x + 600</u>

m = slope = MC = 10. To add every additional units of x, total cost is increased by Br 10.

y = 10x + 600

At x = 0,

y = 10 (0) + 600

y = Birr 600

y = 10x +

600 At x = 30

y = 10(30) + 600

y = 300 + 600

y = Birr 900

Marginal cost is the additional cost for every additional unit made. That is, MC = Birr 10.

1.6.3 Two-Point Form

If two points P_1 (x_1, y_1) and P_2 (x_2, y_2) are known, then the slope is defined as the change in the y-coordinate divided by the change in the .*x*-coordinate given as

$$Slope = \frac{\text{change in y-coordinate}}{\text{change in x-coordinate}} = \frac{\Delta y}{\Delta x} = \frac{y2 - y1}{x2 - x1}, x1 \neq x2$$

There are two steps involved when deducing the equation of a line when given two points, P_1 (x_1, y_1) and P_2 (x_2, y_2)

Step 1: Calculate the slope of the line using the slope formula

Step 2: Determine the equation of the line by substituting the slope calculated in step 1 above and either point into the equation $y - y_1 = m(x - x_1)$.

Example 1: A line passes through the points (2, 4) and (6, 1).

Deduce the equation of the line.

Plot the graph of the line.

Solution:

Step 1: The slope of the line is calculated by substituting the points into the slope formula.

$$Slope = \frac{y2 - y1}{x2 - x1} = \frac{1 - 4}{6 - 2} = \frac{-3}{4} = -0.75$$

Step 2: The equation of the line is deduced by substituting this slope, and either point into equation $y - y_1 = m(x - x_1)$.

$y - y_1 = m(x - x_1)$.

$y - 4 = -0.75(x - 2)$. Using point (2, 4)

$y - 4 = -0.75x + 1.5$

y = <u>5.5 - 0.75x</u>

Example 2: The total cost (y), producing x units of soap is a linear function. Records show that in one occasion 20 units of soap were made at a total cost of Birr 60 and on the other occasion 30 units are made at total cost of Birr 80.

Write the equation for total cost (y) in terms of the number of units of soap (x).

State the slope and interpret the result.

State the y-intercept and interpret the result.

What will total cost be if 100 soaps are produced?

What is the marginal cost of the 100^{th} unit of soap?

Solution:

a) Given the coordinates (soap, total cost), that is, (20, 60) and (30, 80)

$$Slope = \frac{y2 - y1}{x2 - x1} = \frac{80 - 60}{30 - 20} = \frac{20}{10} = 2$$

$y - y_1 = m(x - x_1)$.

$y - 60 = 2(x - 20)$.

$y - 60 = 2x + 40$

y = <u>2x + 20</u>

b) m = slope = MC = Br. 2

Interpretation: For every additional units of soap, total cost increased by Br. 2.

y – intercept is obtained when x =

0 y = 2x + 20

At x = 0

y = 2(0) + 20

y = <u>Br. 20</u>

Interpretation: with no production of soaps, the firm incurred a cost of Br. 20 (which is equal to the fixed cost)

y = 2x + 20

At x = 100

y = 2(100) + 20

y = 200 + 20

y = <u>Br. 220</u>

MC = slope = Br. 2 for 100th unit of soap. Marginal cost remains constant for every additional units of soap under the linear equation.

SELF CHECK EXERCISES 1.2

An agency rents cars for one day and charges 22 birr plus 20 cents per kilo Meter the car is driven.

Write the equation for one day's rental(y) in terms of the number of Kilo Meters driven (x)

Interpret the slope and the y- intercept

What is the renter's cost per mile if a car is driven 100 kilo meters? 200 Kilo Meters?

A publisher asks a printer for quotations on the cost of printing 1000 and 2000 copies of books. The printer quotes 4500 birr for 1000 copies and 7500 birr for 2000 copies. Assume that cost (y) is linearly related to the number of books printed (x)

Write the coordinates of the given points

Write the equation of the line

If the relationship between total cost and number of units made is linear and if cost increases by 3 birr for each additional unit made, and if the total cost of 10 units is 40 birr find the equation of the relationship between total cost (y) and number of units X

Quantity demand of a commodity is linearly related to its price. When price was Br. 100 quantity demanded was 500 units, when price becomes Br. 150 quantity demanded was 300 units.

Write the demand equation (y) in terms of price (x).

State the slope and interpret the number.

1.7 Applications of Linear Equations in Business Problems

Having revised the basic concepts of linear equations, let's now see some application areas of linear equations. Linear equations can be used to explain linear cost functions, linear revenue functions, linear profit functions, and linear supply and demand functions.

1.7.1 Linear Cost Function

Cost of production refers to the total monetary expenditure made for the production of certain level of outputs. Cost of production can be classified in two: fixed and variable costs.

Fixed cost is the sum of all the costs that are independent of the level of production, namely rent, insurance, depreciation and other overhead expenses, interest on loans, and upkeep of the establishment which exist even while there is no production.

Variable cost is the sum of all costs that are dependent on the level of production namely material costs, labor costs, cost of publicity and fuel etc. thus, total cost function is given by:

Total cost = Total Variable cost + Total Fixed cost

C(X) = V(X) + F(X)

Where,

C(X) = Total cost

V(X) = Total Variable cost

F(X) = Total Fixed cost

There are two assumptions that need to be made for the linear cost functions to valid:

Constant returns to scale

Fixed cost of production remains constant for any level of output.

Average costs

Average costs are costs of production per unit. Generally, there are about three different types of Average costs: Average total cost, Average fixed cost and Average variable costs.

ATC (AC) = $\frac{C(X)}{Q}$ and since C(X) = V(X) + F(X)

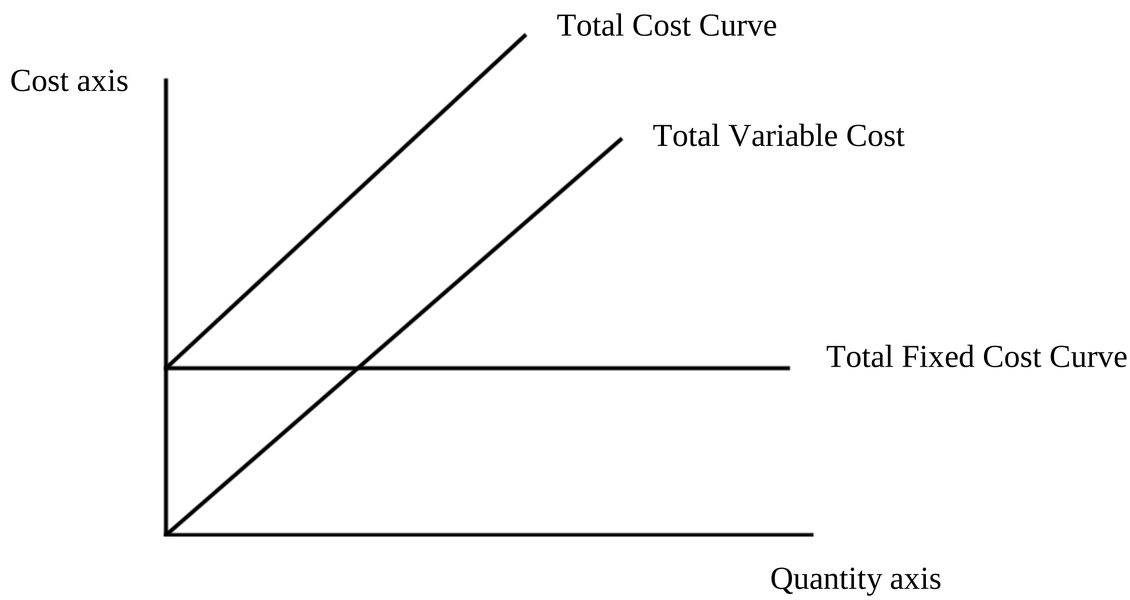
$$ATC = \frac{V(X)}{Q} + \frac{F(X)}{Q}$$

Where; ATC = Average total cost

Q = level of production

$\frac{V(X)}{Q}$ = Average variable cost

$\frac{F(X)}{Q}$ = Average Fixed cost

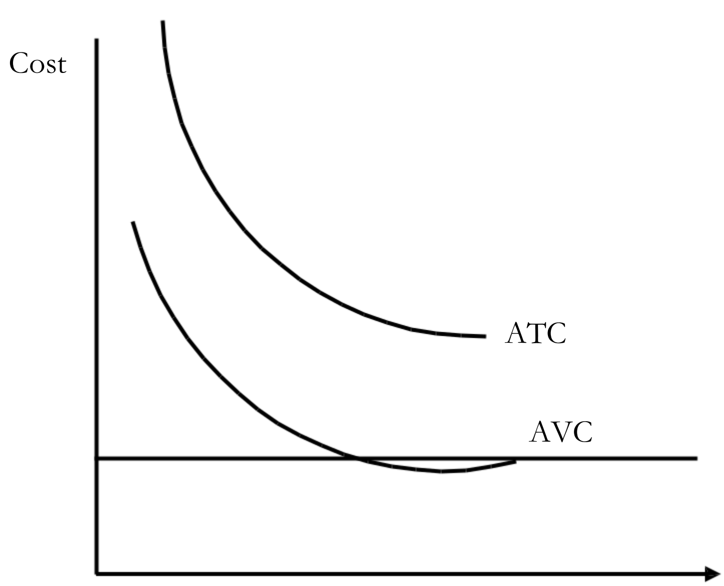


Let's see some characteristics of average costs.

Average Variable Cost remains constant over the whole range of output.

Average Fixed Cost declines as output level increases

Average Total Cost declines as Output level increases.



Quantity

Illustration 1:

Given cost function, Y = 40X + 50

Determine;

a) Average Variable Cost
b) Average Fixed Cost
c) Average Total Cost

For Q = 50, units and Q = 100 units

When Q_1=50

$$AVC = \frac{TVC}{Q} = \frac{40(50)}{50}$$

= **Birr 40**

When Q_2=100

$$\frac{40X100}{100}$$

= **Birr 40**

Look at the constancy of average variable cost per unit regardless of the changes in the level of production.

When Q= 50

$$AFC= \frac{50}{50} = \textbf{Birr 1}$$

when Q= 100

$$= \frac{50}{100} = \textbf{Birr 0.5}$$

Compare the two average costs under the two levels of production. Have you detected that average fixed cost declines as quantity rises?

When Q= 50

$$AC= \frac{40(50)}{50} + \frac{50}{50}$$

= **Birr 41**

when Q= 100

$$\frac{40(100)}{100} + \frac{50}{100}$$

= **Birr 40.5**

1.7.2 LINEAR REVENUE FUNCTIONS

Revenue is the income we get from sale of a commodity. Manufacturers produce and sell products at their respective prices in the market. Would you calculate the revenue earned by the producer when the producer produces and sells from 0-10 units; if one unit of the product is sold at 10 birr per unit?

Since one unit of the product is sold at 10 birr, the producer gets zero birr of revenue when it sells nothing, 10 birr when it produces and sells one unit, 20 birr when it produces and sells 2 units, 30 birr when it produces and sells 3 units, and so. The detail for this question can be presented as follows.

Units sold	price per unit	Revenue (selling price X units sold)
0	10	0
1	10	10
2	10	20
3	10	30
4	10	40
5	10	50
6	10	60
7	10	70
8	10	80
9	10	90
10	10	100

From this, one can understand the relationship between revenue and quantity produced and sold. In linear revenue function, Revenue is positively related to level of sales.

When we use linear revenue function to explain the relationship between revenue and sales volume we are making one assumption: selling Price per unit remains constant. If selling price is variable, the equation becomes nonlinear. Hence, we cannot explain it using linear equation concepts.

The relationship between the quantity produced and revenue earned can be shown mathematically using the following equation.

R=P.Q

Where, R = revenue, Q = units sold and P is constant and is the slope of the equation.

1.7.3 Linear Profit Function

Revenue and cost functions lead to the profit function of the firm, as profit is the excess of revenue over cost of production. The profit function of the firm takes the form:

$$(Q) = R(Q) - C(Q)$$

Where, R $_{(Q)}$ is the revenue generated from selling Q units and C $_{(Q)}$ is cost incurred in production of Q units of product.

Companies either generate profit from their operation or incur a loss from their operation.

Do you know when companies make profit and when they suffer loss?

Refer to the following box to check your answer.

Conditions	Operation result
$(Q) > C(Q)$	Profit (net gain)
$(Q) < C(Q)$	Loss
$R_{(Q)}=C_{(Q)}$	No loss and no gain

Illustration 1:

Z & T company produces and sales a commodity for $100. The company's cost function is given by the equation (y) =50x + 2,500 where x is units of products produced and sold and y is total cost.

Construct the profit function

what profit is earned if the annual sale are 1,00

Solution:

a) Profit = TR -TC

=100Q - (50Q +2,500)

=100Q-50Q-2,500

Profit = **50Q -2,500**

To find the profit level when 100 units are sold, substitute Q= 100 in profit function and solve for profit. Therefore, profit will be

50 (100) -2,500

birr 2,500

1.8. BREAKEVEN ANALYSIS FOR MANUFACTURING ENTERPRISES

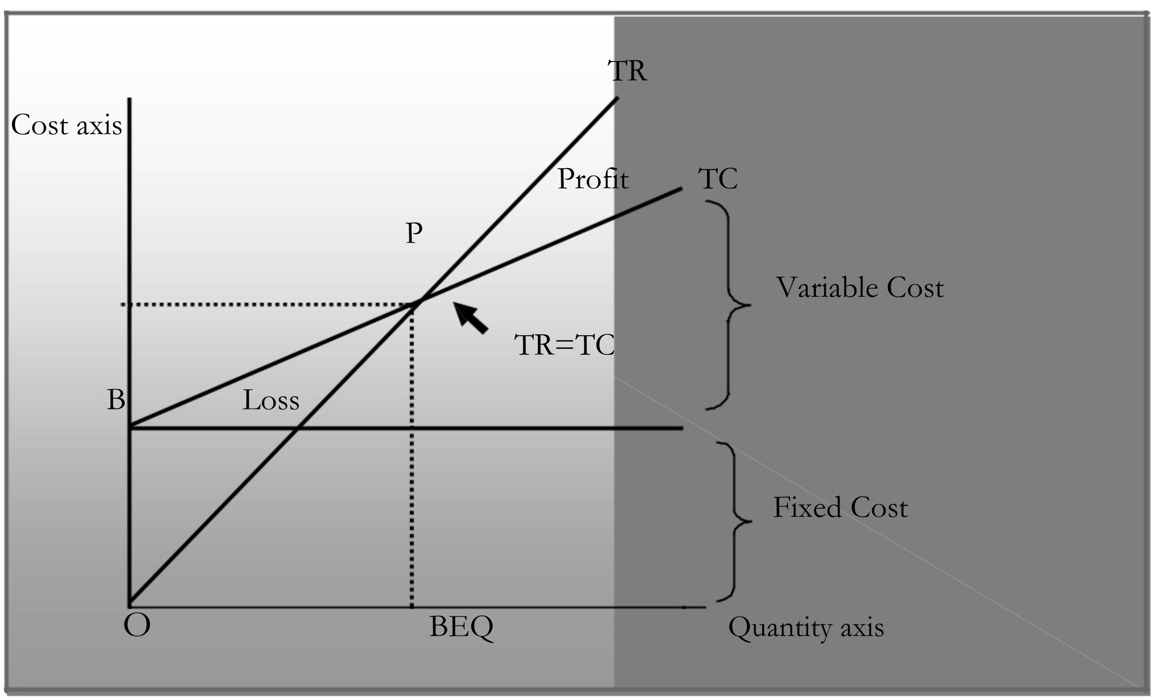
Breakeven point is level of production or a level of sales at which the firm gains no profit and suffers no loss. In another way round, it is the point where total cost of production and total revenue are equal. Knowing breakeven quantity helps firms to determine how many units they need to produce and sell if they have to make profits. When an organization produces any quantity less than the breakeven quantity, it suffers a loss, where as producing more than a breakeven quantity yields a profit. The issue is therefore, how many units to produce to at least get the cost of production back.

Break-even analysis is a technique widely used by production management and management accountants. It is based on categorizing production costs between those which are "variable" (costs that change when the production output changes) and those that are "fixed" (costs not directly related to the volume of production).

Total variable and fixed costs are compared with sales revenue in order to determine the level of sales volume, sales value or production at which the business makes neither a profit nor a loss (the "break-even point").

1.8.1 THE BREAK-EVEN CHART

In its simplest form, the break-even chart is a graphical representation of costs at various levels of activity shown on the same chart as the variation of income (or sales, revenue) with the same variation in activity. The point at which neither profit nor loss is made is known as the "break-even point" and is represented on the chart below by the intersection of the two lines:



In the diagram above, the line OTR represents the variation of income at varying levels of production activity ("output"). OB represents the total fixed costs in the business.

As output increases, variable costs are incurred, meaning that total costs (Fixed + Variable) also increase. At low levels of output, Costs are greater than Income. At the point of intersection, P, costs are exactly equal to income, and hence neither profit nor loss is made.

Mathematically, breakeven point can be obtained as follows:

Literally, breakeven point is the level of production where total cost of production and total revenue are equal

TC = TR (we can rewrite total cost as FC + VCQ and TR as

PQ Therefore,

PQ = FC+VCQ (collecting similar terms to one side yields)

PQ - VCQ = FC (Q is common for the left hand side, factor it out)

Q (P-VC) = FC (divide both sides by P – VC)

$$Q = \frac{FC}{P - VC}$$

Where;

Q = implies breakeven point quantity

P = Price per unit

FC = Fixed Cost

VC = Variable cost per unit

However, the objective of firms is not only to breakeven, they rather want to make profit of a given amount. Wise managers should, therefore, know how many units to produce in order to get a profit of say Π amount, where Π is the amount of target profit. To find how many units to produce to get a profit of π amount, we need to make minimum adjustment to our previous equation. The modification needed is:

$TR - TC = \Pi$ Where Π is the desired profit Level.

$$PQ - FC - VCQ = \Pi$$

$$PQ - VCQ = FC + \Pi$$

$$Q(P - VC) = FC + \Pi$$

$$Q = \frac{FC + \Pi}{P - VC}$$

Illustration

A manufacturer of cassette tapes has a fixed cost of Birr 75,000 and a variable cost of Birr 7 per cassette produced. Selling price is Birr 10 per cassette.

Write the revenue, cost and profit functions

At what number of unit will break even occur?

At what sales volume (revenue) will break even occur?

Compute profit if;

20,000 units are made and sold.

50,000 units are made and sold.

Construct the break even chart

If, because of strike, the company will be able to produce only 20,000 unit, should the company shut down its operation? Why and why not?

How does average cost per unit change if production changes from 5000 units to 9000 units?

Solution

a) R = PQ = 10Q

C = VQ + FC =7Q + 75,000

Π = TR – TC = 10Q – (7Q + 75,000) = 3Q – 75,000

b) At breakeven point, TR = TC

$$10Q = 7Q + 75,000$$

$$3Q = 75,000$$

$$Q = \frac{75,000}{3}$$

Q = 25,000 units

OR $Q = \frac{FC}{P-V} = \frac{75,000}{10-7} = \frac{75,000}{3}$ = **25,000 units (Break even Quantity)**

Sales volume at BEP = PQ

10 (25,000)

Birr 250,000

d)

At Q = 20,000 units

=TR–TC

10Q – (7Q + 75,000)

10(20,000) – (7(20,000) + 75,000)

200,000 – (140,000 + 75,000)

200,000 – (215,000)

- Birr 15,000 (Loss)

Therefore, the manufacturer has incurred a loss of Birr 15,000

At Q = 50,00 units

=TR–TC

10Q – (7Q + 75,000)

10(50,000) – (7(50,000) + 75,000)

500,000 – (350,000 + 75,000)

500,000 – 425,000

Birr 75,000

Therefore, the manufacturer has gained a profit of Birr 75,000

To plot the graph of the break even, consider the following three points

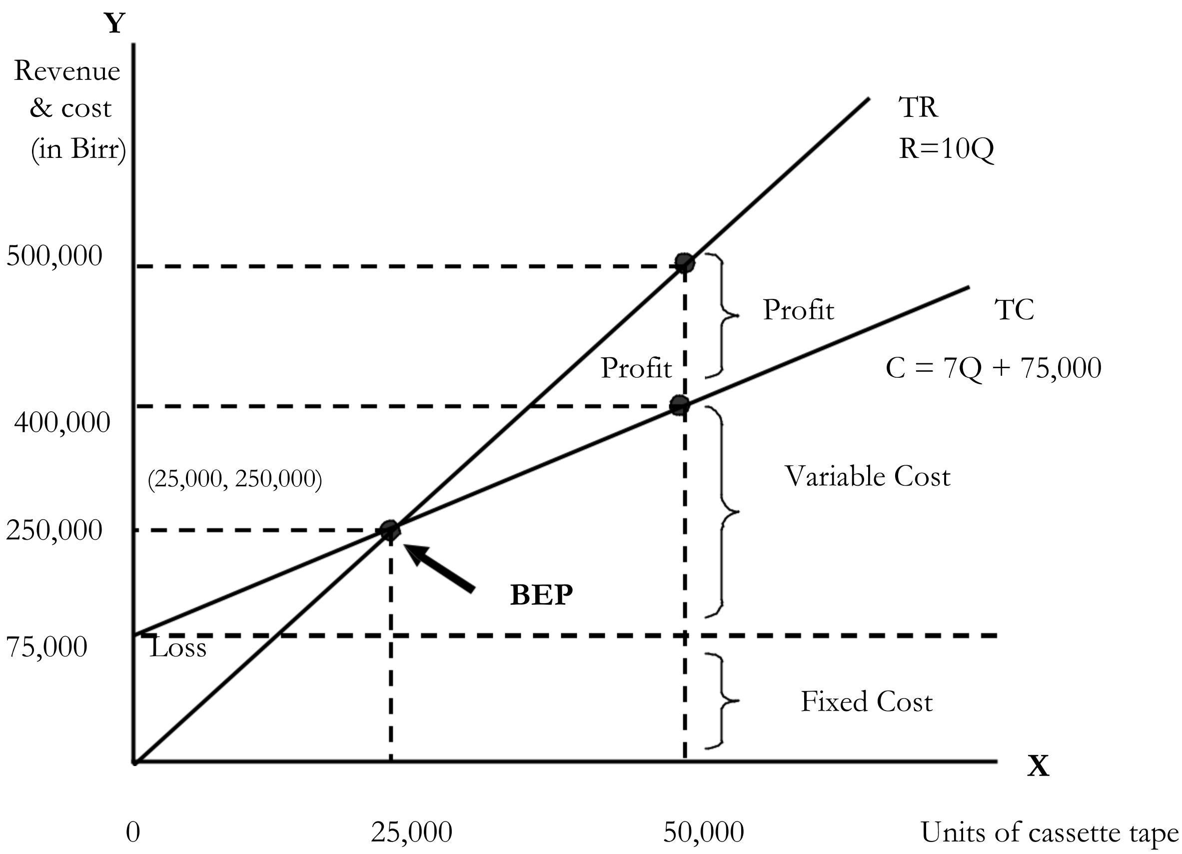
When Q = 0 units, at zero production

When Q = 25,000 units, at breakeven point

When Q > 25,000 units, any units of output greater than the breakeven point.

Recall that we have developed two (Revenue and Cost) equations;

TR = 10Q and TC = 7Q + 75,000



From the above breakeven chart, the revenue goes through the origin and that the cost line has a y – intercept of Birr 75,000. The dotted horizontal line shows that this fixed cost is constant at all levels of operation. The total variable cost, which is 7Q in the equation TC = 7Q + 25,000 (where Birr 7 variable cost per cassette is constant) is the vertical distance from the fixed cost line to the total cost line, which, of course, increases as more units are produced

To the left of breakeven point (25,000, Birr 250,000), the cost line is above the revenue line and the vertical separation at any point represents loss, while to the right of the breakeven the vertical separation represents a profit.

The producer should continue production, since the loss is less than the total fixed cost. In this situation if production is going on the loss will be minimized. In our case, the loss is Birr 15,000 whereas the fixed cost is Birr 75,000 (Loss < TFC). The producer should shut down its operation only if loss is greater than the total fixed cost (Loss > TFC). In this situation if production is going on, the extent of the loss will be increased rather than minimized.

Dear learners, I hope you have understood the extent to which any firm can continue operation and the point where it should shut down operation. Now, let's justify the extent of loss

If Production is going on

If there is no operation

The amount loss if production is going on can be computed as follows

$$= TR - TC$$

$$\Rightarrow 10Q - (7Q + 75{,}000)$$
$$\Rightarrow 10(20{,}000) - (7(20{,}000) + 75{,}000)$$
$$\Rightarrow 200{,}000 - (140{,}000 + 75{,}000)$$
$$\Rightarrow 200{,}000 - 215{,}000$$
$$\Rightarrow -15{,}000$$

The loss is 15,000

ii. The amount of loss with no operation is computed as follows

$$\Rightarrow = TR - TC$$
$$\Rightarrow 10Q - (7Q + 75{,}000)$$
$$\Rightarrow 10(0) - (7(0) + 75{,}000)$$
$$\Rightarrow 0 - (75{,}000)$$
$$\Rightarrow Birr - 75{,}000$$

The loss is 75,000 which equal to TFC

Therefore, it is better to continue production at Q = 20,000 units of cassette tape rather to stop production, because it is better to face a loss of Birr 15,000 than facing a loss of Birr 75,000.

TC = 7Q+75,000

At Q = 5,000

g) TC = 7(5,000)+75,000

35,000+75,000

Br.110,000

$$\text{Average Cost} = \frac{TC}{} = \frac{\text{Birr}110{,}000}{5{,}000} = \text{Birr } 22$$

TC = 7Q + 75,000

At Q = 9000

TC = 7Q + 75,000

7(9000) + 75,000

63,000 + 75,000

Br. 138,000

$$AC = \frac{TC}{Q} = \frac{\text{Br. } 138{,}000}{9000} = \text{Br. } 15.3$$

The average cost per cassette decrease from Br. 22 to Br. 15.3 as the number of cassette tape manufactured increases from 5,000 to 9,000 units.

1.9 BREAKEVEN ANALYSIS FOR MERCHANDISING ENTERPRISES

Merchandizing firms do have different type of cost called cost of retailing. Merchandizing firms do not produce goods they sell; they rather buy it from others and resale it at a profit. Therefore, the only variable costs of merchandizing firms are cost of retailing.

Assume a company that buys a merchandize at 180 birr and sells it at 200 birr. This implies that if there is X birr revenue from sales of merchandize, (180/200) or 90 % of it is cost of goods sold and the remaining 10% is gross margin the company is earning.

This means, from 1birr of sale 90 cents is cost of the goods sold and the remaining 10 cents is the profit margin added by the company.

Gross Profit

Margin = Margin =

$$\text{Margin} = \frac{20}{200} = 0.1 = \frac{\text{Retail Price - Cost}}{\text{Retail Price}}$$

Cost of goods sold proportion = $\frac{180}{200}$ = 0.9 or 90%

$$\frac{\text{Purchase Cost}}{\text{Retail Price}} \text{ or 1- Profit Margin}$$

Formula for computation of breakeven quantity for merchandizing firms is as follows:

Y = mx + b (merchandizing firm's total cost function)

X - Represents merchandiser's total revenue

At breakeven point cost = revenue

i.e. Y = X

we know that Y = mx + b

mx + b = x = at breakeven

if we let the breakeven dollar volume of sales to be equal to X_e

$x_e = x$ hence,

$x_e = x = mx_e + b$

$x_e - mx_e = b$

$x_e (1-m) = b$ dividing both sides by (1- m), results in

$$Xe = \frac{b}{1-m}$$

$$\text{Breakeven point} = \frac{\text{Fixed cost}}{\text{1 - variable cost/dollar of sales}}$$

Illustration

A company expects fixed cost of Br. 100,000. It plans to work on a margin of 45% of retail price and to incur variable costs of Br. 0.05 per Br. of sales.

Write the equation relating total cost to sales volume

What is the breakeven levels of sales volume

What will total cost and net profit before taxes be if the company sales Br. 300,000

Construct the break even chart.

Solution:

Let y = Total Cost

x = sales in Birr

Given:

Margin = 45% of retail price = 45%x = 0.45x

Cost of Goods Sold (CGS) = 100% - margin = 100% - 45%

55% of retail price

0.55x

Other variable cost = Br. 0.05x

Fixed Cost = Br. 100,000

y = 0.55x + 0.05x + 100,000

= 0.60x + 100,000

b) $x = \dfrac{b}{1- m} = \dfrac{100,000}{1-0.6} = \dfrac{100,000}{0.4}$ = Br. 250,000

Check that Birr 250,000 is sales volume at breakeven point. That is;

Cost = Sales

y = x

y = 0.6x - 100,000

0.6 (250,000) + 100,000

150,000 + 100,000

Br. 250,000

Thus, total cost at sales volume of Br. 250,000 is 250,000 Hence, y = x

Br. 250,000 = Br, 250,000

y = 0.6x + 100,000

0.6(300,000) + 100,000

180,000 + 100,000

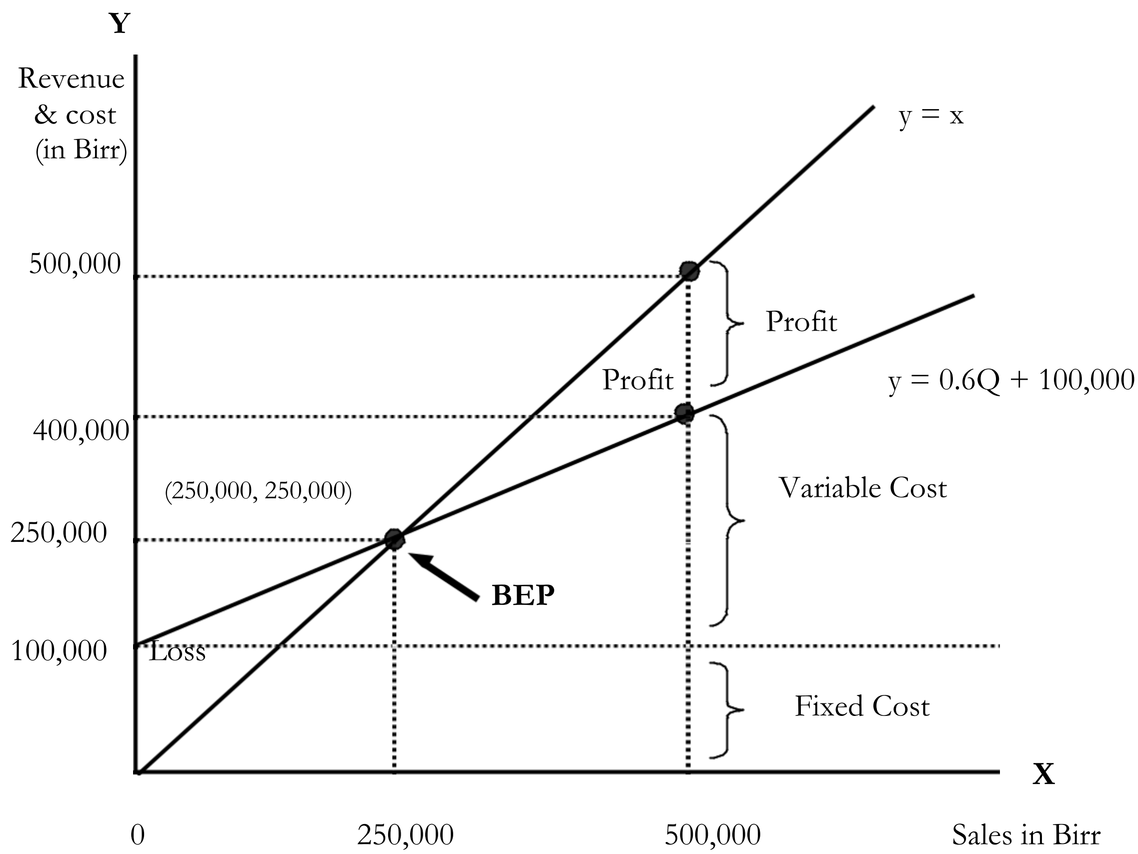
= Br. 280,000

= y - x

300,000 - 280,000

Br. 20,000

A breakeven chart can be constructed by plotting y = 0.6x + 100,000 and the line where cost equals sales (y = x).



The breakeven point is seen to be Br. 250,000. The separation of the lines to the right of the breakeven point indicates profit and to the left it indicates loss.

ELF CHECK EXERCISES 1.3

Alemu cement factory has a fixed cost of Br. 10,000 and variable cost and selling price per quintal of cement is Br. 5 and Br. 10 respectively.

rite the revenue and cost equations

t what number of quintal will breakeven occur?

t what sales volume (revenue) will breakeven occur?

f, because of strike, the factory will be able to produce only 500 units. Should it shut down its operation? Why and why not?

2. A company expects fixed cost of birr 36,000. Margin is to be 0.52 of retail and variable cost in addition to cost of goods sold is estimated to be birr 0.07 per birr of sale

Find the Breakeven point.

write the equation relating sales and cost

What will net profit before tax be on sales of birr 75 ,000

1.10. LINEAR DEMAND AND SUPPLY ANALYSIS

Dear Learners, in this particular section we will introduce you to the economic concept of demand and supply analysis. Demand function relates the quantity of products consumers are willing and able to buy to the unit price of the product. Linear demand functions have negative slopes because demand decreases when price increases.

A supply function, on the other hand, relates the quantity producers are willing and able to supply to the unit selling price of the product. Linear supply functions have positive slope because producers increase supply as price increases.

Consider the following demand and supply functions:

Demand: P = 0.1Q + 40

Supply: P = 0.2Q + 10

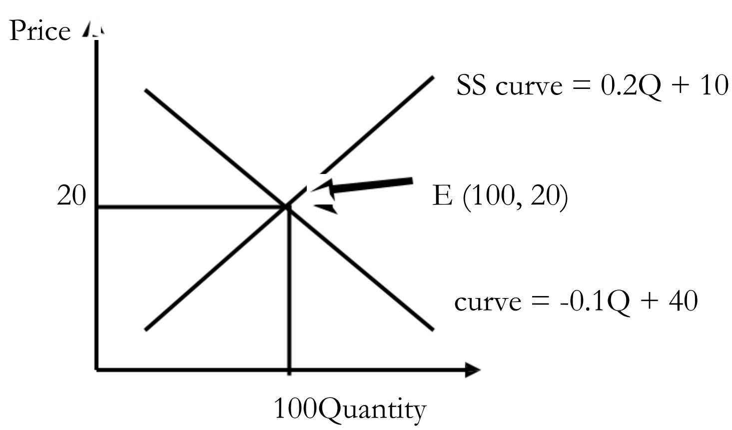
Where; P is price per unit

Q is number of units

The intersection of demand and supply functions results in market equilibrium, E. Market equilibrium is the point where the quantity demanded is equal to quantity supplied. At equilibrium point;

P = P
-0.1Q + 40 = 0.2Q + 10
30 = 0.3Q (dividing both sides by 0.3)
Q=100
Taking one of the linear functions;
P=0.2Q+10
P = 0.2(100) = Birr 20

This implies that, market equilibrium occurs at the point where quantity is 100 units and price is birr 20 per unit. The equilibrium price at which quantity demanded is equal to quantity supplied (100 units) is birr 20, E (100, 20). The same can be plotted on a plane geometry as follows;



SUMMARY

For an equation to be linear sum conditions need to be fulfilled:

All variables should be raised only to the first power.

There should not be cross product of variables.

Linear equations have standard form of Y= ax + b, where, x is independent variable, y is dependent variable, a is the slope of the linear equation and b is y intercept.

Slope is a measure of change in y due to a unit change in x and is denoted by a.

Intercepts are the points at which the graph of an equation cuts the y axis and the x axis. The point where the graph of an equation crosses y axis is referred to as y intercept and the point where the graph crosses the x axis x- intercept.

Given any one of the following situations one can determine the equation of a line using the following formula.

i/ Slope and y-intercept

$$y = mx + b$$

ii/ Slope and one point on the line

$$m (x-x_1) = y – y_1$$

iii/ Two points on a given line; we first compute the slope (m) and then use point slope formula to determine the equation.

Linear functions are always represented by a straight line and used to represent one to one simple relationship.

Linear cost, revenue, profit, demand and supply functions can simply be analyzed based on ones own comprehension of linear function.

Breakeven point is the point where manufacturing and for mechanizing firms make zero profit. That is, the total revenue is equal to the total cost, which yields zero profit.

CHAPTER TWO

MATRIX ALGEBRA AND ITS

APPLICATION 2.1 INTRODUCTION

Most business problems can better be handled using matrix than it can be with other notations. This is the reason behind the need to give students of business (accounting and management) a brief overview of matrix theory and its application in the field. Hence in this particular chapter of this module the necessary matrix theories and applications are emphasized.

Numerical data arranged in a form referred to as a matrix are very common in everyday life. Matrices and the algebra of matrices are long recognized for their usefulness in almost every branch of science and engineering. They are assuming an increasingly important role among the modern techniques used in mathematical analysis of social, business and economic problems. In this chapter, we will first have a review of matrix algebra learned in previous mathematics course. After the review, we will introduce technique of solving system of linear equations. Finally, we will have another application of matrix algebra in concept of Markov chains.

2.2 OBJECTIVE OF THE CHAPTER

After finishing this chapter, trainees will be able to:

- Understand what a matrix is
- Differentiate various types of matrices
- Conduct matrix operation: matrix addition, matrix subtraction, matrix multiplication
- Understand the various properties of matrix operation
- Express systems of linear equations in matrix form and solve it using inverse method.

2.3. DEFINITION OF MATRIX ALGEBRA

BRAIN STORMING QUESTION

What is a matrix?

Dear learners, have you defined matrix as "**A MATRIX** is **RECTANGULAR ARRAY OF REAL NUMBERS ARRANGED IN** m **ROWS AND** n **COLUMNS**.

". Excellent! The numbers in the array are called entries or elements.

$$\mathbf{C} = \begin{bmatrix} 1 & 9 & 8 & 5 \\ -2 & 3 & -7 & 6 \\ -8 & -5 & 4 & 0 \\ 7 & -6 & 2 & -1 \end{bmatrix}$$

This is a 4 x 4 matrix.

Each number appearing in the array is said to be an **ELEMENT**, or **COMPONENT**, of the matrix. Elements of a matrix are designated using a lowercase form of the same letter used to symbolize the matrix itself. These letters are subscripted, as $a_{ij,}$ to give the row and column location of the element within the array. The first subscript always refers to the row location of the element; the second subscript always refers to its column location. Thus, component a_{ij} is the component located at the intersection of the ith row and the jth column.

The number of rows, m, and the number of columns, n, of the array give its **ORDER**, or its **DIMENSIONS**, m x n (read "m by n"). The abbreviated notation; $\mathbf{A}_{mxn}$ or $[a_{ij}]_{(mxn)}$ is often used to represent the matrix and specify its order.

Example 1 The following are examples of matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 7 \\ 5 & 3 \\ 4 & -2 \end{bmatrix}$$

This is a 3x2 matrix.
Element $a_{12} = 7$
Element $a_{21} = 5$

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 6 \end{bmatrix}$$

This is a 2x3 matrix.
Element $b_{13} = -1$
Element $b_{23} = 6$

$$\mathbf{C} = \begin{bmatrix} 1 & 9 & 8 & 5 \\ -2 & 3 & -7 & 6 \\ -8 & -5 & 4 & 0 \\ 7 & -6 & 2 & -1 \end{bmatrix}$$

This is a 4 x 4 matrix.
Element $c_{33} = 4$
Element $c_{44} = -1$
Element $c_{31} = -8$

Matrices provide a most convenient vehicle for organizing and storing large quantities of data. Because the basic idea is to organize the data, we cannot overemphasuze the importance of the location of each number within the matrix. It is not simply a matter of putting numbers into rows and columns; each row-column location within each matrix carries with it a special interpretation.

Generally, a matrix having m rows and n columns can be written as follows;

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{21} & \dots & a_{2n} \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

2.4 TYPES OF MATRICES

2.4.1 VECTORS

Matrices that consist of just one row or just one column may be referred to as **VECTORS**. More specifically, any m x 1 matrix may be called a **COLUMN VECTOR** and any 1 x n matrix may be called a **ROW VECTOR**.

VEXAMPLE[-1-6 1 2]

=

is a 1 x 3 matrix or a three-component **ROW VECTOR**.

$$W = \begin{bmatrix} 1 \\ 3 \\ 5 \\ -7 \end{bmatrix}$$

is a 4 x 1 matrix or a four-component **COLUMN VECTOR**.

2.4.2 SQUARE MATRIX

A matrix that has the same number of rows as it has columns is called a **SQUARE MATRIX** or an nth-**ORDER MATRIX**.

EXAMPLE 2

$$A = \begin{bmatrix} 4 & 5 & 0 \\ 2 & 8 & -5 \\ 2 & -4 & 9 \end{bmatrix}$$

is a square matrix of size 3. The entries on the main diagonal of the matrix are: $a_{11} = 4$, $a_{22} = 8$ and $a_{33} = 9$

2.4.3 IDENTITY MATRIX

A square matrix in which all of the primary diagonal entries are ones and all of the offdiagonal entries are zeros is called an **IDENTITY MATRIX**. The identity matrix, symbolized $\mathbf{I}_n$, is used in matrix algebra in much the same way as the number one is used in regular algebra.

EXAMPLE 3 Each of the following is an **IDENTITY MATRIX**.

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that each identity matrix is a square matrix.

2.4.4 ZERO MATRIX

A ZERO MATRIX is a matrix that has zero for every entry. The zero matrix is generally denoted $\mathbf{0}_{m \times n}$ and is used in matrix algebra in much the same way that the number zero is used in regular algebra.

EXAMPLE 4

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.4.5 DIAGONAL MATRIX

Diagonal matrix is a square matrix with all elements zeros other than the main diagonal.

EXAMPLE 5

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

2.4.6 TRANSPOSE OF A MATRIX

The TRANSPOSE of an m x n matrix **A**, denoted $\mathbf{A}^t$, is the n x m matrix whose rows are the columns in **A** (in the same order) and whose columns are the rows in **A** (in the same order).

EXAMPLE 6

$$\text{If } \mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad \text{then } \mathbf{A}^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Note that $a^t_{ij} = a_{ji}$.

The transpose of a row vector is a column vector and the transpose of a column vector is a row vector.

Two matrices, **A** and **B**, are said to be **EQUAL** only if they are of the same dimenisons and if each element in **A** is identical to its corresponding element in **B**; that is, if and only if $a_{ij} = b_{ij}$ for every pair of subscripts I and j. If **A** = **B**, or if **A**≠**B**, then **B** ≠ **A**.

$A = \begin{bmatrix} & 2 \\ & 4 \end{bmatrix}$ is equal to B = $A = \begin{bmatrix} & 2 \\ 3 & 4 \end{bmatrix}$

However; $A = \begin{bmatrix} & 2 \\ & 4 \end{bmatrix}$ is not equal to C = $\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$

Even though they contain the same set of numerical values, A and C are not equal because their corresponding elements are not equal; that is, $a_{11} \neq C_{11}$ and so on.

2.5 MATRIX OPERATIONS

2.5.1 ADDITION OF MATRICES

Two matrices of the same dimensions are said to be **CONFORMABLE FOR ADDITION**. The addition is performed by adding corresponding elements from the two matrices and entering the result in the same row-column position of a new matrix.

If **A** and **B** are two matrices, each of size m xn, then the **SUM** of **A** and **B** is the m x n matrix **C** whose elements are

$$C_{ij} = a_{ij} + b_{ij} \qquad \text{for } i = 1,2,\ldots, m$$
$$j = 1,2,\ldots, n \qquad (3.1)$$

EXAMPLE 1 (a) $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 14 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} (1+7) & (2+14) \\ (-1+0) & (0+3) \end{bmatrix} = \begin{bmatrix} 8 & 16 \\ -1 & 3 \end{bmatrix}$

(b) [5 8 2] + [3 0 1]=[(5+3) (8+0) (2+1)]=[8 8 3]

$$\begin{bmatrix} 6 & 8 \\ 3 & 4 \\ -9 & 1 \end{bmatrix} + \begin{bmatrix} 7 & 11 & 12 \\ 0 & 3 & 7 \end{bmatrix}$$ is not defined because the two matrices are not of the same dimension

EXAMPLE 2 Theater-on-the-Square is presenting a special production of Faust. Ticket sales for the afternoon matinee were as follows: 331 orchestra seats, 427 loge seats, and 642 balcony seats. For the night performance, 282 orchestra seat tickets, 438 loge seat tickets, and 834 balxony seat by type, for the two performances.

We set up two vectors as follows:

M = [331 427 642] and **N** = [282 438 834]

Then, total ticket sales, by type, is given by

S=**M**+**N**=[613 865 1,476]

LAWS OF MATRIX ADDITION The operation of adding two matrices that are conformable for addition has these basic properties.

1) **A + B = B + A** (The commutative law of matrix addition)
2) **(A + B) + C = A + (B + C)** (The associativ law of matrix addition)

If A and B are matrices of the same size, using the scalar multiplication we can define the subtraction A- B by A - B= A + (-B)

EXAMPLE 3

$$\begin{pmatrix} 4 & -1 \\ 8 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 5 \\ 12 & -9 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 8 & 0 \end{pmatrix} + \begin{pmatrix} -3 & -5 \\ -12 & 9 \end{pmatrix} = \begin{pmatrix} 1 & -6 \\ -4 & 9 \end{pmatrix}$$

Note the following points

The subtraction of matrices can also be obtained by subtracting corresponding entries of A and B.

For matrix addition and subtraction to be conformable, the matrices must have same dimension.

2.5.2 MULTIPLICATION BY CONSTANT

A matrix can be multiplied by a constant by multiplying each composnent in the matrix by the constant. The result is a new matrix of the same dimensions as the original matrix.

> If **K** is any real number and **A** is an m x n matrix, then the product **KA** is defined to be the matrix whose components are given by **K** times the corresponding component of **A**; that is,
>
> $$\mathbf{KA} = [k\, a_{ij}]_{(m \times n)} \qquad (3.2)$$

EXAMPLE 4 (a) If **V** = [1 2 9], then **5V** = [(5 x 1) (5 x 2) (5 x 9)]
=[5 10 45]

(b) If

$$\mathbf{B} = \begin{bmatrix} 9 & -3 \\ 6 & 1 \\ 12 & 4 \end{bmatrix} \quad \text{then}$$

$$\mathbf{(1/3)B} = \begin{bmatrix} (1/3) \times 9 & (1/3) \times -3 \\ (1/3) \times 6 & (1/3) \times 1 \\ (1/3) \times 12 & (1/3) \times 4 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1/3 \\ 4 & 4/3 \end{bmatrix}$$

2.5.3 MULTIPLICATION OF MATRICES

Dear trainee, we can multiply two or more matrices in similar way we do for real numbers. Let’s first see how to multiply a row vector and a column vector. This is termed as a dot product of two special matrices. A 1xn row matrix and an nx1 column matrix which is a real number obtained as follow:

$$(a_1\ a_2\ a_3\ \cdots\ a_n)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n$$

Let us begin by considering the multiplication of two vectors, one a row vector and the other a column vector, but each with the same number of components

EXAMPLE 1

Consider the product of the following two matrices.

$$(2\ 7\ 9\ {-5})\begin{pmatrix} 3 \\ 1 \\ -2 \\ -4 \end{pmatrix} = (2x3 + 7x1 + 9x-2 + -4x-5) = 6 + 7 + -18 + 20 = 15$$

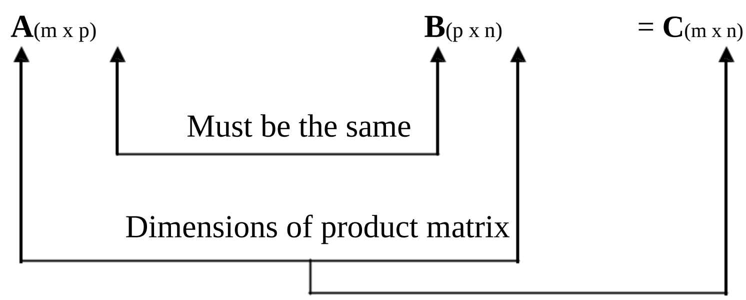
Dear learner, note that he product of a row vector and a column vector is a real number.

The product of an n-component row vector **A** and an n-component column vector **B** is the number **C**, defined as $\mathbf{C} = \mathbf{AB} = [a_1 b_1 + a_2 b_2 + \ldots + a_n b_n]$ (3.3)

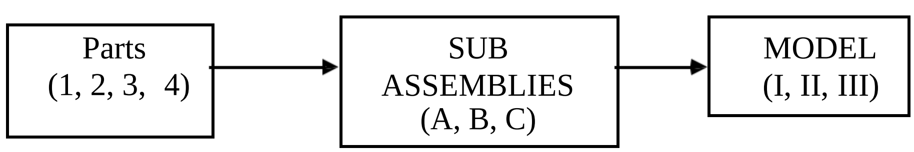
Note these important things about the procedure. **A ROW IS MULTIPLIED IN TO A COLUMN**. In an orderly progression, the first element in the in the row is multiplied times the first element in the column, the second element in the row times the second element in the column, and so on until the nth row element is multiplied times the nth column element. These products ate then summed to obtain the single number that is the product of the two vectors. Clearly, the procedure can be executed only if the row vector contains the same number of components as the column vector.

Now let us multiply two m x n matrices. If **A** and **B** are two matrices, the product **AB** is defined if and only if the number of columns in **A** is equal to the number of rows in **B**. If this requirement is met, **A** is said to be **CONFORMABLE** to **B FOR MULTIPLICATION**. The matrix resulting from the multiplication has dimensions equivalent to the number of rows in **A** and the number of columns in **B**. Thus, for example, a 2 x 3 matrix can be multiplied by a 3 x 4 matrix and the product matrix will be of dimensions 2 x 4. However, multiplication of a 3 x 4 matrix by a 2 x 2 matrix is not defined.

The following schematic illustrates the dimensional requirements for the multiplication = **AB**.



EXAMPLE 2 A manufacturers of microcomputers produces three final models, **I**, **II**, and **III**, which are made from three subassemblies, **A**, **B**, and **C**. Furthermore, each subassembly is made from parts 1, 2, 3, and 4.



Model **I** requires two subassemblies **A** and two subassemblies **C**. Model **II** requires one each of subassemblies **A** and **C** two subassemblies **B**. Model **III** requires four subassemblies **B** and one subassembly **C**. This information can be organized in matrix format, as

$$\mathbf{S} = \begin{array}{c} \begin{array}{ccc} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{array} \\ \left[\begin{array}{ccc} 2 & 0 & 3 \\ 1 & 2 & 1 \\ 0 & 4 & 1 \end{array}\right] \end{array} \begin{array}{c} \\ \mathbf{I} \\ \mathbf{II} \\ \mathbf{III} \end{array}$$

In a similar manner, the number of each different type of part required for each subassembly can be organized in matrix format. This information is as follows:

$$\mathbf{P} = \begin{array}{c} \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 1 & 4 & 0 \\ 0 & 3 & 1 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \end{array}$$

The matrix **SP**, obtained by multiplying the subassembly matrix **S** times the parts matrix **P**, gives the number of units of each part required in the production of one unit of each of the final products.

$$\mathbf{SP} = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 2 & 1 \\ 0 & 4 & 1 \end{bmatrix} \text{x} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 1 & 4 & 0 \\ 0 & 3 & 1 & 2 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{bmatrix} 2 & 13 & 3 & 8 \\ 7 & 7 & 9 & 3 \\ 12 & 7 & 17 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{I} \\ \mathbf{II} \\ \mathbf{III} \end{array}$$

Notice that in multiplying row 1 of **S** rimes column 1 of **P**, these computations are made: [(number of units of subassembly **A** in one unit of product **I**) x (number of units of part 1 in one unit of subassembly **A**)] + [(number of units of subassembly **B** in one unit of product **I**) x (number of units of part 1 in one unit of subassembly **B**)] + [(number of units of subassembly **C** in one unit of product **I**) x (number of units of part 1 in one unit of subassembly **C**)] = number of units of part 1 in each unit of product 1. In matrix multiplications give meaningful result in the context of the problem situation.

Other numbers in the **SP** matrix have a similar meaning. Thus from the first row of this matrix we read that each unit of Model **I** requires 2 units of part 1, 13 units of part 2, 3 units of part 3, and 8 units of part 4. The entries on the second row of the matrix give the number of units of each type of part required in the manufacture of one unit of Model **II**. The third row gives the number of units of each part required by one unit of Model **III**.

Now, the firm has just received an order for six units of Model **I**, three units of Model **II**, and four units of Model **III** and needs to know the number of units of each type of part required to fill the order. An order quantity vector can be established as

$$\begin{array}{cccc} & \mathbf{I} & \mathbf{II} & \mathbf{III} \\ \mathbf{Q}= & [6 & 3 & 4] \end{array}$$

By multiplying **Q** times **SP**, we obtain the desired information, as follows:

$$\mathbf{Q\ (SP)} = \begin{array}{ccc} \mathbf{I} & \mathbf{II} & \mathbf{III} \\ [6 & 3 & 4] \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \end{array} \\ \begin{bmatrix} 2 & 13 & 3 & 8 \\ 7 & 7 & 9 & 3 \\ 12 & 7 & 17 & 2 \end{bmatrix} \end{array} \begin{array}{c} \mathbf{I} \\ \mathbf{II} \\ \mathbf{III} \end{array} = \begin{array}{cccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ [81 & 127 & 113 & 65] \end{array}$$

We see, thus, that 81 units of part 1, 127 units of part 2, 113 units of part 3, and 65 units of part 4 will be required to manufacture the items included in the order.

The latest price catalog shows that each unit of part 1 costs Br.4.50, each unit of part 2 costs Br.9.00, each unit of part 3 costs Br.7.25, and each unit of part 4 costs Br.12.00. With this information we can compute the total parts cost for the order. Costs per unit are set up in a vector, as

$$\mathbf{C} = \begin{bmatrix} 4.50 \\ 9.00 \\ 7.25 \\ 12.00 \end{bmatrix}$$

Then, total cost is determined by

$$\mathbf{T} = \mathbf{(QSP)\ C} = [81 \quad 127 \quad 113 \quad 65] \begin{bmatrix} 4.50 \\ 9.00 \\ 7.25 \\ 12.00 \end{bmatrix} = [3106.75]$$

Parts needed to fill the order will have a total cost of Br.3, 106.75.

AB and BA are the same size matrices, but they are not equal; that is AB # BA. Thus, we say that matrix multiplication is not commutative. But matrix multiplication has some of the same type of properties as multiplication of real numbers. If all products and sums are defined for the indicated matrices A, B and C and K is a real number then;

1. (AB) C = A (BC) Associative property
2. A(B+C)=AB+AC Left- hand distributive property
3. (B+C)A=BA+CA Right – hand distributive property

4. k (AB) = (KA)B = A(AB)

Since matrix multiplication is not commutative, properties 2 and 3 must be listed as distinct properties.

SELF CHECK EXERCISES

2.1 1. Given

$\mathbf{A}=[1 \quad 2 \quad 3]$ and $\mathbf{B}=\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

Find the product such that **AB** = **C**.

2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$

Find i) AB ii) BA iii) CB iv) BC

3. Let $A = \begin{bmatrix} 3 & 4 \\ 2 & -5 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 8 & 1 & 4 \\ 2 & -19 & \end{bmatrix}$

Find i) 3A-2B ii) 2AB iii) AB

Almaz purchased 6 cases of soft drinks, 18 pounds of hot dogs, 12 packages of hotdog buns, 3 jars of mustard, and 4 jars of relish. Soft drinks cost Br.5 a case, hot dogs cost Br. 2.25 a pound, buns cost Br. 0.90 a package, mustard Br.1.90 a jar, and relish Br.1.75 a jar. Use vector multiplication to determine the total amount Almaz spent.

decision problem is a problem in which one must choose between several alternatives. The following is an example of a decision problem. Ato Abebe wants to have kitfo for his lunch and he has four places where he can eat his lunch. The first place charges Birr 11 for kitfo, Birr 3 for a beer and 60c for a coffee. The second place charges him birr 10 for kitfo, birr 4 for birr, and 80 cents for coffee. The third place charges Birr 10 for Kitfo, Birr 3.50 for a beer and 75c for a coffee. The fourth place charges Birr 11 for kitfo, Birr 2.75 for a beer and Birr 1 for a coffee. Express the above information in a 4x3 matrix. If Ato Abebe wishes to have a Kitfo, 2 beers and a coffee for his lunch, use matrix multiplication to find the cost of lunch at each place. Assuming, that Ato Abebe has no preference for any of the places; decide where he should eat to spend the least amount of money.

he following matrix M and S describe an automobile dealer’s inventory on Meskerem 1 and the sales for the month of Meskerem, respectively.

$$M = \begin{bmatrix} 26 & 43 & 52 & 40 \\ 31 & 25 & 50 & 46 \\ 35 & 29 & 63 & 41 \end{bmatrix} \qquad S = \begin{bmatrix} 5 & 9 & 5 & 10 \\ 6 & 5 & 11 & 12 \\ 7 & 9 & 13 & 4 \end{bmatrix}$$

On Tikimt 1 a new shipment arrives. The numbers in this shipment are given by the matrix T

$$T = \begin{bmatrix} 35 & 19 & 50 & 60 \\ 16 & 35 & 52 & 72 \\ 76 & 90 & 33 & 24 \end{bmatrix}$$

Describe the inventory matrix after the Tikemet delivery.

7. A department store has two warehouses and three branch sales centers to be supplied. The transportation costs per quintal from the warehouses to the branches are given in the following table.

	Branch 1	Branch2	Branch 3
Warehouse A	Birr 6	Birr 4	Birr 3
Warehouse B	Birr 2	Birr 3	Birr 8

Suppose if the number of quintals to be transported from warehouse A to branch 1 branch 2 and branch 3, are 16 quintals, 40 quintals, and 23 quintals respectively; and from warehouse B to branch 1 and branch 2 are 20 quintals and , 35 quintals respectively,

Set up the column matrices A and B which shows the number of quintals transported from each warehouse to the branches

Use matrix multiplication to determine the total cost of transportation from each warehouse to the branches.

A manufacturer produces two models of ladies shoes. Model 1 and model 2. The productions of both models of shoes of shoes require 3 different raw materials. A pair of model 1 shoe requires 3 units of material A, two units of raw material B and one unit of material C.A pair of model 2 shoe requires one unit of material A, 2 units of material B, and 1 unit of material C. Using matrix multiplication find the number of materials required to produce 130 model 1 shoes and 98 model 2 shoes.

2.6 INVERSE OF A MATRIX

The material of this section refers to square matrices; that is, matrices which have the same number of rows as columns. A square matrix with n rows and n columns is said to be of order n.

UNIT MATRIX

A UNIT (OR IDENTITY) matrix is a square matrix whose diagonal elements from upper left to lower right are each 1, and whose other elements are all zeros.

Thus,

$$I = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

is a unit matrix of order 3. The unit matrix of order 2 is

$$I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

The most important property of the unit matrix is illustrated by the statements

$$\mathbf{AI = A \quad and \quad IA = A}$$

That is, the product of any given matrix and the unit matrix is the given matrix itself. In other words, the unit matrix behaves like the number 1 in ordinary arithmetic where we say

$$(a)\ (1) = (1)\ (a) = a$$

The reader may verify the unity property of I by carrying out the following multiplication in which the first written matrix, I, when multiplied by the second matrix, yields the second matrix as the product.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} & 3 & 4 \\ & -2 & 0 \\ & 2 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 4 \\ -1 & -2 & 0 \\ 5 & 2 & -3 \end{vmatrix}$$

If for a given square matrix A, another square matrix B of the same order is obtained such that AB = BA = I

Then matrix B is called inverse of A and is denoted by B = A^{-1}. Thus, $AA^{-1} = A^{-1}A = I$

Dear students, at this juncture you need to bear in mind some features of the inverse of square matrix.

If the inverse of square matrix is multiplied by the original matrix, the result is an identity matrix.

If the inverse of square matrix exists, it is unique.

The inverse of the transpose of a matrix is the transpose of its inverse i.e., $(A')^{-1} = (A^{-1})'$

2.6.1 Row Operations

There are **THREE** types of row operations:

1. Multiplying (dividing) a row by a non zero constant,

Example 1: Given the matrix

$$A = \begin{vmatrix} 1 & 2 \\ 3 & 9 \end{vmatrix}$$

We may multiply the first row by -2 to give another matrix

$$D= \begin{vmatrix} -2 & -4 \\ 9 & \end{vmatrix}$$

Multiplying (dividing) a row by a non zero constant and adding (subtracting) the result to (from) another row.

EXAMPLE 2: Consider the following two matrices:

$$E = \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix} ; \quad F = \begin{bmatrix} 1 & 2 \\ 5 & 13 \end{bmatrix}$$

Matrix F is obtained from matrix E by multiplying the first row of matrix E by 2 and adding the result to the second row.

INTERCHANGING OF ANY TWO ROWS.

EXAMPLE 3: Consider the following two matrices:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} ; \quad B = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

Matrix B is obtained from matrix A by Interchanging of first and second rows.

EXAMPLE 4: What row operation performed on

$$A = \begin{bmatrix} 2 & 5 \\ 6 & 13 \end{bmatrix}$$

will lead to another matrix which will have a 0 in place of the 6, and what will the new matrix be?

SOLUTION: Multiply the first row of the given matrix by -3 and add to the second row, obtaining.

$$B = \begin{bmatrix} 2 & 5 \\ 0 & -2 \end{bmatrix}$$

If we have a statement of matrix equality, such as **AB = C** and we perform the same row operations on A and C (not on B and C), we are led to new statements of equality. For example,

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ - & 5 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -3 & 15 \end{bmatrix}$$

is a true statement. If we multiply the first row of the leftmost matrix by 3 and the first row of the right most matrix by 3, we obtain

$$\begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ - & 5 \end{bmatrix} = \begin{bmatrix} 0 & 18 \\ -3 & 15 \end{bmatrix}$$

which the reader may verify is also a true statement,. Or, starting over, if we take the leftmost and the rightmost matrices and add twice the first row to the second row, we obtain

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} & -4 \\ & 5 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -3 & 27 \end{bmatrix}$$

which computation will prove to be a true statement.

2.6.2 INVERSION BY GAUSSIAN ELIMINATION

The inverse of a matrix can be found by the method known as Gaussian elimination. This is particularly suitable when the elements of the matrix are numbers. The method involves augmenting the matrix to be inverted by including the identity matrix next to it. The row operations are applied to both matrices until the original matrix is transformed in to the unit matrix, whereupon the identity matrix will have been transformed in to the inverse matrix. The permitted row operations are called **elementary row operations**. The row operation includes

Row Operation **1**: Interchanging of two rows, denoted by $R_i \leftrightarrow R_j$

Row Operation **2**: Multiply (divide) a row by a nonzero constant, denoted by $kR_i \rightarrow R_i$

Row Operation **3**: Add (subtract) a nonzero multiple of one row to another row, Denoted by, $R_i + KR_j \rightarrow R_i$

$\{$*Note* : *The arrow* $\rightarrow$ *means* "*replace*."$\}$

The **Gauss-Jordan Elimination method** uses these row operations to transform the augmented matrix into an equivalent matrix that represents equivalent systems of equations. For example, to find the inverse of

The inverse of a matrix A is another matrix, written as A^{-1}, such that the product of the two is the unit matrix. Thus,

$$\mathbf{AA^{-1} = I \text{ and } A^{-1} A = I}$$

The symbol A is read as “A inverse.” The -1 superscript is not an exponent in the usual algebraic sense. We can verify that two matrices are inverses by carrying out the multiplication specified in the definition. Thus, the two leftmost matrices in the following are inverses because their product is the unit matrix.

$$\begin{vmatrix} 8 & 5 \\ 3 & 2 \end{vmatrix} \begin{vmatrix} & -5 \\ & 8 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

EXAMPLE 1: Verify that the following matrices are inverses of each other

$$A = \begin{vmatrix} 13 & 3 \\ 4 & 1 \end{vmatrix} \qquad B = \begin{vmatrix} 1 & -3 \\ -4 & 13 \end{vmatrix}$$

SOLUTION: Multiplication of the two matrices leads to the unit matrix

$$A = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

We shall see later that not every matrix has an inverse. However, if the inverse of a does exist, it is unique: that is, a matrix which has an inverse has exactly one inverse. The uniqueness of the inverse may not seem obvious, so, let us see what would happen if in addition to A^{-1}, A had another inverse, call it B. Then, according to the definition of the inverse, the product AB would have to be I; that is,

$$AB=I$$

Now if we multiply both sides of this expression by A^{-1}, we have

$$A^{-1} AB = A^{-1} I$$

Noting that on the left $A^{-1}A$ can be replaced by I, and then IB can be replaced by B, and on the right A^{-1} I can be replaced by A^{-1}, we have,

$$B=A^{-1}$$

So, the supposedly different inverse, B, turns out to be the original inverse, A^{-1}, we shall make use of the uniqueness of the inverse a little later when we will ask what the expression (?) must be to make the following a true statement.

$$I=(?)A$$

The answer will be that (?) must be A^{-1}.

We are now ready to attack the main objective of this section of our study, which is to compute the inverse of a given matrix. After we have learned the computational procedures, we shall discover that they are directly applicable to the solution of n by n systems of linear equations.

Briefly, the inverse of a given matrix can be found by writing the given matrix at the left, and the corresponding unit matrix next to it, at the right.

Then select and carry out row operations which will convert the given matrix in to the unit matrix, and apply the dame operations to the matrix at the right. When the left (given becomes the unit matrix, the matrix on the right will be the desired inverse. To illustrate, let us find the inverse of the matrix.

$$C = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}$$

We start by writing the given matrix to the left and the unit matrix to the right; thus

$$\left|\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right|$$

To change the left matrix into the unit matrix will require several steps (which we perform on both matrices). We start by getting a 1 in the upper left corner. Divided the first row above by 3 to obtain.

$$\left|\begin{array}{cc|cc} 1 & 2/3 & 1/3 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right|$$

Next, get a o in the lower left corner by multiplying the first row above by -1 and adding to the second row to obtain

$$\left|\begin{array}{cc|cc} 1 & 2/3 & 1/3 & 0 \\ 1 & 1/3 & -1/3 & 1 \end{array}\right|$$

multiply the second row, just above, by 3 to obtain

$$\left|\begin{array}{cc|cc} 1 & 2/3 & 1/3 & 0 \\ 0 & 1 & -1 & 3 \end{array}\right|$$

Finally, multiply the second row, just above, by -2/3 and add to the first to obtain

$$\left|\begin{array}{cc|cc} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 3 \end{array}\right|$$

We have found the inverse of

$$C = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}$$

it is the matrix at the right; namely,

$$C^{-1} = \begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix}$$

BASIC GENERAL PROCEDURE RECOMMENDED:

Working one column at time and using the appropriate row operations, first get the 1 into the correct location (with respect to the identify matrix) within that column. The 1 can be obtained in the cell of the matrix by multiplying by the reciprocal of the number currently in that cell. (Row operation type 1).

Then use this 1 to obtain zeros in all other locations within that column. Only after once column is in the correct format should you move to the next column. The zero can be obtained in a cell by using an appropriate multiple of the row with the 1 in that column. (Row Operation type 2).

3. Depending on the objective and the situation we may also interchange two rows. (Row operation type 3).

SELF CHECK EXERCISES 2.2

A firm manufacturing office furniture finds that it has the following variable costs, in dollars

	Desk	Chair	Table	Cabinet	
	50	20	15	25	**Material**
V=	30	15	12	15	**Labor**
	30	15	8	20	**Overhead**

That is, the manufacture of each desk requires materials costing Br.50, labor costing Br.30, and overhead of Br.30. Each chair requires material costing Br.20, and so on.

An order for 5 desks, 6 chairs, 4 tables, and 12 cabinets has just been received. This information is organized in a column vector, as

	5	Desks
Q =	6	Chairs
	4	Tables
	12	Cabinets

REQUIRED: Compute the total material, total labor, and total overhead cost associated with filling this order can be computed as follows:

2. Apply the procedure and find the inverse of the matrix.

C= 7 3

1

Apply the procedure and find the inverse of the matrix.

2 3 1

4 2

6 4

A company packs two types of items, A and B, in three boxes of different sizes: small, medium and large. The small size box can contain 1 unit of item A and 2 units of item B. Each medium size box can contain 3 units of item A and 1 unit of item B. While the large size box can contain 2 units of item A and 4 units of item B. If a company has 30 units of item A and 25 units of item B find the possible number of each size of boxes required to pack the items.

A second hand car dealer orders 8 large cars, 26 intermediate cars, and 36 small cars. There are three types of cars that can be used to transport the cars from the port to the city. Trucks of type A carry 2 large cars, 2 intermediate cars, and 2 small cars. Trucks of type B carry 6 intermediate cars. Trucks of type C carry 7 small cars. How many type A trucks, type B trucks, and type C trucks will be needed to transport the ordered cars.

A grocery supplies three different types of Wine: gudar, Axumite and Crystal. A retailer wants to purchase a total of 600 bottles. The number of Gudar bottles he wishes to purchase is equal to the sum of the number of bottles of Axumite and Crystal. And the number of bottles of Axumite is twice the number of bottles of Crystal. Determine how many bottles of each type of Wine he would buy.

A factory produces two alloys, A and B. Each ton of allowy A requires 4 units of metal 1 and 4 units of metal 2 Each ton of alloy B requires 7 units of metal 1 and 3 units of metal 2. How many tons of each alloy can be produced in a day if the factory obtains and uses;

60 units of metal 1 and 40 units of metal 2

44 units of metal 1 and 36 units of metal 2. (use inverse method)

A company manufactures three different types of calculators and classifies them as small medium, and large according to their calculating capacity. The following table gives the production requirements for each type of calculators.

	Small	Medium	Large
Assembly time	1 hr.	3 hrs.	
Electronic components	5 units	7 units	8 units

If the company has 30 hours of labor and 90 units of electronic components how many of each type of calculators can be produced?

To produce two different items three types of raw materials are needed for each item. The number of units of raw materials needed to produce one unit of each of the two items is given below

	Type 1	Type 2	Type 3
Item A	3	1	2
Item B	2	1	1

How many units of each item can be produced if the numbers of units of raw materials available are 56 of type 1.22 of type 2 and 34 units of type 3.

2.6.3 GAUSS- JORDAN METHOD OF SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS

Using matrix algebra any systems of linear equations can be expressed as, Ax = B where A is a coefficient matrix, X is a variable matrix and B is the constant matrix or constant vector.

Dear trainees, the elimination of variable procedure is generally satisfactory for solving systems of two equations in two variables. However, for systems containing more than two equations and involving more than two variables, this procedure is not as efficient as we would like; so we find ourselves turning to another solution technique.

The alternative solution procedure we shall use employs a **MATRIX FORMAT** to organize the data.

The two linear equations of the system

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

can be represented by the **MATRIX EQUATION AX = B** where

$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is called the **COEFFICIENT MATRIX,**

$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the **SOLUTION VECTOR,** or **VECTOR OF UNKNOWNS,**

$\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is called the **RIGHT-HAND-SIDE VECTOR,** or **VECTOR OF CONSTANTS.**

The vector X = $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ represents a solution of the system if both equations are satisfied when we substitute $x_1 = n_1$ and $x_2 = n_2$.

The Gaussian Method: One “matrix” procedure that can be used to solve systems of linear equations is known as the **GAUSSIAN METHOD.** This procedure was develoed by the mathematician Karl F. Gauss (1777-1855).

We begin with the augmented matrix representation [**A| B**] of the original system of equations. Then we systematically reduce the augmented matrix to equivalent augmented matrices in simpler form. We continue until we have reached the simplest possible augmented matrix representation, one from which the solution of the system can be obtained by inspection. Indeed, we seek to transform the augmented matrix [**A| B**] into the augmented matrix [**I|X**], where **I** is the identity matrix and **X** is the solution vector.

The operations that may be performed on the augmented matrices to convert them to simpler form are the **ELEMENTARY ROW OPERATIONS (ERO$_S$).** The elementary row operation may consist of the following:

Operations permitted on the rows of a matrix are called **ELEMENTARY ROW OPERATIONS.** They are as follows:

Type I. Any pair of rows in a matrix may be interchanged. (**EXCHANGE** operation) **Type II.** A row can be multiplied by any nonzero real number. (**MULTIPLY** operation) **Type III.**A multiple of any row can be added to any other row. (**ADD A-MULTIPLE** operation)

Again, it is recommended that we follow the same step-by-step procedure for obtaining the identity matrix format that we used when calculating an inverse. That is, working one column at a time, obtain a one in the correct cell using **ERO** Type **II**, then; obtain zeros in all other cells of the column using **ERO** Type **III** and multiples of the row which has the one.

Recall that two linear systems are said to be equivalent if they have exactly the same solution set. Similarly, we say that two augmented matrices are equivalent, denoted by the symbol placed between the two matrices, if they are augmented matrices of equivalent systems of equations.

CASE 1: WHEN NUMBER OF EQUATION IS EQUAL TO NUMBER OF UNKNOWN"

Apply elementary row operations to reduce matrix A to an identity matrix. This can be done by working with one column at a time. First to get a '1' in the proper position (a_{ii}) and then by adding a multiple of this row to the other rows to get '0s' in all other positions within that column.

EXAMPLE 1 Using the Gaussian procedure, find the solution to the system of equations

$$2x + y = 60$$
$$x + 3y = 105$$

We set up the augmented matrix tableau

$$\left[\begin{array}{cc|c} 2 & 1 & 60 \\ 1 & 3 & 105 \end{array}\right]$$

To obtain a 1 in the row one-column one cell, we multiply row one by ½, to obtain

$$\left[\begin{array}{cc|c} 1 & ½ & 30 \\ 1 & 3 & 105 \end{array}\right] \quad R_1 \leftarrow (½)R_1$$

Now, a zero is required in the row two-column one cell. Employing **ERO** Type III, we subtract row one from row two, to obtain

$$\left[\begin{array}{cc|c} 1 & ½ & 30 \\ 0 & ½ & 75 \end{array}\right] \quad R_2 \leftarrow R_2 - R_1$$

Column one is now in the identity matrix format, and we turn our attention to column two. To obtain a 1 in the row two-column two cell, we multiply row two by , as

$$\left[\begin{array}{cc|c} 1 & ½ & 30 \\ 0 & 1 & 30 \end{array}\right] \quad R_2 \leftarrow (½)\, R_2$$

Finally, to obtain a 0 in the row one-column two cell, we subtract ½ times row two from row one, as

$$\left[\begin{array}{cc|c} 1 & 0 & 15 \\ 0 & 1 & 30 \end{array}\right] \quad R_1 \leftarrow R_1 - (½)\, R_2$$

The solution vector is

$$X = \begin{bmatrix} 15 \\ 30 \end{bmatrix}$$ which is interpreted as x = 15, y = 30.

EXAMPLE 2: Solve the following system of linear equations Using Gauss Jordan method

$$2x+ 2y + 4y = 13$$

$$X - y + 3z = 8$$

$$3x + 3y + 6z = 20$$

SOLUTION: Apply elementary row operations to the augmented matrix of the system

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 13 \\ 1 & -1 & 3 & 8 \\ 3 & 3 & 6 & 20 \end{array}\right]$$

$$R_1 \leftrightarrow R_2 = \left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 2 & 2 & 4 & 13 \\ 3 & 3 & 6 & 20 \end{array}\right]$$

$$R_2\text{-}2R_1 \rightarrow R_2 = \left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 0 & 4 & -2 & -3 \\ 0 & 6 & -3 & -4 \end{array}\right]$$

$R_3\text{-}3R_1 \rightarrow R_3$

$$\frac{1}{4} R_2 \rightarrow R_2 = \left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{4} \\ 0 & 6 & -3 & -4 \end{array}\right]$$

$$R1+R2 \rightarrow R_1 = \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{2} & \frac{29}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{4} \\ 0 & 0 & 0 & -\frac{1}{2} \end{array}\right]$$

$R3 - 6R2 \rightarrow R_3$

When we write the equation corresponding to the last row of this final augmented matrix we get a false statement 0 = 1/2. Hence, the system has no solution.

CASE 2: WHEN NUMBER OF EQUATIONS IS NOT EQUAL TO NUMBER OF UNKNOWNS

In this case the coefficient matrix A is not a square matrix. However, working one column at a time we can apply elementary row operations to the augmented matrix $[A/B]$, to get the number 1 in the appropriate position (a_{ii}), we then add multiple of the row which contains this 1 to the other rows to get 0s in all other positions within that column. Continue this process, until it is not possible to go further. If at any point in the process we obtain a row that is all zeros except in the constant column, as example 4 above, we can stop, since the system has no solution.

EXAMPLE 1: Solve the following 3 by 2 systems of linear equations Gauss Jordan method

$$2x + 3y = 1$$

$$3x - y = 7$$

$$2x - y = 5$$

SOLUTION: Apply elementary row operations to the augmented matrix of the system

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 3 & -1 & 7 \\ 2 & -1 & 5 \end{array}\right]$$

$$\frac{1}{2} R_1 \rightarrow R_1 = \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{1}{2} \\ 3 & -1 & 7 \\ 2 & -1 & 5 \end{array}\right]$$

$R_2 - 3R_1 \rightarrow R_2 =$

$$R_3 - 2R_1 \rightarrow R3 = \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{11}{2} & \frac{11}{2} \\ 2 & -4 & 4 \end{array}\right]$$

$$-2/11R2 \rightarrow R2 = \left(\begin{array}{cc|c} 1 & 3/2 & 1/2 \\ 0 & 1 & -1 \\ 2 & -4 & 4 \end{array}\right)$$

$$R_1 - \frac{3}{2}R_2 \rightarrow R_1$$

$$R_3 + 4R_2 \rightarrow R_3 \quad = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

At this point we stop the process since there is no entry in the third row third column. In fact, there is no third column. The linear system corresponding to this last augmented matrix is;

$$X + 0y = 2$$

$$0x + y = -1$$

$$0 = 0$$

Thus the system has a unique solution given by x = 2 and y = -1

REMARK:

Dear learner, note that a system of linear equations with more variables than equations can't have a unique solution.

APPLICATION EXAMPLE

A furniture company makes tables, beds and chairs along with other furniture. Each of which requires skilled labor and to be processed on a machine. Each table requires 3 hours on the machine and 1 hour of skilled labor. Each bed requires 2 hours on the machine and 3 hours of skilled labor and; each chair requires 1 hour on the machine and 2 hours of skilled labor. Find the number of each type of furniture to be made if the number of hours available on the machine is 23 hours; and the skilled labor available is 26 hours.

SOLUTION:

First we assign variables to each of the unknown quantities for which we are required to solve. Thus in our example we let

X= Number of tables to be made

Y= Number of beds to be made

Z = Number of chairs to be made

Then we write the equations by noting that the number of hours required on the machine to make x tables is 3x hours, y beds is 2y hours, and z chairs is z hours; thus 3x + 2y + z must equal the time available on the machine, which is 23 hours. Similarly x + 3y + 2z must equal the skilled labor hours available, which is 26 hours. Hence, we have the following 2 by 3 system,

$3x + 2y + z = 23$

$X + 3y + 2z = 26$

Apply elementary row operation to the augmented matrix of the system

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 23 \\ 1 & 3 & 2 & 26 \end{array}\right]$$

$$R1 \leftrightarrow R_2 = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 26 \\ 3 & 2 & 1 & 23 \end{array}\right]$$

$$= -3R1 + R2 = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 26 \\ 0 & -7 & -5 & -55 \end{array}\right]$$

$$-\frac{1}{7}R_2 \rightarrow R_2 = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 26 \\ 0 & 1 & \frac{5}{7} & \frac{55}{7} \end{array}\right]$$

$$-3R_2+R_1 \rightarrow R_1 = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{7} & \frac{17}{7} \\ 0 & 1 & \frac{5}{7} & \frac{55}{7} \end{array}\right]$$

We have an unlimited number of solutions given by

X- 1/7 Z = 17/7 $\leftrightarrow$ X = 17/7 + Z/7

Y+5/7Z=55/7 $\leftrightarrow$ Y=55/7-5/7Z

If we let z = k then (17/7 + k/7, 55/7- 5/7k, k) is the general solution. However, we cannot produce a negative number of chairs, tables or beds. If we also assume we don’t need fractional number of furniture, then k must be a nonnegative integer.

From the first equation, x = 17/7 + k/7 k, can assume the values 4, 11, 18, 25,…

From the second equation k can only assume the values 4 and 11. Hence the values of k that can satisfy both the equations are 4 and 11.

Thus, for k = 4 we have x = 3 tables, y = 5 beds and z = 4 chairs;

For k = 11, we have x = 4 tables, y = 0 beds and z = 11 chairs.

These are the only possible number of furniture that can be produced at the given available time.

2.6.4 USING THE INVERSE TO SOLVE A SYSTEM OF EQUATIONS

Given an n × n system of linear equations, represented in matrix notation as **AX** = **B**, we may be able to find the solution vector using the inverse of **A.** If the coefficient matrix **A** is invertible, the system has a unique solution which is given by

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

The result stems from the fact that, if **A** has an inverse, both sides of the matrix equation can be multiplied by that inverse, as

$\mathbf{AX}=\mathbf{B}$

$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$

$(\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

$\mathbf{IX} = \mathbf{A}^{-1}\mathbf{B}$

$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

EXAMPLE 1 $x_1 + 3x_2 = 16$

$2x_1 + 2x_2 = 16$

The inverse of the coefficient matrix

A= $\begin{bmatrix} 1 & 3 \\ & 2 \end{bmatrix}$ is found by using elementary row operations to covert the augmented matrix [**A** | **I**] into the format [**I** | $\mathbf{A}^{-1}$]. This inverse is

$$\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

This inverse can be used to determine the solution to the system. Accordingly the solution is given as follows:

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 16 \\ 16 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} (-\frac{1}{2} \times 16) + (\frac{3}{4} \times 16) \\ (\frac{1}{2} \times 16) + (-\frac{1}{4} \times 16) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

When the information becomes that 18 for the right hand side value for the first equation and 16 as it is for the second one, the solution can easily be determined by introducing tha new change to the solution procedure as given below.

$$\begin{vmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{vmatrix} \qquad \begin{bmatrix} 18 \\ 16 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{matrix} = \\ = \end{matrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

The inverse could be used in this manner to determine new solution.

SELF CHECK EXERCISES 2.3

Solve the following linear equation system using Gauss Jordan method,

$4x + 6y = 8$

$6x + 9y = 12$

Solve the following 2 by 3 system of linear equations Using Gauss

Jordan $X+Y-2Z=8$

$2X-Y+2Z=7$

A factory produces two types of products: A and B. Each product must pass through three processing machines M-1, M-2 and M-3. The following table shows the number of hours each product requires on each machine.

	M – 1	M – 1	M – 1
Product A	1	1	3
Product B	1	2	1

Using the information in the table, determine the number of products that can be produced, if the number of hours available per day on M-1, M-2 and M-3 are 8 hours, 10 hours, and 20 hours, respectively.

2.7 Markov Chains: Concepts, Models and Solutions

2.7. 1 Markov Chain

Suppose two companies, company A and company B, are producing the same product. Because of a promotion campaign, buyers are switching between the product produced by company A and company B. Assume that 10% of those who buy company A's product buy from company B next time and 40% of those buying from company B buy from company A next time. What will be the market share of each company after a given period of time?

Using Markov chains, we will learn the answers to such question. The answers to the question will help to study the success of these companies in terms of their business.

A Markov chain is a sequence of random values whose probabilities for a next state depend only upon the present state, and not on any past states. Markov chains are named after the Russian mathematician A.A. Markov (1856-1922)

The Markov chain process consists of a finite number of states and some known probabilities p_{ij} is the probability of moving from state i to state j. In the example above, we have two states: buying the product produced by company A and company B. the number p_{ij} represents the probability of moving from buying the product produced by company A to the product produced by company B in the next transition state. The matrix p that represents these transition probabilities is called the transition matrix of the Markov chain. In this matrix, the states are indicated at the side and the top. The transition matrix for the example given above, thus is

$$\begin{array}{cc} & \begin{array}{cc}\text{Company A} & \text{Company B}\end{array} \\ \begin{array}{c}\text{Company A} \\ \text{Company B}\end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

A transition matrix has the following features:

It is square, since all possible states must be used both as rows and as columns.

All entries are between o and 1, inclusive; this is because all entries represent probabilities.

The sum of the entries in any row must be 1, since the numbers give the probability of moving from the state at the left to one of the states indicated across the top and the states are mutually exclusive and collectively exhaustive.

To answer the question, what the market share of each company will be at a given period, we need first to define the state vector. For a Markov Chain, which has k states, the state vector for a certain observation period is a row vector given by

$$X = (x_1, x_2, \ldots x_k)$$

Where xi = probability that the system is in the i^{th} state at the time of observation.

The sum of the entries of the state vector has to be one ($x_1 + x_2 + \ldots + x_k = 1$). Such type of vector is called **a probability vector**. A probability vector is a matrix of only one row, having nonnegative entries, with the sum of the entries equal to one.

Suppose currently company A has 70% and company B has 30% of the market for the product hence, the **initial state** vector is given by, X= $[0.7 \ 0.3]$

In the next observation period, say end of the first month, if we wish to find the market share of each company, the state vector can be found by multiplying the initial state vector X and the transition matrix p

$$XP = [0.7 \quad 0.3] \begin{bmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{bmatrix} = [.75 \ .25]$$

At the end of one month period Company A will have 75% of the market for the product. Note that 90% of the 70% who bought company A's product one month will buy it the next month and an additional 40% of the 30% who bought company B's product one month will buy company A's product next month. This sum can be found by the dot product 0.7(0.9) + 0.3 (0.4) = 0.75%. In the same manner the market share for company B's product is 0.7(0.1) + 0.3 (0.6) = 0.25 = 25%

In a similar way, the market share of the two companies after two months is given by the state vector

$$[0.75 \;\; 0.25] \begin{bmatrix} 0.9 & 0.1 \\ 0.4 & 0.6 \end{bmatrix} = [0.775 \;\; 0.225]$$

This state vector $[0.775 \; 0.225]$ can also be obtained by multiplying the initial probability vector and the square of p (XP^2).

If we wish to find a long – range prediction for the market share of each company, we can continue to compute successive state vectors forming a chain as illustrated in the following table 1.

TABLE 1

After n months	Company A Company B
0(initial)	[0.7 0.3]
1	[0.75 0.25]
2	[0.775 0.225]
3	[0.7875 0.2125]
4	[0.79375 0.20625]
5	[0.796875 0.203125]
6	[0.79843 0.20156]

Although it takes a little longer, the results seem to be approaching the numbers in the state

vector $\begin{bmatrix}0.8 & 0.2\end{bmatrix} = \begin{bmatrix}0.8 & 0.2\end{bmatrix}\begin{bmatrix}0.9 & 0.1\\ 0.4 & 0.6\end{bmatrix}$

The market shares of the two companies remain to be the same (though the transition from buying product of one company to buying product of the other company continues). Therefore, in the long run the market shares of company A and company B will be 80% and 20% respectively.

One of many applications of Markov chains is in finding long-range predictions. It is not possible to make long-range predictions with all transition matrices, but for a large set of transition matrices long range predictions are possible. Such predictions are always possible with some transition matrices.

For some matrices there exists a unique probability vector V such that VP = V. Such probability vector is called the steady state vector or the fixed probability vector. This Vector V is unique and gives the long-range trend of the Markov chain. To find V, solve a system of equation obtained from the matrix equation Vp= V, and from the fact that the sum of the entries of V is 1. Finding the fixed probability vector by tabulating successive state vectors and guessing the numbers from the sequence of results can be very tedious, and is prone to error.

In our example we found vector V to be (0.8 0.2). Vector V can be determined by solving

$$(X\ \ Y)\begin{pmatrix}0.9 & \\ 0.4 & 0.6\end{pmatrix} = (X\ Y)$$

Use matrix multiplication on the left.

$$\begin{bmatrix}0.9x+ 0.4y & 0.1x + 0.6y\end{bmatrix} = \begin{bmatrix}x\ y\end{bmatrix}$$

Set corresponding entries from the two matrices equal to

get 0.9x + 0.4y = x

0.1x + 0.6y = y

Simplify these equations

$$-0.1x + 0.4y = 0$$

$$0.1x - 0.4y = 0$$

It is clear that both equations are the same. The second equation is simply the first equation multiplied by -1, so we will drop the second equation. Recall that V= $[x\ y]$ is a probability vector, so, x + y = 1. Hence to find values of x and y substitute y = 1- x in the first equation,

$$-0.1x + 0.4(1 - x) = 0$$

Solving this equation for x and then finding value of y we obtain, x = 0.8 and y = 0.2

ILLUSTRATION

Currently it is known that 15% of the population own a car and 85% do not. Based upon past experience every year 1 out of 100 of the non owners will become owners and 1 out of 1000 of the owners will become non owner.

Find the percentage of car owners after two years

What will be the predicted proportion of car owners and non owners in the long run?

$$P = \begin{array}{ll} & \begin{array}{cc} \text{Owner} & \text{non owner} \end{array} \\ \begin{array}{l} \text{Owner} \\ \text{Non owner} \end{array} & \begin{bmatrix} 0.999 & 0.001 \\ 0.01 & 0.99 \end{bmatrix} \end{array}$$

And the initial probability vector : owner non owner

$$[.15 \quad .85]$$

To find the proportion after two years, we first find the proportion at the end of one year as,

$$[.15 \ \ .85]\begin{bmatrix} .999 & .001 \\ .01 & .99 \end{bmatrix} = [0.15835 \ \ 0.84165]$$

Then at the end of two years the percentage of car owners and non owners is given by,

$$[0.15838 \ \ 0.84165]\begin{bmatrix} .999 & .001 \\ .01 & .99 \end{bmatrix} = [0.16661 \ 0.83340]$$

Hence after two years 16.661% of the population will be car owners.

To find the long range prediction we find the steady state vector V = $[x\ y]$ by solving the equation

$$[x\ y]\begin{bmatrix} 0.999 & 0.001 \\ 0.01 & 0.99 \end{bmatrix} = [x\ y]$$

Using the procedure discussed above the solution is x = 0.909 and y = 0.091. Hence, in the long run 90.9% of the population will own a car.

SELF CHECK EXERCISES 2.4

Suppose three companies: A, B and C dominate the market for a certain product and are competing against each other for a large share of the market. Currently company A has 2/9 of the market, company B has 4/9 of the market and company C has 1/3 of the market. The market survey indicates that every 6 months company A retains ¾ of its customer and loss 1/6 to company B and 1/12 to company C. Company B retains ½ of its customer and loss 1/3 to company A and 1/6 to company C. Company C retains 3/8 of its customer and loss ¼ to company A and 3/8 to company C retains 3/8 to company C. Find the share of the market that each company will have i) one year later

Currently 1% of the populations in a certain city are smokers. Based on a survey conducted each year 2 out of 100 non –smokers become smokers; and 1 out of 10 smokers become non-smokers. How many percent of the population will become smokers after two years?

A division of the ministry of public health has conducted a sample survey on the public attitudes towards the use of condoms. From the results of the survey the department concluded that currently only 20% of the population uses condoms and every month 10% of non-users become users, where as 5% of users discontinue using.

Write the current transition matrices.

What will be the percentage of users from total population just after two months?

What will be the proportion of the non users and users in the long run?

SUMMARY

Matrix is a rectangular array of numbers represented in parenthesis or bracket. Matrix operation is used to solve different real life problems: engineering, Agriculture, medicine, economics, business and others.

Matrices are represented by capital letters. The entries with in the matrix are called elements.

Matrices are classified by the number of rows and columns they have. A matrix having m rows and n columns is said to have dimension of m by n written as mxn.

There are different types of matrices:

Square - A matrix having the same number of rows as columns is called a **square matrix**

Vector A matrix with only one column is called a **Column matrix**, and one with only one row is called a **row matrix**.

Zero - The matrix with entries that are all zeros is called the zero matrix

Identity matrix- Is a square matrix with all its elements along the main diagonal (that extends from the upper left to the lower right) one (unit) and zero in all other positions.

Diagonal - Diagonal matrix is a square matrix with all elements zeros other than the main diagonal.

Scalar matrix- it is a matrix with constant number on the main diagonal and zero elsewhere.

Transpose- If we interchange the rows and the columns of an mxn matrix A, we get new nxm matrix; which is the transpose of matrix A. it is normally denoted by A'.

Two matrices are said to be equal if they have the same size and each element aij = bij for all ijs.

Two matrices A and B are conformable for addition and subtraction if they the same dimension.

Sum of two matrices is obtained by adding the corresponding elements of the two matrices.

The product of a matrix and a constant number is called scalar multiplication and is obtained by multiplying all entries of the matrix by the constant number.

Two matrices, A and B are conformable for multiplication if the number of columns of the first matrix is equal to the number of rows of the second matrix. The product matrix (AB) will have a dimension which is the same as number of rows of the first matrix and number of columns of the second matrix.

If for a given square matrix A, another square matrix B of the same order is obtained such that AB = BA = I Then matrix B is called inverse of A and is denoted by $B = A^{-1}$. Thus, $AA^{-1} = A^{-1}A = I$. Characteristics of inverse of a matrix include: if the inverse of square matrix is multiplied by the original matrix, the result is an identity matrix; If the inverse of square matrix exists, it is unique; the inverse of the transpose of a matrix is the transpose of its inverse i.e., $(A')^{-1} = (A^{-1})'$

The inverse of a matrix can be found by Gaussian elimination which involves application elementary row operations on augmented matrix.

The elementary row operations include:

Interchanging of two rows

Multiply (divide) a row by a nonzero constant

Add (subtract) a nonzero multiple of one row to another row

Solution of a system of linear equation can be obtained using Gaussian elimination method by expressing the matrix as coefficient matrix, variable matrix and constant matrix.

Writing the matrix as (A/B) and converting matrix A to identity matrix by applying the row operations converts the constant matrix B in to the solution matrix. The solution for X and Y can then be read from the matrix.

A system of linear equations with equal number of equations and variables may have unique solution, no solution, or multiple solutions.

A system of linear equations with more variables than equations can not have unique solution.

The Markov model is concerned with the brand switching behaviour of consumers show are essentially repeat buyers of a product. These models are used to forecast the market share of the product at time period n if the market share at the time period (n-

is known. In Markov Chain model, the current state and transition matrix are used to compute the states after a given period of time. State after the end of n period is given by the product of state at the end of n-1 period multiplied by the transition matrix. In a long run though the transition keeps on the states of nature remains unchanged. This long run state is called steady state matrix.

CHAPTER THREE

3. INTRODUCTION TO LINEAR PROGRAMMING

3.1 INTRODUCTION

Dear Learners, Many major decisions faced by a manager of a business focus on the best way to achieve the objectives of the firm, subject to the restrictions placed on the manager by the operating environment. These restrictions can take the form of limited resources, such as time, labor, energy, material, or money; or they can be in the form of restrictive guidelines, such as a recipe for making cereal or engineering specifications. One of the most frequent objectives of business firms is to gain the most profit possible or, in other words, to maximize profit. The objective of individual organizational units within a firm (such as a production or packaging department) is often to minimize cost. When a manager attempts to solve a general type of problem by seeking an objective that is subject to restrictions, the management science technique called **linear programming** is frequently used.

Objectives of a business frequently are to maximize profit or minimize cost.

CHAPTER OBJECTIVE

After completing this chapter students will be able to:

- Define what linear programming is
- Identify application areas of linear programming
- Formulate linear programming model
- Solve linear programming model using graphical method

3.2 OVERVIEW OF LINEAR PROGRAMMING

Every organization faces the problem of allocation of scarce resources. The resources include men, machine, material, capital, information etc. most of these decisions are made subject to constraints. For example, production from factory is limited due to capacity constraints, organizations face working capital constraint, technical constraints, demand constraint, and others. If the decision is to be made after long time, some of these resources can be expanded; fund can be raised, capacity can be added, additional resources can be acquired. However, when the decision making is short term the resources cannot be expanded. Therefore, the decision is to go for what is possible. The best solution cannot be achieved, but given the constraint (limitation) the decision makers try to optimize (maximize or minimize) the objective.

BRAIN STORMING QUESTION

What does the term linear imply?

What is linear programming? And what does optimization mean?

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Linear programming is a mathematical technique designed to aid managers in allocating scarce resources (such as labor, capital, or energy) among competing activities. It reflects, in the form of a model, the organization's attempt to achieve some objective (frequently, maximizing profit contribution, maximizing rate of return, minimizing cots) in view of limited or constrained resources (available capital or labor, service levels, available machine time, budgets).

The linear programming technique can be said to have a linear objective function that is to be optimized (either maximized or minimized) subject to linear equality or inequality constraints and sign restrictions on the variables.

Linear Programming is a technique for making decisions under certainty i.e.; when all the courses of options available to an organization are known & the objective of the firm along with its constraints are quantified. That course of action is chosen out of all possible alternatives which yields the optimal results. Linear Programming can also be used as a verification and checking mechanism to ascertain the accuracy and the reliability of the decisions which are taken solely on the basis of manager's experience-without the aid of a mathematical model.

The term linear describes the proportionate relationship of two or more variables. Thus, a given change in one variable will always cause a resulting proportional change in another variable.

> DEFINITION 3.1: ***Linear programming*** is a model that consists of linear relationships representing a firm's decision(s), given an objective and resource constraints.

3.3MAIN APPLICATION AREAS OF LINEAR PROGRAMMING

In the last few decades since 1960s, no other mathematical tool or technique has had such a profound impact on the management's decision making criterion as Linear Programming well and truly it is one of the most important decision making tools of the last century which has transformed the way in which decisions are made and businesses are conducted. Starting with the Second World War till the Y - 2K problem in computer applications, it has covered a great distance.

Discussed below are some of the important application areas of Linear Programming:

I. Military Applications. Paradoxically the most appropriate example of an organization is the military and worldwide, Second World War is considered to be one of the best managed or organized events in the history of the mankind.

Linear Programming is extensively used in military operations. Such application include the problem of selecting an air weapon system against the enemy so as to keep them pinned down and at the same time minimizes the amount of fuel used. Other examples are dropping of bombs on pre-identified targets from aircraft and military assaults against localized terrorist outfits.

Agriculture. Agriculture applications fall into two broad categories, farm economics and farm management. The former deals with the agricultural economy of a nation or a region, while the latter is concerned with the problems of the individual form. Linear Programming can be gainfully utilized for agricultural planning e:g. allocating scarce limited resources such as capital, factors of production like labour, raw material etc. in such a way 'so as to maximize the net revenue.

Environmental Protection. Linear programming is used to evaluate the various possible a1temative for handling wastes and hazardous materials so as to satisfy the stringent provisions laid down by the countries for environmental protection. This technique also finds applications in the analysis of alternative sources of energy, paper recycling and air cleaner designs.

IV. Facilities Location. Facilities location refers to the location nonpublic health care facilities (hospitals, primary health centers) and' public recreational facilities (parks, community hal1s) and other important facilities pertaining to infrastructure such as telecommunication booths etc. The analysis of facilities location can easily be done with the help of Linear Programming.

Apart from these applications, LP can also be used to plan for public expenditure and drug control. '

Product-Mix. The product-mix of a company is the existence of various products that the company can produce and sell. However, each product in the mix requires finite amount of limited resources. Hence it is vital to determine accurately the quantity of each product to be produced knowing their profit margins and the inputs required for producing them. The primary objective is to maximize the profits of the firm subject to the limiting factors within which it has to operate.

VI. Production. A manufacturing company is quite often faced with the situation where it can manufacture several products (in different quantities) with the use of several different machines. The problem in such a situation is to decide which course of action will maximize output and minimize the costs.

Another application area of Linear Programming in production is the assembly by-line balancing - where a component or an item can be manufactured by assembling different parts. In such situations, the objective of a Linear Programming model is to set the assembly process in the optimal (best possible) sequence so that the total elapsed time could be minimized.

VII. Mixing or Blending. Such problems arise when the same product can be produced with the help of a different variety of available raw-materials each having a fixed composition and cost. Here the objective is to determine the minimum cost blend or mix (cost minimizations) and the various constraints that operate are the availability of raw materials and restrictions on some of the product constituents.

VIII. Transportation & Trans-Shipment. Linear Programming models are employed to determine the optimal distribution system i.e.; the best possible channels of distribution available to an organization for its finished product sat minimum total cost of transportation or shipping from company's go down to the respective markets. Sometimes the products are not transported as finished products but are required to be manufactured at various sources. In such a situation, Linear Programming helps in ascertaining the minimum cost of producing or manufacturing at the source and shipping it from there.

IX. Portfolio Selection. Selection of desired and specific investments out of a large number of investment' options available to the managers (in the form of financial institutions such as banks, non-financial institutions such as mutual funds, insurance companies and investment services etc.) is a very difficult task, since it requires careful evaluation of all the existing options before arriving at decision. The objective of Linear Programming, in such cases, is to find out the allocation/which maximizes the total expected return or minimizes the total risk under different situations.

X. Profit Planning & Contract. Linear Programming is also quite useful in profit planning and control. The objective is to maximize the profit margin from investment in the plant facilities and machinery, cash on hand and stocking-hand.

XI. Traveling Salesmen Problem. Traveling salesmen problem refers to the problem of a salesman to find the shortest route originating from a particular city, visiting each of the specified cities and then returning back to the originating point of departure. The restriction being that no city must be visited more than once during a particular tour. Such types of problems can quite easily be solved with the help of Linear Programming.

XII. Media Selection/Evaluation. Media selection means the selection of the optimal media-mix so as to maximize the effective exposure. The various constraints in this case are: Budget limitation, different rates for different media (i.e.; print media, electronic media like radio and T.V. etc.) and the minimum number of repeated advertisements (runs) in the various media. The use of Linear Programming facilities like the decision making process.

XIV. Staffing. Staffing or the man-power costs are substantial for a typical organization which make its products or services very costly. Linear Programming techniques help in allocating the optimum employees (man-power or man-hours) to the job at hand. The overall objective is to minimize the total man-power or overtime costs.

XV. Job Analysis. Linear Programming is frequently used for evaluation of jobs in an organization and also for matching the right job with the right worker.

XVI. Wages and Salary Administration. Determination of equitable salaries and various incentives and perks becomes easier with the application of Linear Programming. LP tools" can also be utilized to provide optimal solutions in other areas of personnel management such as training and development and recruitment etc.

Some areas in which linear programming has been applied will be helpful in setting the climate for learning about this important technique.

A company produces agricultural fertilizers. It is interested in minimizing costs while meeting certain specified levels of nitrogen, phosphate, and potash by blending together a number of raw materials.

An investor wants to maximize his or her rate of return by investing in stocks and bonds. The investor can set specific conditions that have to be met including availability of capital.

A company wants the best possible advertising exposure among a number of national magazines, and radio and television commercials within its available capital requirements.

An oil refinery blends several raw gasoline and additives to meet a car manufacturer's specifications while still maximizing its profits.

A city wants to maximize the daytime use of recreational properties being proposed for purchase with a limited capital available.

This technique, called linear programming (L.P), is solved in a step-by-step manner called iterations. Each step of the procedure is an attempt to improve on the solution until the "best answer" is obtained or until it is shown that no feasible answer exists.

3.4 LINEAR PROGRAMMING MODELS (LPM)

Linear Programming (LP) models are mathematical representations of LP problems. Some LP models have a specialized format, whereas others have a more generalized format. Despite this, LP Models have certain characteristics in common. Knowledge of these characteristics enables us to recognize problems that are amenable to a solution using LP models and to correctly formulate an LP model. The characteristics can be grouped into two categories: ***Components*** and ***Assumptions***. The *components* relate to the *structure* of a model, whereas the *assumptions* reveal the *conditions under which the model is valid*.

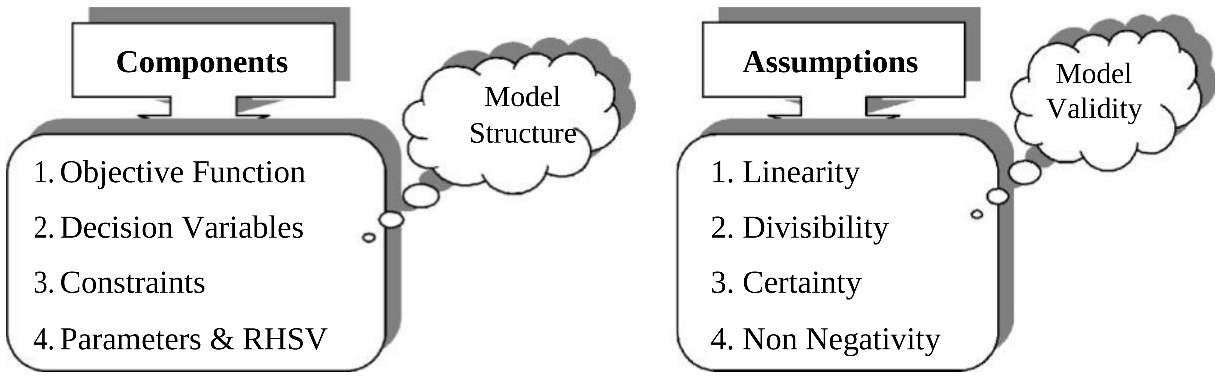

FIGURE 1: Linear Programming Model Components & Assumptions

3.4.1 COMPONENTS OF LINEAR PROGRAMMING MODEL

The Objective Function: is a linear mathematical relationship that describes the objective of the firm in terms of the decision variables. The objective function always consists of either maximizing or minimizing some value (e.g., maximize the profit or minimize the cost of producing radios). Moreover, because we are dealing with optimization, the objective will be either maximization or minimization, but not both at a time.

> DEFINITION 3.2: The ***Objective Function*** is a linear relationship that reflects the objective of an operation.

The Decision Variables – are mathematical symbols that represent levels of activity by the firm. For example, an electrical manufacturing firm desires to produce x_1 radios, x_2 toasters, and x_3 clocks, where x_1, x_2, and x_3 are symbols representing unknown variable quantities of each item. The final values of x_1, x_2, and x_3, as determined by the firm, constitute a decision (e.g., the equation $x_1 = 100$ radios is a decision by the firm to produce 100 radios).

> DEFINITION 3.3: ***Decision variables*** are mathematical symbols that represent levels of activity.

The constraints - are restrictions which define or limit the feasibility of a proposed course of action. They limit the degree to which the objective can be pursued.

A typical restriction embodies *scarce resources* (such as labor supply, raw materials, production capacity, machine time, storage space), l*egal or contractual requirements* (e.g. Product standards, work standards), or they may reflect other limits based on forecasts, customer orders, company policies etc.

> DEFINITION 3.4: A ***Constraint*** is a linear relationship that represents a restriction on decision making.

Parameters - are fixed values that specify the impact that one unit of each decision variable will have on the objective and on any constraint it pertains to as well as to the numerical value of each constraint.

> DEFINITION 3.5: ***Parameters*** are numerical values that are included in the objective functions and constraints.

3.5 ASSUMPTIONS OF LINEAR PROGRAMMING MODELS

Proportionality-proportionality is assumed to exist in the objective and constraint. The measure of effectiveness in the objective function and amount of each resource used must be proportional to the value of each decision variable.

Additivity- sum of resources used by each activity must be equal to the total quantity of resources used by each activity for all resources individually and collectively.

Linearity. The linearity requirement is that each decision variable has a linear impact on the objective function and in each constraint in which it appears. Following the above example, producing one more unit of product 1 adds Br. 4 to the total profit. This is true over the entire range of possible values of X_1. The same applies to each of the constraints. It is required that the same coefficient (fro example, 2 lb. per unit) apply over the entire range of possible value so the decision variable.

Divisibility. The divisibility requirement pertains to potential values of decision variables. It is assumed that non-integer values are acceptable. For example: 3.5 TV sets/hr would be acceptable → 7 TV sets/2hrs.

Certainty. The certainty requirement involves two aspects of LP models.

With respect to model parameters (i.e., the numerical values) – It in assumed that these values are known and constant e.g. in the above example each unit of product 1 requires 2lab his is known and remain constant, and also9 the 300 lab/hr available is deemed to be known and constant.
All the relevant constraints identified and represented in the model are as they are.

Non-negativity. The non-negativity constraint is that negative values of variables are unrealistic and, therefore, will not be considered in any potential solution; only positive values and zero will be allowed.

SELF CHECK EXERCISES 3.1

1. *What is linear programming?*

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2. *What are the application areas for linear programming model?*

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3. *What the advantages and disadvantages of linear programming?*

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4.What are the basic requirements to use linear programming model?

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5.What are the basic assumptions of linear programming?

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3.6 Formulating LP Models

Once a problem has been defined, the attention of the analyst shifts to formulating a model. Just as it is important to carefully define a problem, it is important to carefully formulate the model that will be used to solve the problem. If the LP model is ill formulated, ill-structured, it can easily lead to poor decisions.

Formulating linear programming models involves the following steps:

STEP I: Define the problem/problem definition

To determine the # of type 1 and type 2 products to be produced per mother so as to maximize the monthly profit given the restrictions.

STEP II: Identify the decision variables or represent unknown quantities

Let X_1 and X_2 be the monthly qualities of Type 1 and type 2 products

STEP III: Determine the objective function

Once the variables have been identified, the objective function can be specified. It is necessary to decide if the problem is maximization or a minimization problem and the coefficients of each decision variable.

NOTE:

The units of all the coefficients in the objective function must be the same. E.g. If the contribution of product 1 is in terms of Birr, so does for product 2.

All terms in the objective function must include a variable each term have to have 1 variable.

All decision variables must be represented in the objective function.

STEP IV: Identifying the constraints

System constraints (more than one variable)

Individual constraints (one variable)

Non-negative constraints

Dear Learners, all the above discussed components, assumptions, and steps are the building blocks of an LP model. Next, we will try to better understand their meaning by examining a simple LP model as follows.

3.6.1 General Model of Linear Programming Problem

Suppose an L.P problem involve the determination of the values of n decision variables x_1, X_2 ,…… X_n such that the linear objective function of these variables assumes an optimum value (maximum or minimum) when these variables are subject to set of m constraints which are also linear functions of the variables. The standard form of linear programming model can be written as:

Optimize (maximize or Minimize)

Decision variables

$$Z = C_1X_1 + C_2X_2 + \ldots\ldots C_nX_n$$ Objective function

Cost or Profit Coefficient

Subject to

Structural Coefficients

$a_{11} X_1 + a_{12}X_2 + \ldots\ldots a_{1n}X_n \ (\leq \geq =)\ b_1$

$a_{21}X_1 + a_{22}X_2 + \ldots\ldots a_{2n}X_n \ (\leq \geq =)\ b_2$ Requirement (Availability of Resource)

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$a_{m1}X_1 + a_{m2}X_2 + \ldots\ a_{mn}X_n \ (\leq \geq =)\ b_n$

Constraints

$X_1, X_2, \ldots\ldots X_n \geq 0$ Non-negativity constraint

Equality or inequality

A Maximization Model Example 1

Beaver Creek Pottery Company is a small crafts operation run by a Native American tribal council. The company employs skilled artisans to produce clay bowls and mugs with authentic Native American designs and colors. The two primary resources used by the company are special pottery clay and skilled labor. Given these limited resources, the company desires to know how many bowls and mugs to produce each day in order to maximize profit. This is generally referred to as a product mix problem type. This scenario is illustrated in Figure 3.1.

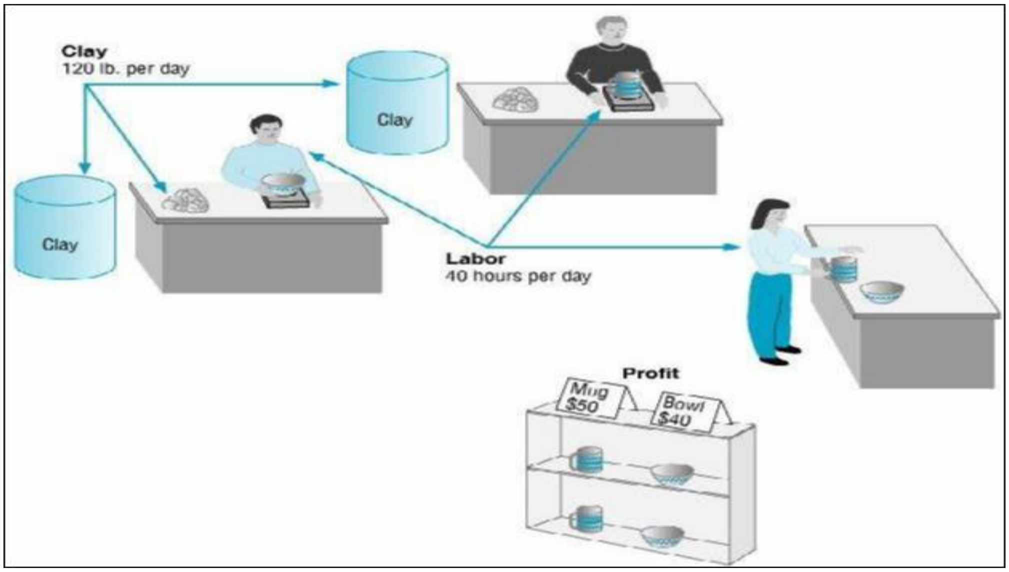


Figure 3.1: Beaver Creek Pottery Company

The two products have the following resource requirements for production and profit per item produced (i.e., the model parameters):

Resource Requirement			
Product	Labor (hr./unit)	Clay (lb./unit)	Profit ($/unit)
Bowl	1	4	40
Mug	2	3	50

There are 40 hours of labor and 120 pounds of clay available each day for production. We will formulate this problem as a linear programming model by defining each component of the model separately and then combining the components into a single model. The steps in this formulation process are summarized as follows:

SUMMARY OF LP MODEL FORMULATION STEPS

STEP 1. Define the decision variables

How many bowls and mugs to produce

STEP 2. Define the objective function

Maximize profit

STEP 3. Define the constraints

The resources (clay and labor) available

DECISION VARIABLES

The decision confronting management in this problem is how many bowls and mugs to produce. The two decision variables represent the number of bowls and mugs to be produced on a daily basis. The quantities to be produced can be represented symbolically as

x_1 = number of bowls to produce

x_2 = number of mugs to produce

THE OBJECTIVE FUNCTION

The objective of the company is to maximize total profit. The company's profit is the sum of the individual profits gained from each bowl and mug. Profit derived from bowls is determined by multiplying the unit profit of each bowl, \$40, by the number of bowls produced, x_1. Likewise, profit derived from mugs is derived from the unit profit of a mug, \$50, multiplied by the number of mugs produced, x_2. Thus, total profit, which we will define symbolically as Z, can be expressed mathematically as $\$40x_1 + \$50x_2$. By placing the term maximize in front of the profit function, we express the objective of the firm to maximize total profit:

Maximize Z = $\$40x_1 + 50x_2$

where

= total profit per day

$\$40x_1$ = profit from bowls

$\$50x_2$ = profit from mugs

MODEL CONSTRAINTS

In this problem two resources are used for production labor and clay both of which are limited. Production of bowls and mugs requires both labor and clay. For each bowl produced, 1 hour of labor is required. Therefore, the labor used for the production of bowls is $1x_1$ hours. Similarly, each mug requires 2 hours of labor; thus, the labor used to produce mugs every day is $2x_2$ hours. The total labor used by the company is the sum of the individual amounts of labor used for each product:

$1x_1 + 2x_2$

However, the amount of labor represented by $1x_1 + 2x_2$ is limited to 40 hours per day; thus, the complete labor constraint is

$1x_1 + 2x_2 \leq 40$ hr.

The "less than or equal to" ($\leq$) inequality is employed instead of an equality (=) because the 40 hours of labor is a maximum limitation that can be used, not an amount that must be used. This constraint allows the company some flexibility; the company is not restricted to using exactly 40 hours but can use whatever amount is necessary to maximize profit, up to and including 40 hours. This means that it is possible to have idle, or excess, capacity (i.e., some of the 40 hours may not be used).

The constraint for clay is formulated in the same way as the labor constraint. Because each bowl requires 4 pounds of clay, the amount of clay used daily for the production of bowls is $4x_1$ pounds; and because each mug requires 3 pounds of clay, the amount of clay used daily for mugs is $3x_2$. Given that the amount of clay available for production each day is 120 pounds, the material constraint can be formulated as

$4x_1 + 3x_2 \leq 120$ lb.

A final restriction is that the number of bowls and mugs produced must be either zero or a positive value because it is impossible to produce negative items. These restrictions are referred to as non negativity constraints and are expressed mathematically as

$x_1 \geq 0, x_2 \geq 0$

The complete linear programming model for this problem can now be summarized as follows:

Maximize $Z = \$40x_1 + 50x_2$
Subject to

$$1x_1 + 2x_2 \leq 40$$
$$4x_1 + 3x_2 \leq 120$$
$$x_1, x_2 \geq 0$$

The solution of this model will result in numeric values for x_1 and x_2 that will maximize total profit, Z. As one possible solution, consider $x_1 = 5$ bowls and $x_2 = 10$ mugs. First, we will substitute this hypothetical solution into each of the constraints in order to make sure that the solution does not require more resources than the constraints show are available:

$1(5) + 2(10) \leq 40$ $\quad\quad$ $4(5) + 3(10) \leq 120$

$25 \leq 40$ $\quad$ and $\quad$ $50 \leq 120$

Because neither of the constraints is violated by this hypothetical solution, we say the solution is **feasible** (i.e., it is possible). Substituting these solution values in the objective function gives Z = 40(5) + 50(10) = \$700.

However, for the time being, we do not have any way of knowing whether $700 is the maximum profit.

A feasible solution does not violate any of the constraints.

Now consider a solution of x_1 = 10 bowls and x_2 = 20 mugs. This solution results in a profit of

Z = $40(10) + 50(20)
400 + 1,000
$1,400

Although this is certainly a better solution in terms of profit, it is **infeasible** (i.e., not possible) because it violates the resource constraint for labor:

$1(10) + 2(20) \leq 40$
$50 \leq 40 /$

An infeasible problem violates at least one of the constraints.

The solution to this problem must maximize profit without violating the constraints. The solution that achieves this objective is x_1 = 24 bowls and x_2 = 8 mugs, with a corresponding profit of $1,360. The determination of this solution is shown using the graphical solution approach in the following section.

A MAXIMIZATION MODEL EXAMPLE 2

Assume Adey Company produces two products; both of which are made on the same two machines. Since there are two different products, there are two independent activities producing product A and product B. producing a unit of A requires 1 hour on the first machine and 4 hours on second machine. Producing a unit of B requires 4 hours on the first machine and 1 hour on the second machine. Each week, there are 20 hours available on the first machine and 35 hours available on the second machine.

The firm makes profit of 60 birr on a unit of product A and 90 birr on each unit of product B. it is assumed that all units produced will be sold. Formulate this problem as a linear programming problem to determine as to how many units of each of the products should be produced per week so that the firm can earn the maximum profit.

SOLUTION

To express the above word problem in linear programming model, we can use the steps mentioned above.

STEP - 1. IDENTIFYING THE DECISION VARIABLES

Let X be the number of product A to be produced

Let Y be the number of product B to be produced

STEP - 2. DETERMINING OBJECTIVE FUNCTION

The last sentence of the word problem clearly shows that the objective of the company is maximizing profit. Moreover, it is clearly stated that profit contribution of Product A is birr 60 per unit and that of B is birr 90 per unit. Therefore, the profit earned when the company produces and sales X unit of A and Y units of B is given as: 60X + 90Y

The objective function thus can be expressed as Maximize π = 60X + 90 Y

STEP - 3. IDENTIFICATION OF CONSTRAINTS

A close examination of the word problem reveals that there are two constraints: availability of machine hour on the first machine and the second machine. There is only 20 hours of machine one and 35 hours of machine two. Therefore the company can not use more than 20 hours of the first machine and more than 35 hours of the second machine.

STEP - 4. IDENTIFICATION OF PARAMETERS

The resource consumption and availability can be summarized using the following table.

	Machine I	**Machine II**
Product A	1	4
Product B	4	1
Total available hours	20	35

This table clearly shows that 1 hour of Machine I and 4 hours of machine I are used for production of product A and B.

Therefore since the total resource consumption cannot be greater than the total resource availability

$X+4Y\leq 20$

The left hand side of the inequality shows the total amount of machine hour I used to produce X units of product A and Y units of Product B. whereas the right hand side shows the total amount of resource available for the production of Product A and product B. For the second resource,

$4X+ Y\leq 35$

STEP - 5. CONSTRUCTING THE MODEL

The above model can now be summarized as

Maximize $\pi = 60X + 90Y$

Subject to $X +4Y \leq 20$

$4X+Y\leq 35$

$X,Y\geq 0$

The last constraint (non negativity constraint) is added since we can not produce negative units of a product.

A MINIMIZATION MODEL EXAMPLE 1

Fauji Foundation produces a cereal SUNFLOWER, which they advertise as meeting the minimum daily requirements for vitamins A and D. The mixing department of the company uses three main ingredients in making the cereal-wheat, oats, and rice, all three of which contain amounts of vitamin A and D. Given that each box of cereal must contain minimum amounts of vitamin A and D, the company has instructed the mixing department determine how many ounces of each ingredient should go into each box of cereal in order to minimize total cost.

This problem differs from the previous one in that its objective is to minimize cost, rather than Maximize profit.

Each ingredient has the following vitamin contribution and requirement per box.

VITAMIN CONTRIBUTION

Vitamin	Wheat (mg./oz.)	Oats (mg./oz)	Rice (mg./oz.)	Milligrams Required/Box
A	10	20	08	100
D	07	14	12	70

The cost of one ounce of wheat is Rs. 0.4, the cost of an ounce of oats is Rs. 0.6, and the cost of one ounce of rice is Rs. 0.2.

DECISION VARIABLES

This problem contains three decision variables for the number of ounces of each ingredient in a box of cereal:

X1 = ounces of wheat

X2 = ounces of oats

X3 = ounces of rice

THE OBJECTIVE FUNCTION

The objective of the mixing department of the Fauji Foundation is to minimize the cost of each box of cereal. The total cost is the sum of the individual costs resulting from each ingredient. Thus, the objective function that is to minimize total cos, Z, is expressed as

Minimize Z = Rs. 0.4X1 + 0.6X2 + 0.2X3

where Z = total cost per box

Rs. 0.4 X1 = cost of wheat per box

0.6 X2 = cost of rice per box

0.2 X3 = cost of rice per box

MODEL CONSTRAINTS

In this problem the constraints reflect the requirements for vitamin consistency of the cereal. Each ingredient contributes a number of milligrams of the vitamin to the cereal.

The constraint for vitamin A is 10 X1 + 20 X2 + 8 X3 > 100 milligrams where 10 X1 = vitamin A contribution (in mg.) for wheat

20 X2 = vitamin A contribution (in mg.) for oats

8X3 = vitamin A contribution (in mg.) for rice

Notice that rather than an (<) inequality, as used in the previous example, this constraint requires a > (greater than or minimum requirement specifying that at least 100 mg of vitamin A must be in a box. If a minimum cost solution results so that more than 100 mg is in the cereal mix, which is acceptable, however, the amount cannot be less than 100 mg.

The constraint for vitamin D is constructed like the constraint for vitamin A.

$7X_1 + 14X_2 + 12X_3 > 70$ milligrams

As in the previous problem there are also nonnegative constraints indicating that negative amounts of each ingredient cannot be in the cereal.

$X_1, X_2, X_3 > 0$

The L.P. model for this problem can be summarized as

Minimize $Z =$ Rs. $0.4X_1 + 0.6X_2 + 0.2X_3$

Subject to $10X_1 + 20X_2 + 8X_3 > 100$

$7X_1+14X_2+12X_3>70$

$X_1, X_2, X_3 > 0$

A MINIMIZATION MODEL EXAMPLE 2

The Philips Television Company produces and ships televisions from three warehouses to three retail stores on a monthly basis. Each warehouse has a fixed demand per month. The manufacturer wants to know the number of television sets to ship from each warehouse to each store in order to minimize the total cost of transportation.

Each warehouse has the following supply of televisions available for shipment each month.

Warehouse	**Supply (sets)**
1. Karachi	300
2. Lahore	100
3. Islamabad	200
	600

Each retail store has the following monthly demand for television sets:

Store	Demand (sets)
A. Faisalabad	150
B. Peshawar	250
C. Hyderabad	200
	600

The costs for transporting television sets from each warehouse to each retail store are different as a result of different modes of transportation and distances. The shiping cost per television set for each route are,

DECISION VARIABLES

The model for this problem consists of nine decision variables representing the number of television sets transported from each of the three warehouses to each of the three stores,

Xij = the No. of television sets shipped from warehouse "i" to store "j" where i = 1, 2, 3 and j =A,B,C.

Xij is referred to as a double subscripted variable. However, the subscript, whether double or single simply gives a "name" to the variable (i.e., distinguishes it from other decision variables). As such, the reader should not view it as more complex than it actually is. For example, the decision variable X3A Islamabad to store A in Faisalabad.

THE OBJECTIVE FUNCTION

The objective function of the television manufacturer is to minimize the total transportation costs for all shipments. Thus, the objective function is the sum of the individual shipping costs from each warehouse to each store.

Minimize Z = Rs. 6X1A + 8X1B + 1X1C + 4X2A + 2X2B + 3X2C + 3X3A + 5X3B + 7X3C

MODEL CONSTRAINTS

The constraints in this model are available television sets at each warehouse and the number of sets demanded at each store. As such, six constraints exist -- one for each warehouse's supply and one for each store's demand. For example, warehouse 1 retail stores. Since the amount shipped to the three stores is the sum of X1A, X1B, and X1C the constraint for warehouse 1 is

X1A + X1B + X1C = 300

This constrain is an equality (=) for two reasons. First, more than 300 television sets cannot be shipped, because that is cannot be shipped, because all 300 are needed at the three stores, the three warehouses must supply all that can be supplied. Thus, since the total shipped from warehouse 1 cannot exceed 300 or be less than 300 the constraint is equality. Similarly, the other two supply constraints for warehouse 2 and 3 are also equalities. X2A + X2B +X2C = 100

X3A + X3B + X3C = 200

The three demand constraints are developed in the same way except that television sets can be supplied from any of the three warehouses. Thus, the amount shipped to one store is the sum of the shipments from the three warehouses:

X1A + X2A + X3A = 150

X1B + X2B + X3B = 250

X1C + X2C + X3C = 200

The complete linear programming model for this problem is summarized as:

Minimize Z = Rs. 6X1A+8X1B+1X1C + 4X2A + 2X2B + 3X2C + 3X3A + 5X3B + 7X3C subject to

X1A + X1B + X1C = 300

X2A + X2B + X2C = 100

X3A + X3B + X3C = 200

X1A + X2A + X3A = 150

X1B + X2B + X3B = 250

X1C + X2C + X3C = 200

Xij > 0

SELF CHECK EXERCISE 3.2

A company manufactures 3 types of parts which use precious metals, platinum and gold. Due to the shortage of these metals, the government regulates the amount tha may be used per day. The relevant data with respect to supply, requirements and profi are summarized in the table below.

Product	Platinum required/unit (gms)	Gold required/Unit (gms)	Profit/unit
A	2	3	500
B	4	2	600
C	6	4	1200

Daily allotment of platinum and gold are 160 gm and 120 gm respectively. How should the company divide the supply of scarce precious metals? Formulate the linear programming model.

A dietician wishes to mix two types of food in such a way that the vitamin contends of the mixture contains at least 8 units of vitamin A and 10 units of vitamin B. Food I contains 2 units per Kg of vitamin A and 1 unit per Kg of vitamin B while food II contains 1 unit per Kg of vitamin A and 2 units of vitamin B. It costs 5 birr per Kg to purchase food I and birr 8 per Kg to purchase food II. Prepare the linear programming model for the problem.

A farmer has 1000 acres o f land on which he can grow corn, wheat or soya beans. Each acre of corn costs birr 100 for preparation, requires 7 man days of work and yields a profit of birr 30. An acre of wheat costs birr 120 for preparation, requires 10 man days of work and yields a profit of birr 40. Soya beans cost birr 70 to prepare require 8 man days of work and yields a profit of birr 20. If the farmer has birr 100,000 and can count on 80 man days work, formulate the linear programming model.

Manufacturers produce tow products A and B each of which gives a net profit of birr 4 and 3 respectively per unit. These are made either in plant I or plan II. Plant I and II have capacities of 72 and 48 hrs a day. Plant I take 2hrs and 1hr to produce A and B respectively. The corresponding figures of plant II are 1 and 2hrs. formulate linear programming model

A firm makes two types of furniture: chairs and tables. The contribution for each product as calculated by the accounting department is 20 birr per chair and 30 birr per table. Both products are processed on three machines M_1, M_2 and M_3. the time required by each product and total time available per week on each machine are as follows;

Machines	**Chair**	**Table**	**Available hrs**
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

How should the manufacturer schedule his production in order to maximize contribution? Formulate the above problem as a linear programming model.

3.7 GRAPHICAL SOLUTIONS OF LINEAR PROGRAMMING MODELS

Following the formulation of a mathematical model, the next stage in the application of linear programming to a decision-making problem is to find the solution of the model. A common solution approach is to solve algebraically the set of mathematical relationships that form the model either manually or using a computer program, thus determining the values for the decision variables. However, because the relationships are linear, some models and solutions can be illustrated graphically.

Graphical solutions are limited to linear programming problems with only two decision variables.

The graphical method is realistically limited to models with only two decision variables, which can be represented on a graph of two dimensions. Models with three decision variables can be graphed in three dimensions, but the process is quite cumbersome, and models of four or more decision variables cannot be graphed at all.

Although the graphical method is limited as a solution approach, it is very useful at this point in our presentation of linear programming in that it gives a picture of how a solution is derived. Graphs can provide a clearer understanding of how the computer and mathematical solution approaches presented in subsequent chapters work and, thus, a better understanding of the solutions.

The graphical method provides a picture of how a solution is obtained for a linear programming problem.

3.7.1 Graphical Solution of a Maximization Model

Graphical solution method involves drawing the graphs of the inequalities and finding the point that maximizes the objective function. Graphical solution method involves the following steps:

Read them carefully and attentively.

Draw the constraint equations by converting them in to equalities and shading the region that satisfies the inequality.

For a less than or equal to constraint the feasible region is lower side or the region to the left of the constraint line

For a greater than or equal to constraint the feasible region is upper side of the constraint or the region to the right of the constraint.

Identify the region that satisfies all the constraints including the non negativity constraint.

The region that satisfies all the constraints is the intersection of the regions shaded on step 1 above.

Identify the combination of activities that optimizes the objective function using either corner point approach or objective function approach.

After we find the feasible region, we need to find the combination of activities that optimizes our objective function; which can be done using two approaches:

Corner point approach

Iso-profit/ Iso-cost approach

1. CORNER POINT APPROACH

Corner point approach involves:

Identifying each of the corners of the feasible region either by visual inspection or by simultaneous equations

Computing the profit/cost at each corner point by substituting the coordinates of that point in the objective function.

Identifying the optimal solution at that corner point which shows highest profit for maximization or lowest cost in the case of minimization.

ISO PROFIT OR ISO COST METHOD

Iso profit/Isocost approach involves the following activities:

Choosing specific profit or cost figure and drawing an iso profit or Isocost line that falls within the shaded region.

Move the iso profit (iso-cost) line parallel to itself and farther (closer) from (to) the origin until further movement would make the iso profit or iso cost line to fall completely outside of the feasible region.

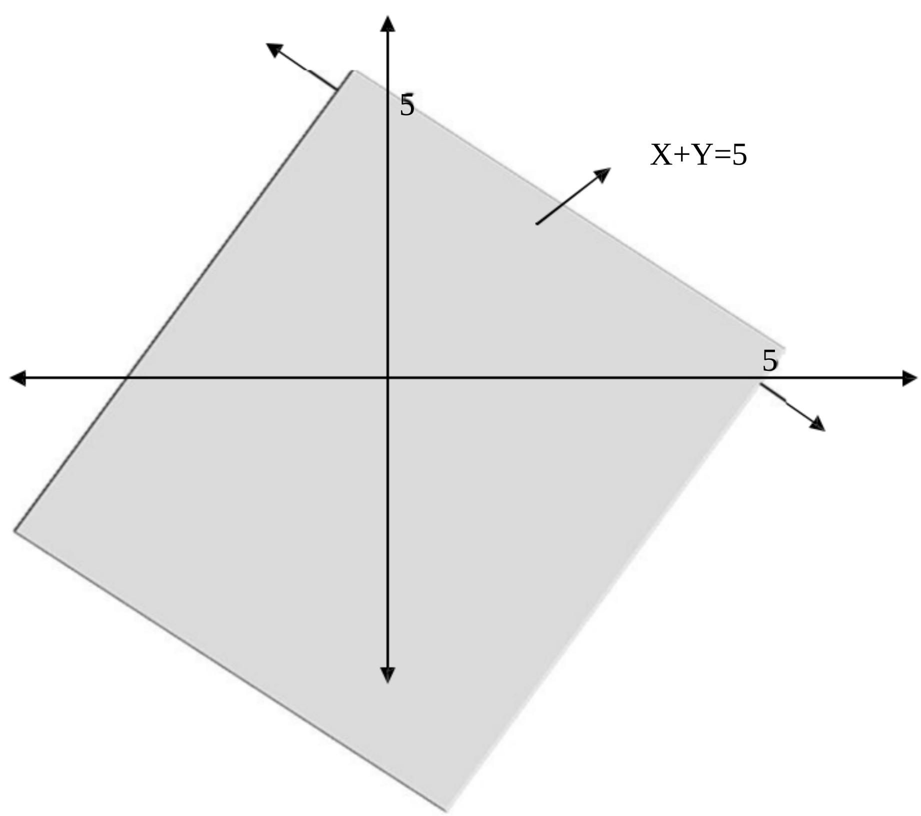
Identify the optimal solution as the coordinates of that point on the feasible region touched by the highest possible iso profit line (lowest possible iso cost line)

Read the coordinates of the optimal point either directly form the graph or compute it using simultaneous equations.

Compute the optimal profit (cost)

ILLUSTRATION

Dear learner, we will see what this steps mean one by one. Do you remember how to draw a graph of inequalities? Let’s say you want to draw a graph of $x + y \leq 5$. Can you draw feasible region for this inequality? The graph looks like this.



The straight line shows equation X + Y = 5. But our focus of interest is not the equation itself but with inequality, X + Y ≤ 5. Therefore we have to select either side of the equation line to show the inequality. Follow the following steps to identify the feasible region.

Draw the equation

Select a point on either side and check if satisfies the constraint. If it satisfies the constraint shade the region otherwise, select a point on other side of the equation line and check for feasibility

MAXIMIZATION EXAMPLE 1:

Consider two models of color TV sets; ***Model*** **A** and **B**, are produced by a company to maximize profit. The profit realized is $300 from A and $250 from set B. The limitations are

Availability of only 40hrs of labor each day in the production department.

a daily availability of only 45 hrs on machine time

Ability to sale 12 set of model A.

How many sets of each model will be produced each day so that the total profit will be as large as possible?

	Resources used per unit		
Constraints	**Model A (X_1)**	**Model B (X_2)**	**Maximum Available hrs.**
Labor/Hr.	2	1	40
Machine/Hr.	1	3	45
Marketing/Hr.	1	0	12
Profit	**$300**	**$250**	

SOLUTION

Formulation of mathematical modeling of LPP

Max Z=300X_1 +250X_2

St:

$$\left.\begin{array}{l} 2X_1 + X_2 \le 40 \\ X_1 + 3X_2 \le 45 \\ X_1 \le 12 \\ X_1, X_2 \ge 0 \end{array}\right\} \text{LPP MODEL}$$

2. Convert constraints inequalities into equalities

$2X_1 + X_2 = 40$

$X_1 + 3X_2 = 45$

$X_1 = 12$

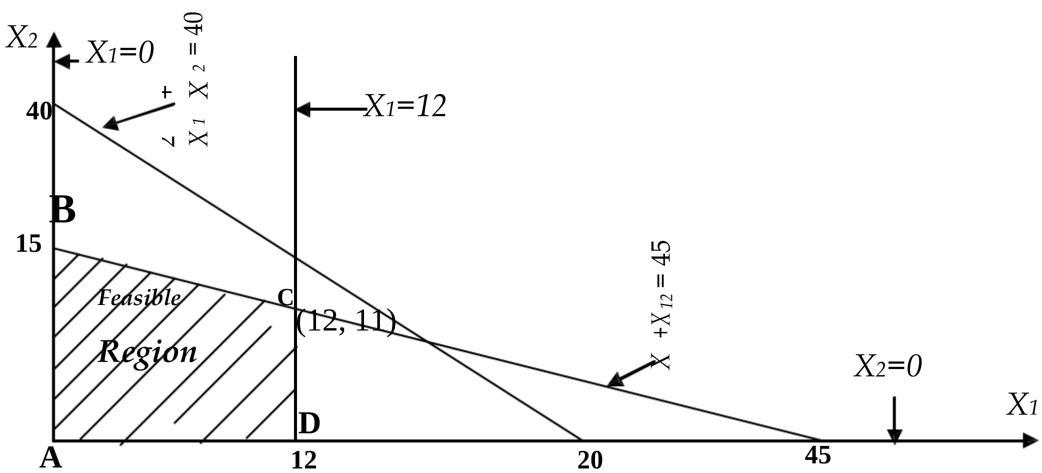
3. Draw the graph by intercepts

$2X_1 + X_2 = 40$ ==> (0, 40) and (20, 0)

$X_1 + 3X_2 = 45$==> (0, 15) and (45, 0)

$X_1 = 12$==> (12, 0)

$X_1, X_2 = 0$



Identify the feasible area of the solution which satisfies all constrains.

Identify the corner points in the feasible region

A (0, 0), B (0, 15), C (12, 11) and D (12, 0)

Identify the optimal point

Interpret the result

Corner points	X1, X2 Coordinates	The value of Objective Function (Max Z=300 X_1 +250X_2)
A	(0, 0)	$0
B	(0, 15)	$3750
C	**(12, 11)**	**$6350**
D	(12, 0)	$3600

INTERPRETATION:

12 units of product A and 11 units of product B should be produced so that the total profit will be $6350.

MINIMIZATION EXAMPLE 1:

Suppose that a machine shop has two different types of machines; machine 1 and machine 2, which can be used to make a single product .These machines vary in the amount of product produced per hr., in the amount of labor used and in the cost of operation.

Assume that at least a certain amount of product must be produced and that we would like to utilize at least the regular labor force. How much should we utilize each machine in order to utilize total costs and still meets the requirement?

	Resources used		
	Machine 1 (X_1)	**Machine 2 (X_2)**	**Minimum Required hrs.**
Product produced/Hr	20	15	100
Labor/Hr	2	3	15
Operation Cost	**$25**	**$30**	

Min.Z=25X_1 +30X_2

St:

20X_1 +15X_2 $\geq$ 100

2X_1 +3X_2 $\geq$ 15

X_1,X_2 ≥ 0

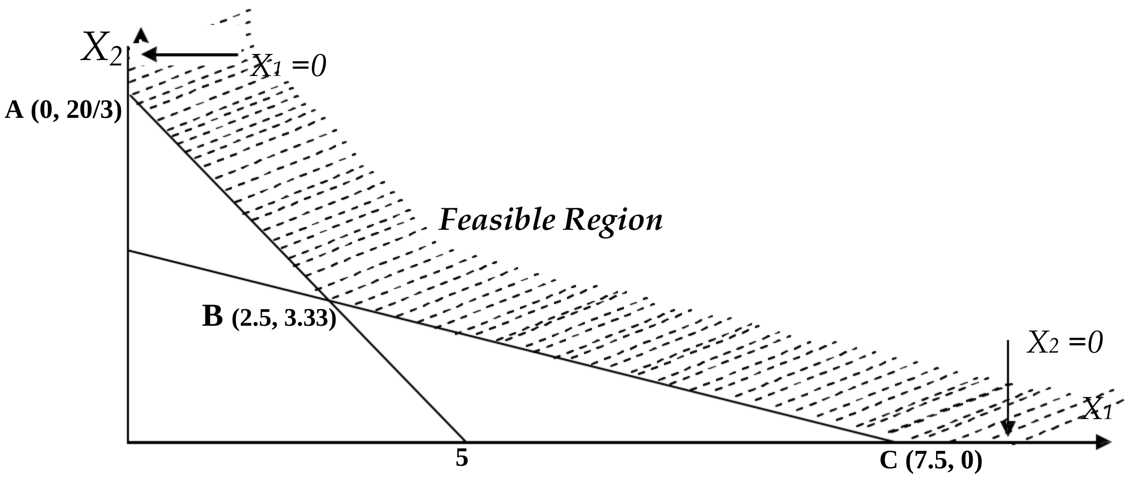
LPP MODEL

Constraint equation:

$20X_1 + 15X_2 = 100$ ==> (0, 20/3) and (5, 0)

$2X_1 + 3X_2 = 15$ ==> (0, 5) and (7.5, 0)

$X_1\ X_2 \geq 0$



Corner points	X1, X2 Coordinates	The value of Objective Function (Max Z=300 **X1** +250**X2**)
A	(0, 20/3)	$200
B	**(2.5, 3.33)**	**$162.5**
C	**(7.5, 0)**	**$**187.5

$\mathbf{X_1}$ =2.5

$\mathbf{X_2}$=3.33 and

Min Z= **162.5**

SELF CHECK EXERCISE 3.3

Solve the following Linear programming models using graphical method.

1. Maximize $Z = 7x_1 + 3x_2$

   Subject to: $2x_1 + 6x_2 \le 24$

   $6x_1 + 2x_2 \le 24$

   $x_1, x_2 \ge 0$

2. Maximize $Z = 30x_1 + 20x_2$

   Subject to: $10x_1 + 6x_2 \le 1000$

   $5x_1 + 4x_2 \le 600$

   $x_1, x_2 \ge$

3. Maximize $Z = 40x_1 + 50x_2$

   Subject to: $3x_1 + x_2 \le 9$

   $x_1 + 2x_2 \le 8$

   $x_1, x_2 \ge 0$

4. Minimize $Z = 4x_1 + 5x_2$

   Subject to: $2x_1 + 7x_2 \ge 31$

   $5x_1 + 3x_2 \ge 34$

   $x_1, x_2 \ge 0$

5. Minimize $Z = 2x_1 + 10x_2$

   Subject to: $2x_1 + x_2 \le 6$

   $5x_1 + 4x_2 \ge 20$

   $x_1, x_2 \ge 0$

6. Minimize $Z = 10x_1 + 6x_2$

   Subject to: $4x_1 + x_2 \ge 80$

   $2x_1 + x_2 \ge 60$

   $x_1, x_2 \ge 0$

A company owns two flour mills (A and B) which have different production capacities for ***HIGH, MEDIUM*** and ***LOW*** grade flour. This company has entered contract supply flour to a firm every week with 12, 8, and 24 quintals of ***HIGH, MEDIUM*** and ***LOW*** grade respectively. It costs the Co. $1000 and $800 per day to run mill A and mill B respectively. On a day, mill A produces 6, 2, and 4 quintals of ***HIGH, MEDIUM*** and ***LOW*** grade flour respectively.

Mill B produces 2, 2 and 12 quintals of ***HIGH, MEDIUM*** and ***LOW*** grade flour respectively. How many days per week should each mill be operated in order to meet the contract order most economically standardize? Solve graphically.

A manufacturer of light weight mountain tents makes two types of tents, ***REGULAR*** tent and ***SUPER*** tent. Each ***REGULAR*** tent requires 1 labor-hour from the cutting department and 3labor-hours from the assembly department. Each ***SUPER*** tent requires 2 labor-hours from the cutting department and 4 labor-hours from the assembly department .The maximum labor hours available per week in the cutting department and the assembly department are 32 and 84 respectively. Moreover, the distributor, because of demand, *will not take more than* 12 ***SUPER*** tents per week. The manufacturer sales each ***REGULAR*** tents for $160 and costs$110 per tent to make. Whereas ***SUPER*** tent ales for $210 per tent and costs $130 per tent to make.

Required:

A. Formulate the mathematical model of the problem

B. Using the graphic method, determine how many of each tent the company should manufacture each tent the company should manufacture each week so as to maximize its profit?

C. What is this maximum profit assuming that all the tents manufactured in each week are sold in that week?

3.7.2 SPECIAL CASES IN GRAPHICS METHODS

1. REDUNDANT CONSTRAINT

If a constraint when plotted on a graph doesn't form part of the boundary making the feasible region of the problem that constraint is said to be redundant.

Example:

A firm is engaged in producing two products A and B .Each unit of product A requires 2Kg of raw material and 4 labor-hrs for processing. Where as each unit of product B requires 3Kg of raw materials and 3hrs of labor. Every unit of product A needs 4hrs to packaging and every unit of product B needs 3.5hrs for packaging. Every week the firm has availability of 60Kg of raw material, 96 labor-hours and 105 hrs I the packaging department.

1 unit of product A sold yields $40 profit and 1 unit of B sod yields $35 profit.

Required:

a. Formulate this problem as a LPP

b. Find the optimal solution

Solution

	Products		
Resources	**Product A (X_1)**	**Product B (X_2)**	**Resource Available/week**
Raw Materials (kg)	2	3	60
Labor (hr)	4	3	96
Packaging (hr)	**4**	**3.5**	105
Profit per unit	**$40**	**$35**	

Let X_1 =The N$^{\underline{O}}$ of units of product A produced per week

X_2 =The N$^{\underline{O}}$ of units of product B produced per week

a. LPP Model

$Max.Z = 40X_1 + 35X_2$

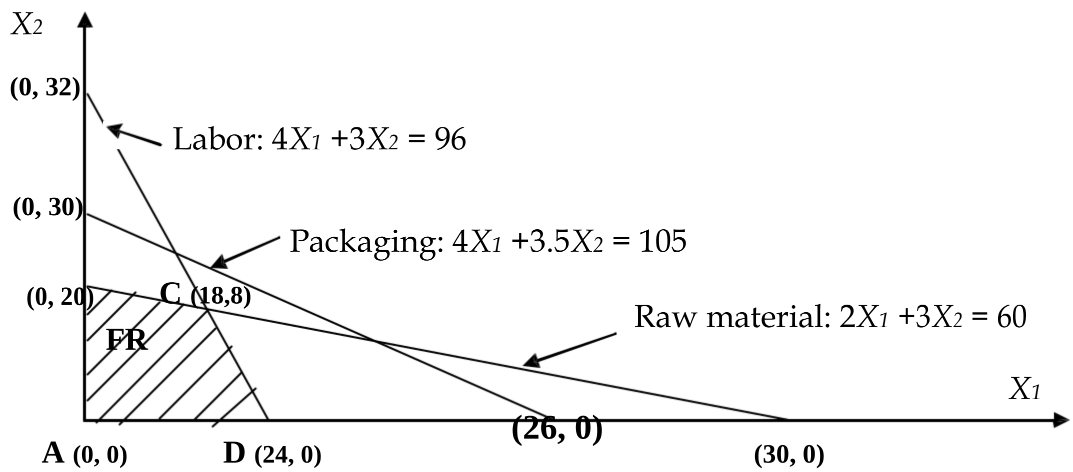
$St:$

$2X_1 + 3X_2 \leq 60$

$4X_1 + 3X_2 \leq 96$

$4X_1 + 3.5X_2 \leq 105$

$X_1, X_2 \geq 0$



The packaging hr is redundant.

Corner points	X1, X2 Coordinates	The value of Objective Function (Min Z=40 X_1 + 35X_2)
A	(0, 0)	0
B	(0, 20)	700
C	**(18, 8)**	**1000**
D	(24, 0)	960

X_1=18 and

X_2=8 and

MinZ= ***1000***

INTERPRETATION:

The company should produce and sale 18 units of product A and 8 units of product B per week so as to get a maximum profit of 1000.

By this production plan the entire raw material will be consumed.

$2X_1 + 3X_2 \leq 60$

$2(18) + 3(8) = 60$

$\underline{60 = 60}$ (No idle or unused raw material)

$4X_1 + 3X_2 \leq 96$

$4(18) + 3(8) \leq 96$

$\underline{96=96}$ (the entire labor hour will be consumed)

$4X_1 + 3.5X_2 \leq 105$

$100 \leq 105$ (There is to be idle or unused capacity of 5hrs in the packaging department)

NOTE:

The packaging hour's constraint does not form part of the boundary making the feasible region. Thus, this constraint is of no consequence and is therefore, redundant. The inclusion or exclusion of a redundant constraint does not affect the **optimal solution** of the problem.

MULTIPLE OPTIMAL SOLUTIONS /ALTERNATIVE OPTIMAL SOLUTIONS/

This is a situation where by a LPP has more than one optimal solution. Multiple optimal Solutions will be found if two corners give optimal solution, then the line segment joining these points will be the solution. We have unlimited number of optimal solution without increasing or decreasing the objective function.

EXAMPLE:

The information given below is for the products A and B.

	Machine hours per week		
	Product A	**Product B**	**Maximum available**
Cutting (Hr)	3	6	900
Assembly (Hr)	1	1	200
Profit per unit	**$8**	**$16**	

Assume that the company has a marketing constraint on selling products B and therefore it can sale a maximum of 125units of this product.

REQUIRED:

Formulate the LPP of this problem

Find the optimal solution

SOLUTION:

Let $\mathbf{X_1}$ =The $N^{\underline{o}}$ of units f product A produced per week

$\mathbf{X_2}$ =The $N^{\underline{o}}$ of units f product B produced per week

A. The LPP Model of the problem is:

$Max.Z=8X_1+16X_2$

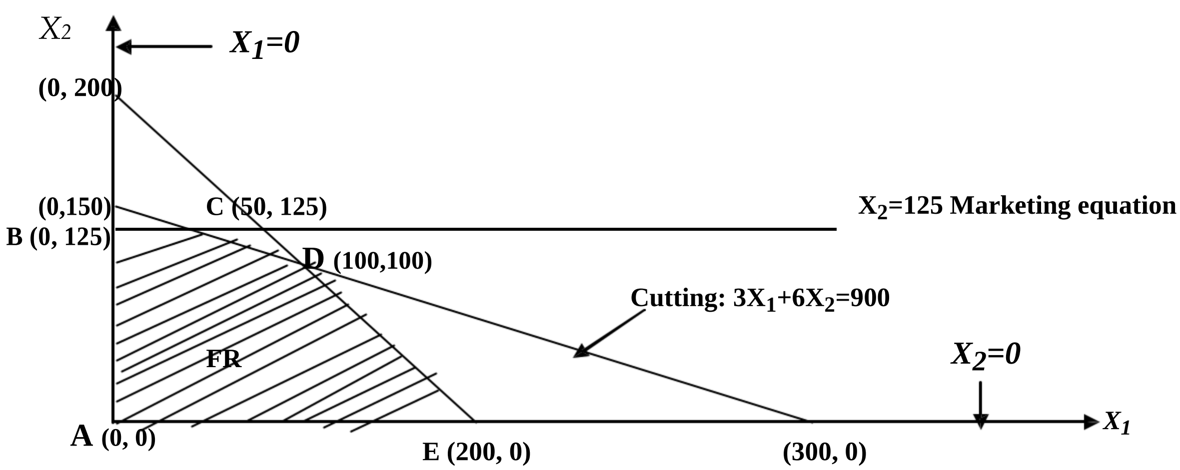
St:

$3X_1+6X_2 \le 900$

$_1+X_2 \le 200$

$X_2 \le 125$

$X_1,X_2 \ge 0$



Corners	X1, X2 Coordinates	The value of Objective Function
A	(0, 0)	0
B	(0, 125)	2000
C	**(50, 125)**	**2400**
D	**(100, 100)**	**2400**
E	(200, 100)	1600

INTERPRETATION:

Both ***C*** and ***D*** are optimal solutions. Any point on the line segment ***CD*** will also lead to the same optimal solution.

Multiple optimal solutions provide more choices for management to reach their objectives.

3. INFEASIBLE SOLUTION

A solution is called feasible if it satisfies all the constraints and the constraints and non-negativity condition. However, it is sometimes possible that the constraints may be inconsistent so that there is no feasible solution to the problem. Such a situation is called ***infeasibility.***

EXAMPLE:

Max $Z=20X_1+30X_2$

St:

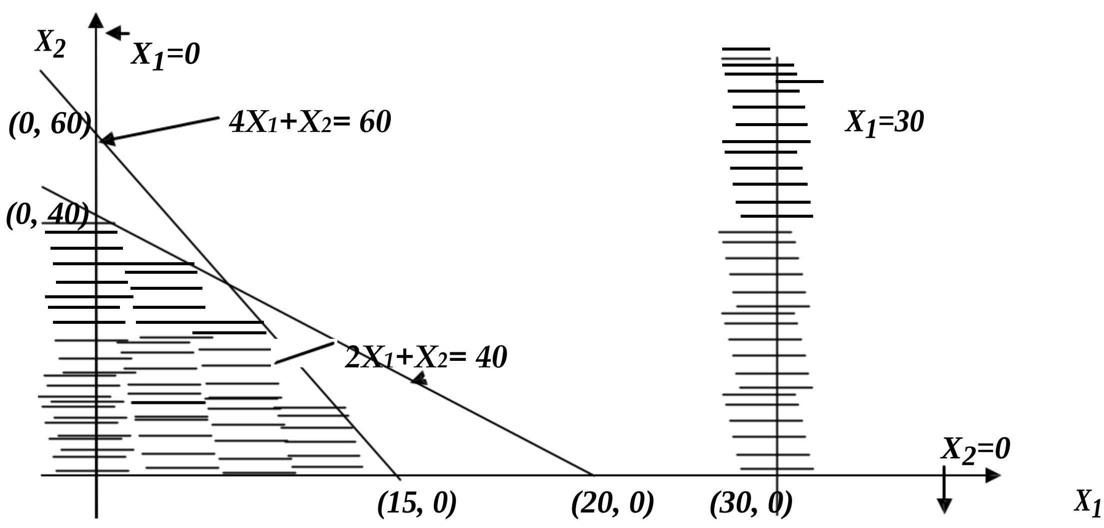
$2X_1+X_2 \leq 40$

$4X_1+X_2 \leq 60$

$X_1 \geq 30$

$X_1, X_2 \geq 0$

Solution:



NOTE:

In the above graph, there is no common point in the shaded area.

All constraints cannot be satisfied simultaneously and there is no feasible solution to the problem.

MIX OF CONSTRAINTS

Example:

ABC Gasoline Company has two refineries with different production capacities. Refinery A can produce 4,000gallons per day of ***SUPER UNLEADD GASOLINE***, 2000 gallons per day of ***REGULAR UNLEADED GASOLINE*** and 1000 gallons per day of ***LEADED GASOLINE***. On the other hand, refinery B can produce 1000 gallons per day of ***SUPER UNLEADED***, 3000 gallons per day of ***REGULAR UNLEADED*** and 4,000 gallons per day of ***LEADED***.

The company has made a contract with an automobile manufacturer to provide 24000 gasoline of ***SUPER UNLEADED***, 42000 gallons of ***REGULAR UNLEADED*** and 36000 gallons of ***LEADED*** .The automobile manufacturer wants delivery in not more than 14 days. The cost of running refinery A is $1500 per day and refinery B is $2400 per day.

Required:

Formulate this problem as a LPP

Determine the number of days the gasoline company should operate each refinery in order to meet the terms of the above contract most economical.(i.e. At a minimum running cost)

Which grade of gasoline would be over produced?

Solution:

Production per day (in gallons)	**Grade of gasoline**		**Contract with an automobile manufacturer**
	A	B	
SUPER UNLEADED	4000	1000	24,000
REGULAR UNLEADED	2000	3000	42,000
LEADED	1000	4000	36,000
RUNNING COST PER DAY	**$1,500**	**$2,400**	

The automobile manufacturer wants delivery in not more than 14 days.

Let X_1 =The $N^{\underline{o}}$ of days refinery A should work.

X_2 =The $N^{\underline{o}}$ of days refinery B should work.

A. LPP of the problem

Min $Z=1500X_1+2400X_2$

St:

$4000X_1+1000X_2\geq 24000$

$2000X_1+3000X_2\geq 42000$

$1000X_1+2000X_2\geq 36000$

$X_1 \leq 14$

$X_2\leq 14$

$X_1,X_2 \geq 0$

T o simplify the problem divide by 1000 the constraints

Min $Z=1500X_1+2400X_2$

St:

$4X_1+1X_2\geq 24$

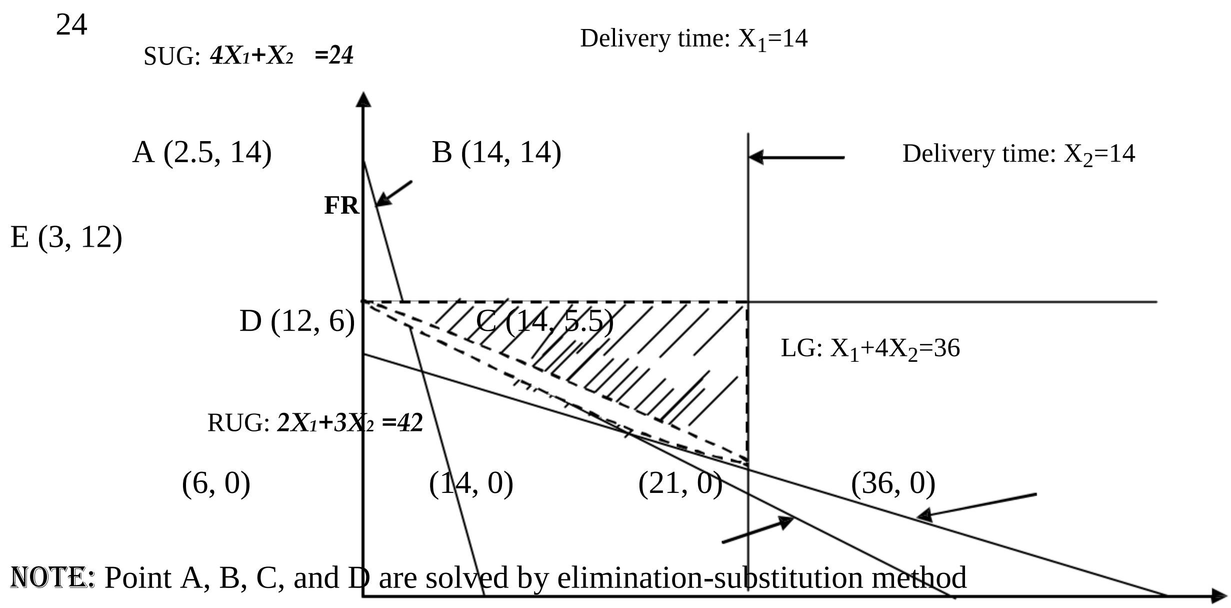
$2X_1+3X_2\geq 42$

$X_1+4X_2 \geq 36$

$X_1 \leq 14$

$X_2\leq 14$

$X_1,X_2 \geq 0$



NOTE: Point A, B, C, and D are solved by elimination-substitution method

Corners	X1, X2 Coordinates	The value of Objective Function (Max $Z=1500X_1 + 2400X_2$)
A	(2.5, 14)	37350
B	(14, 14)	54600
C	**(14, 5.5)**	**34200**
D	**(12, 6)**	**32400**
E	(3, 12)	33300

INTERPRETATION:

The oil company should operate refinery A for 12 days and refinery B for 6 days at a minimum operating cost of $32,400.

c. Is there any over production

SUG: $4000X_1+1000X_2 \geq 24000$

$4000(12) +1000(6) \geq 24000$

$\underline{54000 > 24000}$

Therefore, 30,000 gallons over production

RUG: $2000X_1+3000X_2 \geq 42000$

$2000(12) +3000(6) \geq 42000$

$\underline{42000 > 42000}$

Therefore, there is no over production of RUG

LG: $1000X_1+4000X_2 \geq 36000$

$1000(12)+1000(6) \geq 36000$

$\underline{36000 > 36000}$

Therefore, No over production of LG

5. UNBOUNDED SOLUTION

When the value of decision variables in LP is permitted to increase infinitely without violating the feasibility condition, then the solution is said to be unbounded .Here, the objective function value can also be increased infinitely. However, an unbounded feasible region may yield some definite value of the objective function.

EXAMPLE:

Use the graphical method to solve the following LPP.

1. Max $.Z=3X_1+4X_2$

St: $X_1-X_2 \leq -1 \Longrightarrow -X_1+X_2 \geq 1$ since the quantity solution is positive

$-X_1+X_2 \leq 0$

$X_1, X_2 \geq 0$

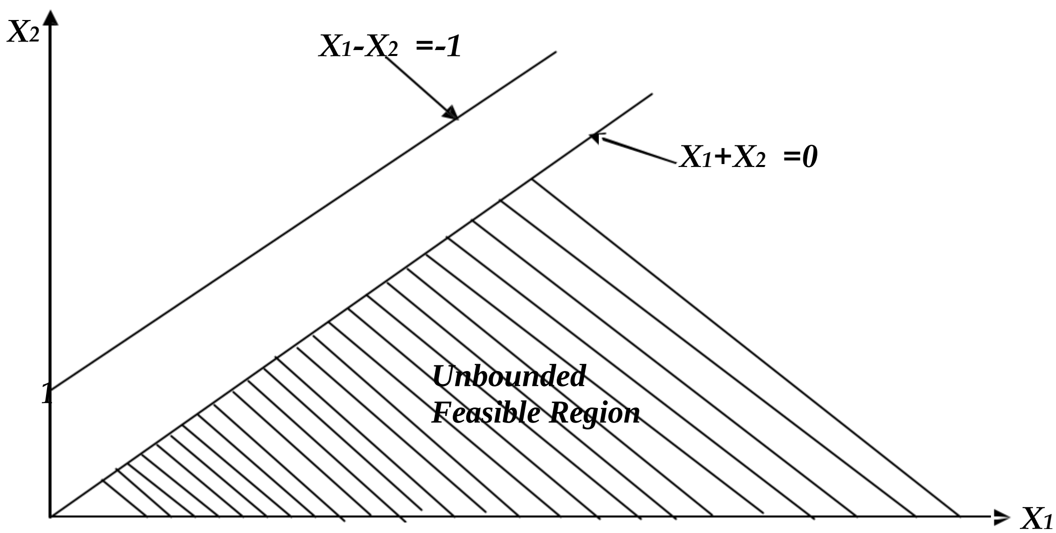


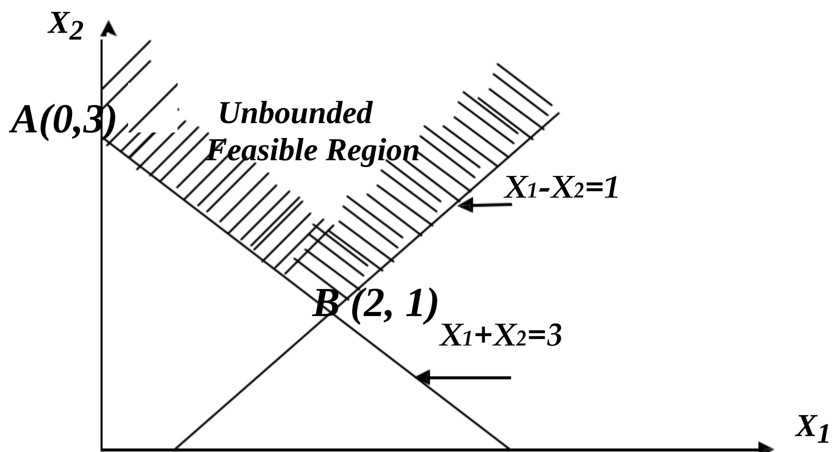
Fig: Unbounded Solution

2. Max. $Z=3X_1+2X_2$

St: $X_1-X_2 \leq 1$

$X_1+X_2 \leq 3$

$X_1, X_2 \geq 0$



Note here that the two corners of the region are A(0,3) and .B(2,1).The value of Max Z(A)=6 and Max Z(B)=8. But there exist number of points in the shaded region for which the value of the objective function is more than 8.For example, the point (10, 12) lies in the region and the function value at this point is 70 which is more than 8.

REMARK:

An unbounded solution does not mean that there is no solution to the given LPP, but implies that there exits an infinite number of solutions.

SELF CHECK EXERCISE 3.4

I. Use graphical method to solve the following LPP.

1. Max. $Z = 7/4X_1 + 3/2X_2$

   Subject to:

   $8X_1 + 5X_2 \leq 320$

   $4X_1 + 5X_2 \leq 20$

   $X_1 \geq 15$

   $X_2 \geq 10$

   $X_1, X_2 \geq 0$

2. Max. $Z=3X_1+2X_2$

   Subject to:

   $-2X_1 + 3X_2 \leq 9$

   $X_1 - 5X_2 \geq -20$

   $X_1, X_2 \geq 0$

3. Max. $Z = 3X_1 + 2X_2$

Subject to:

$X_1 - X_2 \leq 1$

$X_1 + X_2 \geq 3$

$X_1, X_2 \geq 2$

4. Max. $Z = X_1 + X_2$

Subject to:

$X_1 + X_2 \leq 1$

$-3X_1 + X_2 \geq 3$

$X_1, X_2 \geq 0$

5. Max. $Z = 6X_1 - 4X_2$

Subject to:

$2X_1 + 4X_2 \leq 4$

$4X_1 + 8X_2 \geq 16$

$X_1, X_2 \geq 0$

6 .Max. $Z = X_1 + 1/2X_2$

Subject to:

$3X_1 + 3X_2 \leq 12$

$5X_1 \leq 10$

$X_1 + X_2 \geq 8$

$-X_1 + X_2 \geq 4$

$X_1, X_2 \geq 0$

II. Solve the following LP problems using the graphical method.

1. Max. $Z = 15X_1 - 10X_2$

Subject to:

$4X_1 + 6X_2 \leq 360$

$3X_1 + 0X_2 \leq 180$

$0X_1 + 5X_2 \leq 280$

$X_1, X_2 \geq 0$

2. Max. $Z = 2X_1 + X_2$

Subject to:

$X_1 + 2X_2 \leq 10$

$X_1 + X_2 \leq 6$

$X_1 - X_2 \leq 2$

$X_1 - 2X_2 \leq 1$

$X_1, X_2 \geq 0$

3. Max. $Z = 10X_1 + 15X_2$

Subject to:

$2X_1 + X_2 \leq 26$

$2X_1 + 4X_2 \leq 56$

$-X_1 + X_2 \leq 5$

$X_1, X_2 \geq 0$

4. Min. $Z=3X_1+2X_2$

Subject to:

$5X_1 + X_2 \geq 10$

$X_1+X_2 \geq 6$

$X_1+4X_2 \geq 12$

$X_1, X_2 \geq 0$

5. Min. $Z = -X_1 + 2X_2$

Subject to:

$-X_1 + 3X_2 \leq 26$

40

$X_1+X_2 \leq 6$

30

$X_1-X_2 \leq 2$

$X_1, X_2 \geq 0$

6. Min. $Z = 20X_1 + 10X_2$

Subject to:

$X_1+2X_2 \leq$

$3X_1+4X_2 \geq$

$4X_1 + 3X_2 \geq 60$

$X_1, X_2 \geq 0$

III. Formulate the following problems into LP and solve Graphically

1. A manufacturer produces two different models; **X** and **Y**, of the same product .The raw materials r_1 and r_2 are required for production. At least 18 Kg of $\mathbf{r_1}$ and 12 Kg of $\mathbf{r_2}$ must be used daily. Almost at most 34 hours of labor are to be utilized .2Kg of $\mathbf{r_1}$ are needed for each model **X** and 1Kg of $\mathbf{r_1}$ for each model **Y**. For each model of **X** and **Y**, 1Kg of $\mathbf{r_2}$ is required. It takes 3 hours to manufacture a model **X** and 2 hours to manufacture a model **Y**. The profit realized is $50 per unit from model **X** and $30 per unit from model **Y**. How many units of each model should be produced to maximize the profit?

2. A manufacturing firm produces two machine parts P_1 and P_2 using milling and grinding machines .The different machining times required for each part, the machining times

	Manufacturing time Required (min)		
Machine	**P1**	**P2**	**Maximum Available hrs.**
Lathe	10	5	**25,000**
Milling Machine	4	10	**2000**
Grinding Machine	1	1.5	**450**
Profit per unit ($)	**$50**	**$100**	

Determine the number of pieces of P_1 and P_2 to be manufactured per week to maximize profit.

A person requires 10, 12 and 12 units of chemicals ***A***, B and ***C*** respectively for his garden. A liquid product contains 5, 2 and 1 units of ***A***, ***B*** and ***C*** respectively per jar. A dry product contains 1, 2 and 4 units of ***A***, ***B*** and ***C*** per carton. If the liquid product sells for $3 per jar and the dry product sells $2 per carton, how many of each should be purchased in order to minimize cost and meet the requirement?

CHAPTER FOUR: MATHEMATICS OF FINANCE

4.1. OBJECTIVES OF THE CHAPTER

Up on completion of this chapter, you should be able to:

- Understand the key aspects of financial mathematics
- Know what is simple and compound interests are and their computation
- Know the principles of compounding and discounting
- Know the characteristics and objectives of ordinary annuities
- Perform computations relating to future value and present value of an annuity
- Understand the nature and computations related to mortgage loans.

4.2. OVERVIEW OF MATHEMATICS OF FINANCE

In this chapter we will introduce you to the mathematical methods and formulas that are useful in operation of one's own business.

From the financial stand point business transactions may be considered as inflows and outflows of funds over time. Funds must be borrowed on long term, a year or longer basis to replace improve, or add to existing facilities such as buildings, equipment, and machinery. Funds must also be borrowed on short term, less than a year basis to meet immediate needs such as raw materials or obligations such as payroll. But money has time value, that is, a birr today is worth more than a birr tomorrow, which is expressed in terms of interest charges. Since the use of money bears interest, management needs to optimize the employment of investible funds; it must evaluate a wide array of investment opportunities and choose the one which is most profitable.

Optimal allocation of investible funds requires the use of interest formulas. Dear learners, at this juncture, we shall deal with simple interest, future and present value under simple interest, compound interest, compound interest future and present value, annuity and other uniform payment formulas. Finally, we shall discuss a few special cases relating to the sinking fund and mortgages.

4.3. COMMON TERMS

To facilitate further studies in mathematics of finance, it is a must to have common understanding of what the following terms mean.

PRINCIPAL AMOUNT(P)

Is the amount of money that is initially being considered to be invested or deposited. It might be an amount to be invested or loaned or the initial value or cost of plant or machinery. Thus, if an individual was considering a bank loan of say Birr 50,000, this would be referred to as the principal amount to be borrowed.

NUMBER OF TIME PERIODS (n)

The number of time periods over which an amount of money are being invested or borrowed is normally a number of years, it could represent other time periods, such as a number of quarters and months.

RATE OF INTEREST (i)

Interest is the name given to proportionate amount of money which is added to some principal amount (invested or borrowed). It is normally denoted by symbol r and expressed as a percentage rate per annum. For instance, if Birr 100 is invested at an interest rate of 10% per annum, it will accrue birr 100 + (10% x 100) = Birr 110 at the end of one year.

ACCRUED AMOUNT (A)

Generally refers to a principal amount after some time has elapsed for which interest has been calculated and added.

4.4. KINDS OF INTEREST

When an amount of money is invested over a number of years, the interest earned van be seen from two perspectives.

4.4.1. SIMPLE INTEREST

Simple interest is a fixed percentage of the principal, P_0 , which is paid to an investor each year, irrespective of the number of years the principal has been left on deposit; that is, money invested at simple interest will increase in value by the same amount each year. So, if the investor is paid a fixed annual amount, i% of P_0 , then the amount of simple interest, I, received over t years is given by the formula

I = P_0 x r x t

Therefore, the total value after t years, Pt is the principal plus interest and is given by

$P_t = P_0 + (P \times r \times t)$

$P_1 = P_0 (1 + rt)$ 4.1

When the total value (future value), the interest rate and time are known, the principal (Present value) may be calculated by rewriting formula (4.1) as

$P_0 (1 + rt) = Pt$

$P_0 = \frac{Pt}{1 + rt}$ 4.2

Simple interest is where any interest earned is not added back to the principal amount invested. If for instance, birr 1000 is invested at 10% simple interest per annum. The following table shows the state of the investment, year by year:

Table 4.1 Computation of Simple interest

Year	Amount on which interest is calculated	Interest earned	Cumulative amount
1	Birr 1,000	10%of Birr 1000 = 100	Birr 1,100
2	Birr 1,000	10%of Birr 1000 = 100	Birr 1,100
3	Birr 1,000	10%of Birr 1000 = 100	Birr 1,100
4	Birr 1,000	10%of Birr 1000 = 100	Birr 1,100

A) COMPUTATION OF THE AMOUNT OF SIMPLE INTEREST

Assume that a person deposit a sum of money p in a saving account or borrow a sum of money p from a lending institution. Then p is referred to as the principal. When money is borrowed a fee is charged which is called interest. Interest is usually computed as a percentage of the principal over a given period of time. Dear learners, unless otherwise stated, the interest rate is an annual rate.

SIMPLE INTEREST

$$I = P \times r \times t$$

Where

I = amount of interest

P = principal

r = annual simple interest rate

t = time in years

In general, if a principal p is borrowed at a rate r, then after t years the borrower will owe the lender an amount A that will include the principal p plus the interest I. Since p is the amount that is borrowed now and A is the amount that must be paid back in the future, p is often referred to as the present value and A as the future value and is given by the following formulas respectively.

$A = P + I$

$\quad = P + P \times r \times t$

$A = P(1 + rt)$ (Simple interest future value amount)

$$\frac{}{1 + rt} = \frac{P(1 + rt)}{1 + rt}$$

$P = \dfrac{A}{1 + rt}$ (Simple interest present value formula)

Given any three of the four variables A, P, r, and t, we can solve for the fourth. At this juncture, however, you need to bear in mind that the time may be given in months, weeks or days. In simple interest formula, t must be in years, and so conversion must be made.

EXAMPLE 1

Mr. Abule borrowed a principal of Birr 1000 at 8% per year simple interest. Find the future value after;

two yers
three months
180 days

SOLUTION

Given

P =Birr 1,000

r = 0.08 per annum (8/100)

t = 2 years

A = P (1 + rt)

1,000 (1 + (0.08 x 2))

1,000 (1 + 0.16)

1,000 (1.16)

1,000 (1.16) = **Birr 1,160**

A = P (1 + rt)

Here the time period is three months. So, conversion into years must be made and hence three months corresponds to 3/12 = 0.25 year.

A = 1,000 (1 + (0.08 x 0.25))

1,000 (1 + 0.02)

1,000 (1.02)

1,000 (1.02) = **Birr 1,020**

iii) A = P (1 + rt)

Once more again, we have to convert 180 days into years. By assuming a year to have 360 days, 180 days corresponds to t = 180/360 = 0.5 year. Thus, from the simple interest formula

A = 1,000 (1 + (0.08 x 0.5))

1,000 (1 + 0.04)

1,000 (1.04)

1,000 (1.04) = **Birr 1,040**

SELF CHECK EXERCISES 4.1

What rate of interest let a sum of money to double in 5 years?

How long does it take for a sum of money to triple at a 5% simple interest rate?

A saving account opened 3 months ago now has a balance of Birr 20,400. If the bank pays 8% simple interest, how much money was deposited?

Find the amount that an investor should deposit in a bank today if he needs Birr 20,000 in 3 months at a simple interest rate of 9%.

What percentage of simple interest return must a firm get on their deposit if it wants its Birr 25,000 to grow to birr 26,500 in 6 months?

Mr. Bitew wanted to buy a new car for his family. The cost of the car was Birr 250,000. He was in short of cash and went to his local bank and borrowed Birr 150,000 for 6 years at an annual interest rate of 12%. Find the total simple interest and the maturity value of the loan.

4.4.2. COMPOUND INTEREST

Dear learners, at this juncture we need to have a look at another kind of interest computation.

If the interest which is due is added to the principal at the end of each interest period, then this interest as well as the principal will earn interest during the next period. In such a case the interest is said to be compounded.

The result of compounding interest is that starting with the second compounding the account earns interest on interest in addition to earning interest on principal.

The sum of the original principal and all the interest earned is the **Compound Amount**. The difference between compound amount and the original principal is the **Compound interest.**

EXAMPLE

Suppose that Birr 1,000 is invested at 10% compound interest. This situation can be described using the following table, which shows the state of the investment year by year.

Table 4.2 Computation of Compound interest

Year	Amount on which interest is calculated	Interest earned	Cumulative amount accrued
1	Birr 1,000	10% of Birr 1,000 = 100	Birr 1,100
2	Birr 1,100	10% of Birr 1,000 = 110	Birr 1,210
3	Birr 1,210	10% of Birr 1,000 = 121	Birr 1,331
4	Birr 1,331	10% of Birr 1,000 = 133.10	**Birr 1,464.10**

The difference between the methods of interest computation can easily be seen by comparing the above two tables. By looking at the table, you can recognize that the amount on which simple interest is calculated is always the same, while the amount varies year after year for compound interest.

A) Amount of Compound Interest

If at the end of a payment period, the interest due is reinvested at the same rate, then the interest as well as the original principal will earn interest during the next payment period. Interest paid on interest reinvested is called compound interest. Since interest rates are generally quoted as annual nominal rates, the rate per compounding period is found by dividing the annual nominal rate by the number of compounding periods per year.

The rate per period is equivalent to a nominal rate of 12% compounded for various periods is given in the following table.

Table 4.3 Determination of conversion periods

12% interest compounded	m (conversion period)	i = r/m%
Annually	1	12 or 0.12
Semi – annually	2	6 or 0.06
Quarterly	4	3 or 0.03
Monthly	12	1 or 0.01
Weekly	52	12/52 = 0.23 or 0.0023
Daily	365	12/365 = 0.033 or 0.00033

In general, if P is the principal earning interest compounded m times a year at an annual rate of r, then i = r/m, the rate per period, the amount A at the end of each period is

$A = P(1 + i)^{1}$ at the end of 1st period

$P(1 + i)^{2}$ at the end of 2nd period

$P(1 + i)^{3}$ at the end of 3rd period

$P(1 + i)^{n}$ at the end of nth period

Compound interest future value can be computed as follows

$A = P(1 + r/m)^{tm} = P(1 + i)^{n}$ **$i = r/m$**

Where

r = annual nominal rate

m = number of compounding periods per year

i = rate per compound period

n = total number of compounding periods

p = principal or present value

A = amount or future value at the end of n periods

EXAMPLE

What are the compound amount and compound interest at the end of one year if Birr 10,000 is borrowed at 8% compound quarterly?

SOLUTION

P = Birr 10,000 Total number of conversions = 4

r = 8% t = one year

Total number of conversion periods (m) = 4 times = quarter

$i = \frac{r}{\ } . = \frac{8\%}{4} = 2\%$

Original principal……………………………………………..Birr 10,000
Add: interest for the 1st quarter, I = Br. 10,000 x .02……………Br. <u>200</u>
Principal at the end of 1st quarter……………………………Br.10,200 = Br.10,000$(1.02)^1$
Add: Interest for the 2nd quarter, 10,200 x .02…………………..Br. <u>204</u>
Principal at the end of 2nd quarter……………………….......Br.10,404 = Br.10,000 $(1.02)^2$
Add: Interest for the 3rd quarter, 10,404x.02……………………Br. <u>208.08</u>
Principal at the end of 3rd quarter……………………………Br.10,612.08 = Br.10,000$(1.02)^3$
Add: Interest for the 4th quarter, 10,612.08 x .02………………..Br. <u>212.2416</u>
Principal at the end of 4th quarter…………………………Br.10,824.3216 = Br.10,000$(1.02)^4$
(Amount at the end of the year)

If we summarize the above computations, we will come up with a pattern that leads to a general formula for computing compound interest:

1st quarter: S = 10,000 $(1.02)^1$
2nd quarter: S = 10,000 (1.02) (1.02) = 10,000 $(1.02)^2$
3rd quarter: S = 10,000 (1.02) (1.02) (1.02) = 10,000 $(1.02)^3$
4th quarter: S = 10,000 (1.02) (1.02) (1.02) (1.02) = 10,000 $(1.02)^4$

Therefore, for the above question, the amount is equal to

$$A = P (1 + I)^n$$

$$10,000\ (1.02)^4$$

<u>**10,824.3216 Birr**</u>

Compound Interest = Compound Amount - Original Principal

10,824.3216 - 10,000

Birr 824.3216

Dear learners, compound amount can also be computed using **Logarithm** as follows.

$A = 10{,}000\ (1.02)^4$

$\log A = \log 10{,}000 + \log^{(1.02)4}$

$\log 10{,}000 + 4 \log 1.02$

$4 + 4\ (0.0086)$

$4 + 0.0344$

$\log A = 4.0344$

$A = \text{antilog } 4.0344$

Br.10,824.30

Rules of Logarithms

$\log a^a = 1$

$\log^{mp} = p \log m$

$\log^{mn} = \log^m + \log^n$

$\log^{m/n} = \log^m - \log^n$

EXAMPLE

Find the compound amount compound interest resulting from the investment of Birr 1000 at 6% for 10 years,

A. Compounded annually.
B. Compounded semiannually.
C. Compounded quarterly.
D. Compounded monthly.
E. compounded weekly
F. Compounded daily
G. Compounded hourly
H. Compounded continuously

SOLUTION

P = Birr

1,000 t = 10

years m = 1

r = 6%

A =? i

= 6%

n = 10

$A = p\ (1+i)^{n}$

$1{,}000\ (1.06)^{10}$

Birr 1,790.85

Compound interest = Compound amount - Principal

1,790.85 - 1000

Birr 790.85 Birr

SOLUTION

P = Birr
1,000 r = 6%

m = 2

t = 10 years

i = 3%

n = 20

$A = p\ (1+i)^{n}$

$1{,}000\ (1.03)^{20}$

Birr 1,806.11

Compound interest = Compound amount - Principal

1,806.11 - 1000

Birr 806.11

SOLUTION

C. P = Birr 1,000 $A = 1,000 (1.06)^{40}$

r = 6% = **Birr 1,814.02**

m = 4

t = 10 years **Compound interest = Compound amount - Principal**

i = .015 = 1814.02 - 1000

n = 40 = **Birr 814.02**

SOLUTION

P = Birr 1000A = 1,000 (1.005) 120

r = 6% = **Birr 1,819.40**

t = 10 years

m=12 **Compound interest = Compound amount - Principal**

i = .005 = 1,819.40 - 1000

n = 120 = **Birr 819.40**

SOLUTION

E. P = Birr 1000 $A = 1,000 (1.0012)^{520}$

r = 6% = **1,821.49 Birr**

t = 10 years

m=52 **Compound interest = Compound amount - Principal**

i = .0012 = 1821.49 - 1000

n = 520 = **Birr 821.49**

SOLUTION

P = Birr 1000
r = 6%

t = 10 years

m=365

i = .01644%

n = 3650

$A = 1,000\ (1.0001644)^{3650}$

Birr 1,822.03

Compound interest = Compound amount - Principal

1,822.03 - 1000

Birr 822.03

SOLUTION

P = Birr

1000 r = 6%

t = 10 years

m= 8760

i = .000685%

n = 87600

$A = 1,000\ (1.00000685)^{687,600}$

Birr 1,822.12

Compound interest = Compound amount - Principal

1,822.12 - 1000

Birr 822.12

SOLUTION

H . $(1+i)^n = (1+r/m)^{mt}$

$f(x) = \left(1 + \frac{1}{x}\right)^x$ if x approaches infinity $\left(1 + \frac{1}{x}\right)^x$ becomes closer to

2.71828 = e

Let m/r = x as m↑ x ↑

r/m = 1/x = m = rx

$$\left(1 + \frac{r}{m}\right)^{mt} = \left(1 + \frac{1}{x}\right)^{mt}$$

$$= \left(1 + \frac{1}{x}\right)^{rxt}$$

$$= \left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt}$$

$$= e^{rt}$$

$A = Pe^{rt}$

P = Birr 1000	$= 1{,}000\, e^{.06x10} = 1000\, e^{.6}$
r = 6%	= **<u>Birr 1,822.12</u>**
t = 10 years	**Compound interest = Compound amount - Principal**
m = infinite	1,822.12 - 1000 = **<u>Birr 822.12</u>**

TIME COMPUTATION

EXAMPLE

How many years will it take for Birr 12,000 to grow to 24,000 when interest is 7% per annum compounded annually?

GIVEN

P = 12,000

A = 24,000

i = r/m = 0.07/1 0.07

n = ?

SOLUTION

$A = P(1 + i)^n$

$24{,}000 = 12{,}000(1 + 0.07)^n$ dividing both sides by 12,000, we obtain,

$2 = (1 + 0.07)^n$ ⇨ taking logarithm of both sides

$\text{Log } 2 = \log 1.07^n$ ⇨ using log rule that $\log a^b = b \log a$

$\text{Log } 2 = n \log 1.07$ ⇨ dividing both sides by log 1.07, we obtain

$\text{Log } 2/\log 1.07 = n$

$n = \underline{0.3010} = 10.24$ years

INTEREST

RATE EXAMPLE

What annual rate of interest compounded quarterly should one obtain if he/she wants to double his/her investment in five years?

Here, if P is the principal, then A = 2P and n = 4 x 5 = 20, we want to find i/4 in the equation. i.e. interest for a quarter.

$$\frac{2P}{P} = \frac{P(1+i)\ 20}{P}$$

$2 = (1 + i)^{20}$ ⇨ taking logarithm of both sides

$\text{Log } 2 = \log (1 + i)^{20}$ ⇨ using log rule that $\log a^{b} = b \log a$

$\text{Log } 2 = 20 \log (1 + i)$ ⇨ dividing both sides by 20, we obtain

$\text{Log } 2/20 = \text{Log } (1 + i)$

$$\frac{0.3010}{20} = \text{Log } (1 + i)$$

$0.01505 \log (1 + i)$

$1 + i = \text{Antilog } 0.01505$

$1 + i = 1.0353$

$i = 1.0353 – 1 = 0.0353$

Hence, annual rate of interest needed to double the principal in 5 years is 4i = 4 x 0.0353 = 0.1412 or 14.12%.

SELF CHECK EXERCISES 4.2

How much money should be deposited in a bank paying interest at a rate of 6% compounded monthly so that at the end of 3 years accumulated amount is Birr 20,000?

How long will it take for Birr 40,000 to amount to Birr 70,000 if it is invested at 7% compounded quarterly?

A firm wishing to have enough money for a retiring manager 17 years from now decided to buy a Birr 30,000 face value zero coupon bond. If the money is worth 10% compounded annually, what should they pay for the bond?

How long will it take to accumulate Birr 650 if Birr 500 is invested at 10% compound quarterly?

Birr 2000 is deposited in an account. After one year of monthly compounding, the balance in the account is Birr 2,166. What is the annual percentage rate for this account?

A person deposits Birr 10,000 in a savings account that pays 6% compounded semi annually. Three years later, this person deposits an additional Birr 8,000 in the savings account. Also, at this time, the interest rates changes to 8% compounded quarterly. How much money is in the account 5 years after the original Birr 10,000 is deposited?

4.5. PRESENT VALUE

Frequently it is necessary to determine the principal P which must be invested now at a given rate of interest per conversion period in order that the compound amount A be accumulated at the end of n conversion periods. This process is called discounting and the principal is now a discounted value of a future income A.

$A = P(1+i)^n$ dividing both sides by $(1+i)^n$ leads to

$$P = \frac{A}{(1+i)^n} = p = A(1+i)^{-n}$$

Present values of a compound amount:

$$P = A(1+i)^{-n}$$

Where:

p = principal / present value

A = compound amount (or future value)

i = interest rate per conversion period

n = total number of conversion periods

EXAMPLE

Find the present value of a loan that will amount to Birr 5,000 in four years if money is worth 10% compounded semi annually.

SOLUTION

A = 5,000 Birr

t = 4 years

m = 2

r = 10%

P =?

= A $(1+i)^{-n}$

5,000 $(1.05)^{-8}$

Birr 3,384.20

Self Check Exercises 4.3

How much must be deposited now in an account paying 6% compounded monthly in order to have just 20,000 Birr in the account 4 years from now?

If money worth 14% compounded semi-annually, would it be better to discharge a debt by paying Birr 500 now or Birr 600 eighteen months from now?

How much should be deposited in an account paying 8% compounded quarterly in order to have a balance of Birr 8,000 nine years from now?

4.6. Effective Rate

Obviously, for a stated annual interest rate, the amount of interest accumulated depends upon the frequency of conversion. This is because interest which has been earned subsequently earns interest itself. When interest is compounded more than once a year, the stated annual rate is called a Nominal Rate. The effective rate corresponding to a given nominal rate r converted m times a year is the simple interest rate that would produce an equivalent amount of interest in one year.

Effective rates are, therefore, the simple interest rates that would produce the same return in one year had the same principal been invested at simple interest without compounding.

If P = Principal, A = Amount, r = nominal rate, m = number of conversion periods per year, the compound interest for ***one year*** on principal p is,

I=A-P

$= p(1 + r/m)^{m} - p$

The effective rate of interest is (re)= $\frac{\text{Compound interest}}{\text{principal}} = \frac{I}{P}$. From the above statement:

$I = p(1 + r/m)^{m} - p$

$= P[(1+r/m)^{m} - 1)$ Divide both sides by p

$I/p = P\left[\left(1+\frac{r}{m}\right)^{m} - 1\right]$

$re = \left(1+\frac{r}{m}\right)^{m} - 1$

$= (1+i)^{m} - 1$

In continuous compounding case:

$A = Pe^{rt}$ for one year $A = Pe^{r}$

I=A-P

$Pe^{r} - p$, multiplying both sides by

1/p $1/p \times I = P(e^{r}-1) \times 1/p$

<u>$re = e^{r}-1$</u>

Effective rates are used to compare competing interest rates offered by banks and other financial institutions.

EXAMPLE

1. What is the effective rate of money invested at 6% compounded quarterly?

SOLUTION

R=6%

m = 4

$$re = (1+r/m)^{m}-1$$

$$= \left(1+\frac{.06}{4}\right)^{4} - 1$$

$$= (1.015)^{4} - 1$$

re = **6.14%**

An investor has two opportunities to invest his money. The first investment opportunity (opportunity A) pays 15% compounded monthly and the second investment opportunity (opportunity B) pays 15.2% compounded semiannually. Which is the better investment, assuming all other things constant?

SOLUTION

Nominal rates with different compounding periods cannot be compared directly. We must first find the effective rate of each nominal rate and then compare the effective rates to determine which investment will yield the larger return.

Effective rate for investment opportunity. A	Effective rate for investment opportunity B
$re= (1+r/m)^{m} - 1$	$re= (1+r/m)^{m} - 1$
$\left(1+\frac{115}{12}\right)^{12} - 1$	$\left(1+\frac{115}{12}\right)^{2} - 1$
$= (1.0125)^{12} - 1$	$= (1.076)^{2} - 1$
16.075%	15.778%

Since the effective rate for A is greater than the effective rate for B, Investment opportunity A is the preferred investment.

SELF CHECK EXERCISES 4.4

A bank states that the effective interest on savings accounts that earn continuous interest is 10%. Find the nominal rate.

Find the effective rate of interest corresponding to a nominal rate of 9% per year compounded;

- Semi annually
- Quarterly
- Monthly

A saving account opened 3 months ago now has a balance of Birr 20,400. If the bank pays 8% simple interest, how much money was deposited?

4.7. ANNUITIES

The term annuity is derived from the Latin word annum which means annual payment of a fixed amount. But as a matter of practice, it means any periodical payment of a fixed amount made at a regular interval of say, one year, half year, a quarter or a month, to discharge one's obligation or to meet some amount of accumulations.

4.7.1. Characteristics of Annuity

Annuity is recurring payment of fixed amount payable either annually, semi annually, quarterly, monthly or the like. It warrants accounting for compound interest at an agreed time. Annuity is paid against a lump sum received at present or a compounded amount receivable after a certain period. It is paid either for a certain number of periods or till the happening of certain contingent event or forever. Its payment starts either immediately or remains deferred for a certain period for fulfillment of certain conditions. The payment is made at the end of every period and sometimes at the beginning.

4.7.2. Objectives of an Annuity

An annuity is meant to charge an existing debt in certain number of installments inclusive of compound interest at a stipulated rate. Similarly, annuity might also be used to create a fund to redeem a liability or to replace an asset in the future without disturbing the financial position of a person. An annuity is also applicable for someone or firm to secure a regular income for one's bread or any other purpose. Finally, an annuity is a good mechanism for determining the value of the regular installments payable or receivable under a contract or to determine the present value to be set aside under a contract.

4.7.3. Amount (Future Value) of an Ordinary Annuity

Dear learners, in this we will consider ordinary annuities that are certain and simple, with periodic payments that are equal in size in other words, we will deal with annuities that are subject to the following conditions.

- The terms are given by fixed time intervals
- The periodic payments are equal in size
- The payments are made at the end of the payment periods
- The payment periods coincide with the interest conversion periods

EXAMPLE 1

What is the amount of an annuity if the size of each payment is Birr 100 payable at the end of each quarter for one year at an interest rate of 4% compounded quarterly?

SOLUTION

Periodic payment (R) = Birr 100

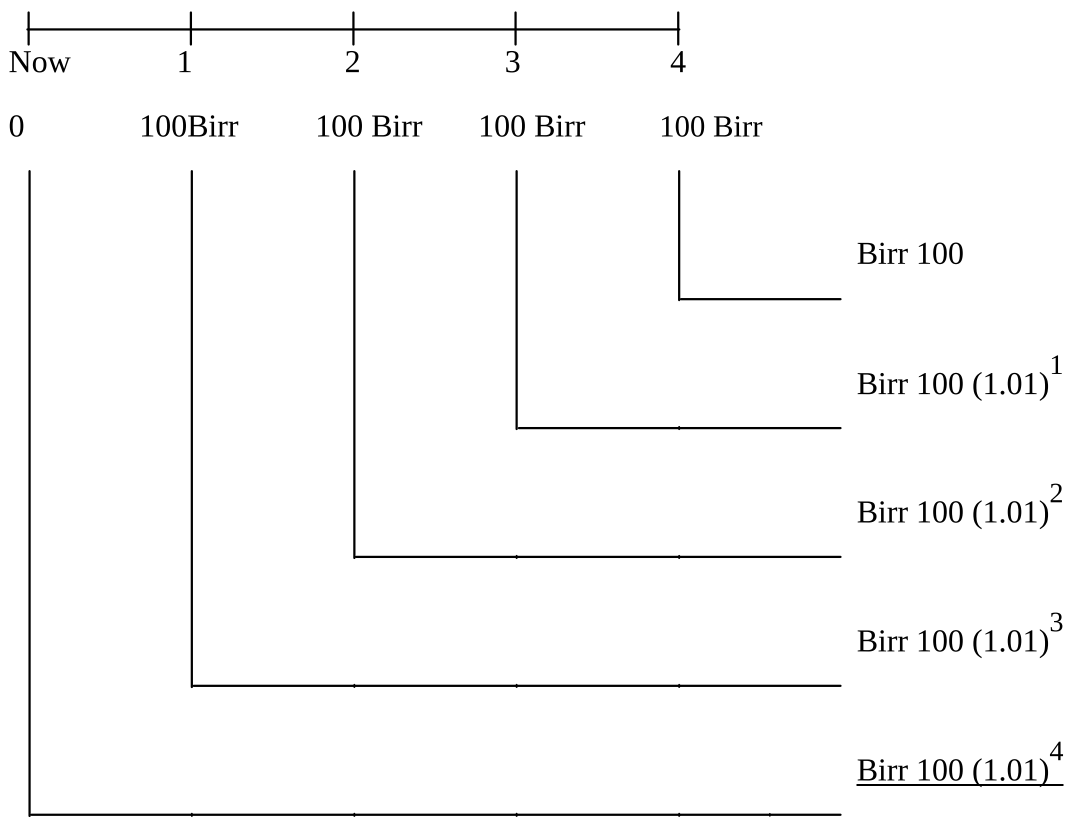
Payment interval = conversion period = quarter

Nominal (annual rate), r = 4%

Interest rate per conversion period (i) = r/m = 4%/4 = 1%

Future value of (sum of) an annuity =?

Term one year



A = Birr 406.04

Compound interest = Amount - R (n)

= 406.04 - 100(4) = **Birr 6.04**

Taking

R = amount of periodic payment

I = interest rate per payment period

n = total number of payment periods

A = Future value (Amount) of an Ordinary Annuity at the end of its term.

Last payment	R
The second payment from the last	$R(1+i)^1$
The third payment from the last-	$R(1+i)^2$
The second payment	$R(1+i)^{n-2}$
The first payment	$R(1+i)^{n-1}$

$A = R + R(1+i)^1 + R(1+i)^2 + \text{------} + R(1+i)^{n-2} + R(1+i)^{n-1}$ ……………………. (1)

Multiplying each side of the equation by 1+i, we obtain

$A(1+i) = R(1+i) + R(1+i)^2 + R(1+i)^3 + \text{----} + R(1+i)^{n-1} + R(1+i)^n$ ……………………. (2)

Then subtracting the first equation from the second equation, we have

$A(1+i) = R(1+i) + R(1+i)^2 + \text{------} + R(1+i)^{n-1} + R(1+i)^n$

$- \quad A = R + R(1+I)^1 + R(+i)^2 + \text{----} + R(1+i)^{n-1}$

$A(1+i) - A = R(1+i)^n - R$

$A[1+i-1] = R[(1+i)^n - 1]$

A (i) = R [(1+i) n-1] dividing both sides by i, we have, $A = R\left[\frac{(1+i)^n - 1}{i}\right]$

$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$ $\frac{(1+i)^n \quad -1]}{i}$ = future value factor.

For the above example: $A = 100 \frac{[(1.01)^4 - 1]}{0.01}$ = **<u>Birr 406.04</u>**

Example 2

A newly married couple are both working and decide to have Birr 1000 at the end of a month for a down payment on a home. The account earns 12% compound monthly. How large a down payment will they have saved in three years?

Solution

R = Birr 1000

t = 3 years.

m = 12

n = 36

r = 12%

i=1%

A =?

$$A = \frac{[(1 + i)^n - 1]}{i}$$

$$\frac{1000\,[(1.01)^{36} - 1]}{0.01}$$

= Birr 43,076.88

Compound interest = A - R(n)

43,076.88 - 36,000

Birr <u>7,076,88</u>

Example 3

Awash International Bank (AIB) pays interest at 8% per annum compounded semi annually. If Mr. Shemsu places Birr 5,000 in AIB at the end of every 6 months, how much will be in his account after 5 years?

Solution

The case is an ordinary annuity consisting of 10 payments of Birr 5,000 each earning 0.04 interest per period. Thus, the required amount is given by,

$$A = \frac{[(1+i)^n - 1]}{i}$$

Where, R = 5,000

n = 10 and

$I = \frac{0.08}{2} = 0.04$

$$\text{Therefore, } A = \left[\frac{(1+0.04)^{10} - 1}{0.04}\right] = 5{,}000\left(\frac{1.04^{10} - 1}{0.04}\right)$$

= 5,000 (12.006107) = Birr 60, 030.535

4.7.4. Sinking Fund (Increasing Annuity)

A Sinking fund is a fund in to which equal periodic payments are made in order to accumulate a specified amount at some point in the future. Sinking funds are generally established in order to satisfy some financial obligation or to reach some financial goal. If the payments are to be made in the form of an ordinary annuity, then the required periodic payment into the sinking fund can be determined by reference to the formula for the amount of an ordinary annuity. That is, if

$$A = R\left[\frac{(1+i)^n - 1}{i}\right]$$

Then,

$$R = \frac{A}{\left[\frac{(1+i)^n - 1}{i}\right]}$$

$$R = A\left[\frac{i}{(1+i)^n - 1}\right]$$

EXAMPLE

What monthly deposit will produce a balance of Birr 100,000 after 10 years? Assume that the annual percentage rate is 6% compounded monthly. What is the total amount deposited over the 10-year period?

SOLUTION

A = Birr 100,000

t = 10years

m = 12

r = 6%

R =?

$$R = A\left[\frac{i}{(1+i)^n - 1}\right]$$

$$R = 100{,}000\left[\frac{0.005}{(1.005)^{120} - 1}\right]$$

$$R = 100{,}000\,(0.006102)$$

$$R = \text{Birr } 610.21$$

The total amount deposited over the 10-yr period is 120 (610.21) = **Birr 73,225.**

SELF CHECK EXERCISES 4.5

A newly married couple are both working and decide to have Birr 1000 at the end of a month for a down payment on a home. The account earns 12% compound monthly. How large a down payment will they have saved in three years?

A person deposits Birr 500 a year for 10 years in to an account that pays 6% compounded annually. After 10 years the person transfers the money into another account that pays 8% compounded quarterly. The money is left in the second account for 8 years. What is the balance after the 18-year period?

Debebe wants to settle a debt of Birr 12,000 to be paid 10 years from now. If deposit earns interest at the rate of 5% compounded annually, how much money should he deposit at the end of each year for the coming 10 years?

A loan of 15,000 with interest rate of 10% compounded semi annually must be paid off in one payment after 3 years. In order to settle this amount, quarterly payments are to be placed in sinking fund which pays 16% compounded quarterly how much should each quarterly payment be?

Mrs. Hiwot has planned to save Birr 25,000 which she would like to reach 10 years from now. During the first 5 years, she is financially able to deposit only Birr 100 each month into the saving account. What must her monthly deposit over the last 5 years be if she is to reach her goals? The account pays 12% interest compounded monthly.

20 Birr is deposited at the end of each quarter for 5 years in Commercial Bank of Ethiopia (CBE) which pays 12% interest compounded quarterly. If CBE decides to increase its nominal interest rate to 16% compounded quarterly after 3 years,

A.) What will be the amount of the annuity at the end of the 5 year
term? B.) What is the total interest that is earned?

An employee deposits Birr 150 at the end of each month in a credit union saving plan that pays an interest rate of 12% compounded monthly. If the monthly periodic payments have increased to Birr 200 after the 8^{th} payment,

What is the balance after one year?
What is the amount of interest that is earned?

4.7.5. SINKING FUND SCHEDULE

The accumulation of value in a sinking fund is illustrated by a ***sinking fund schedule.***

EXAMPLE

Mr. Gedion wishes to set aside semiannual payments to purchase machinery after two years (two years from now). The machinery's estimated cost is Birr 5000. Each payment earns interest at 12%compounded semiannually.

Find the semiannual payment

Find the total interest earned

Prepare a sinking fund schedule

SOLUTION

A = Birr5000

t = 2 years

m = 2

r = 12%

i= 6%

n= 4

R=?

A) $$R = A\left[\frac{i}{(1+i)^n - 1}\right]$$

$$= 5000\left[\frac{.06}{(1.06)^4 - 1}\right]$$

<u>Birr 1,142.96</u>

Interest = Amount - R (n)

5000 - 1,142.96(4)

<u>Birr 428.16</u>

C)

Payment number	Payment (R)	Interest*	Total
1	Birr 1,142.96	Birr 0	Birr 1,142.96
2	Birr 1,142.96	Birr 68.58	Birr 2,354.50
3	Birr 1,142.96	Birr 141.27	Birr 3,638.73
4	Birr 1,142.96	**Birr 218.32**	Birr 5000.01
		Birr 428.16	

* Interest = balance xi

4.7.6. PRESENT VALUE OF AN ORDINARY ANNUITY

The present value of an ordinary annuity is the amount of money today, which is equivalent to the sum of a series of equal payment in the future. It is the sum of the present values of the periodic payments of an annuity, each discounted to the beginning of an annuity. The present value represents the amount that must be invested now to purchase the payment due in the future.

In short, PV of an ordinary annuity can be computed in two ways:

Discounting all periodic payment to the beginning of the term individually.
Discounting the amount of an ordinary annuity to the beginning of the term.

EXAMPLE

What is the PV of an annuity if the size of each payment is Birr 200 payable at the end of each quarter for one year and the interest rate is 8% compounded quarterly?

SOLUTION

R = Birr 200 r = 8%

m = 4 t = 1yr

P =?

P ______________________________ A

0 1 2 3 4

196.10 = 200 $(1.02)^{-1}$ ……...200

192.23 = 200 $(1.02)^{-2}$……………………200

188.46 = 200 $(1.02)^{-3}$……………………………….200

184.77 = 200 $(1.02)^{-4}$………………………………………… 200

Birr 761.56

Equivalently: Find the FV of the ordinary annuity using the formula A = R [$(1+i)^{n}$ - 1]

$$A = R\left[\frac{(1+i)^n - 1}{i}\right]$$

$$200\left[\frac{(1.02)^4 - 1}{0.02}\right]$$

Birr 824.32

Discount this future value to the present value taking it as single FV.

= 824.32 $(1.02)^{-4}$

Birr 761.56

We have seen that the future value of an ordinary annuity after n payment periods is $A = R[(1+i)^n - 1]$, and also we have seen that the PV of a lamp sum investment after n periods with interest rate i per period is: $P(1+i)^n$. The future value of an annuity and the future value of the lamp sum payment should be equal at the end of n periods; thus,

$$(1+i)^n = \frac{R\left[\left((1+i)^n - 1\right)\right]}{i}$$

Devidingbothsidesby$(1+i)^n$gives

$$P = \frac{R\left[1-(1+i)^{-n}\right]}{i} \quad \text{(OR)}$$

If we multiply the FV of an Ordinary annuity by the compound discount factor we have the present value of an annuity.

$$P = \frac{R\left[(1+i)^{-n} - 1\right](1+i)^{-n}}{i}$$

$$= \frac{R\left[(1+i)^{n}(1+i)^{-n} - 1(1+i)^{-n}\right]}{i}$$

$$= \frac{R\left[(1+i)^{0} - 1(1+i)^{-n}\right]}{i}$$

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Using the above formula; the PV of the former example is computed as:

R = Birr 200

r = 8%

m = 4

t = 1yr

P =?

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$P = 200\left[\frac{1-(1.02)^{-4}}{0.02}\right]$$

200 (3.08773)

Birr 761.55

4.7.7. Amortization (Decreasing Annuity)

Amortization means retiring a debt in a given length of time by equal periodic payments that include compound interest. After the last payment, the obligation ceases to exist-it is dead-and it is said to have been amortized by the payments.

In amortization our interest is to determine the periodic payment, R, so as to amortize (retire) a debt at the end of the last payment. Solving the PV of ordinary annuity formula for R in terms of the other variables, we obtain the following amortization formula:

$$R = P\left[\frac{i}{1-(1+i)^{-n}}\right]$$

Where:

R = periodic payment

P = PV of loan

i= interest rate per period

n = number of payment periods

Example

Suppose you borrow Birr 5000 from a bank and agree to repay the loan in five equal installments including all interests due. The bank's interest charges are 5% compounded annually. How much should each annual payment be in order to retire the debt including the interest in 5 years?

Solution

PV = Birr 5000 r = 5%

t = 5years R =?

m = 1

$$R = 5{,}000\left[\frac{0.05}{1-(1.05)^{-5}}\right]$$

= 5000(.230975)

Interest = (1,154.87 X 5) – 5000

Birr 774.35

EXAMPLE 2

Maritu signed a loan for Birr 10,000. The loan is to be repaid with equal yearly payments for the first three years and equal yearly payments twice as large for the next four years. If the interest rate is 12% compounded annually, find the yearly payments. Assume each payment is made at the end of each year.

SOLUTION

Draw the time line as follows.

0	1	2	3	4	5	6	7
	x	x	x	2x	2x	2x	2x

Dear learners, here, you can observe that x denotes the first three annual payments and 2x denotes the remaining payments. Then we choose a **comparison payment**. If the comparison point chosen is to be the end of the third year, then the annuity consisting of three payments of x Birr each must be brought forward to the comparison point. This is done by multiplying x by the future value factor of an ordinary annuity. Also, the annuity consisting of four payments of 2x Birr each must be brought back to the comparison point. This done by multiplying 2x by the present value factor of an ordinary annuity. Finally, Birr 10,000 must be brought forward to the comparison point by multiplying it by future value factor of a single deposit. Hence, the equation of value is,

$$x\left[\frac{(1+0.12)^{3}-1}{0.12}\right] + 2x\left[\frac{1-(1+0.12)^{-4}}{0.12}\right] = 10{,}000\,(1+0.12)^{3}$$

$$x\left[\frac{(1+12)3-1}{.12}\right] + 2x\left[\frac{1-(1+.12)^{-4}}{.12}\right] = 10{,}000(1+.12)3$$

x (3.374400) + 2x (3.037349) = 10,000 (1.404928)

Solving for x, we have

3.3744x + 6.074698x = 14,049.28

x = Birr 1,486.84

$$x\left[\frac{(1+0.12)^{3}-1}{0.12}\right] + 2x\left[\frac{1-(1+0.12)^{-4}}{0.12}\right] = 10{,}000\,(1+0.12)^{3}$$

x (3.374400) + 2x (3.037349) = 10,000 (1.404928)

Solving for x, we have

3.3744x + 6.074698x = 14,049.28

x = Birr 1,486.84

4.7.8. Amortization Schedule

Example

Ato Kebebew borrowed Birr 7000. The loan plus the interest is to be repaid in equal quarterly installments made at the end of each quarter during a 2 year interval. The interest rate is 16% compounded quarterly.

Find the quarterly payment

Find the interest accumulated

Prepare an amortization schedule

SOLUTION

P = Birr 7000

r = 16%, i = 4%

m = 4

t = 2 years, n = 8

R =?

A)

$$R = P\left[\frac{i}{1-(1+i)^{-n}}\right]$$

$$R = 7000\left[\frac{.04}{1-(1.04)^{-8}}\right]$$

R = Birr 1,039.69

Interest accumulated = R (n) -

A = 1,039.69(8) - 7,000

= Birr 1317.52

AMORTIZATION SCHEDULE

Payment No.	Payment	Interest	Principal reduction	Balance
0				Birr 7,000.00
1	Birr 1,039.69	Birr 280.00	Birr 759.69	6,420.31
2	1,039.69	249.61	790.08	5,450.23
3	1,039.69	218.01	821.68	4628.55
4	1,039.69	185.14	854.55	3774.00
5	1,039.69	150.96	888.73	2885.27
6	1,039.69	115.41	924.28	1960.99
7	1,039.69	78.44	961.25	999.74
8	1,039.69	39.99	999.74	0.00
		Birr 1,317.56		

4.7.9. Mortgage Payments

In atypical house purchase transaction, the home-buyer pays part of the cost in cash and borrows the remained needed, usually from a bank or a savings and loan association. The buyer amortizes the indebtedness by periodic payments over a period of time. Typically, payments are monthly and the time period is long-30 years is not unusual.

Mortgage payment and amortization are similar. The only differences are

The time period in which the debt/loan is amortized/repaid

The amount borrowed.

In mortgage payments m is equal to 12 because the loan is repaid from monthly salary, but in amortization m may take other values.

In Mortgage payments we are interested in the determination of monthly payments.

Taking A = total debt

R = monthly mortgage payment

r = stated nominal rate per annum

n = 12 x t

R can be determined as follows:

$$R = A\left[\frac{r/12}{1-(1+r/12)^{-n}}\right] \quad \text{(OR)} \quad R = A\left[\frac{i}{1-(1+i)^{-n}}\right]$$

Similarly $A = R\left[\frac{1-(1+i)^{-n}}{i}\right]$

EXAMPLE

Mr. X purchased a house for Birr 115,000. He made a 20% down payment with the balance amortized by a 30 yr mortgage at an annual interest of 12% compounded monthly.

What is the amount that Mr. X should pay monthly so as to retire the debt at the end of the 30^{th} yr?

Find the interest charged.

SOLUTION

Selling price	= Birr 115,000	r = 12%	i= 1%
Down payment	= (20%) 23,000	m = 12	t = 30yrs
Mortgage (A)	= Birr 92,000	n= 360	
		R =?	

$$R = A\left[\frac{i}{1 - (1+i)^{-n}}\right]$$

$$R = 92{,}000\left[\frac{.01}{1 - (1.01)^{-360}}\right]$$

92,000 (.010286125)

Birr 946.32

Interest = Actual payment - Mortgage (loan)

(946.32 X 360) - 92000

Birr 340,675.20 - 92,000

Birr 248,675.20

SELF CHECK EXERCISES 4.6

What is the cash value of a car that can be bought for Birr 200 down payment and Birr 82 a month for 18 months, if money is worth 12% interest compounded monthly?

How much should you deposit in an account paying 6% compounded quarterly in order to be able to withdraw Birr 1000 every 3 months for the next 3 years?

What is the present value of an annuity of seven payments of Birr 1000 each made at the end of each quarter with an interest rate of 12% compounded monthly?

At the time of retirement, a person has Birr 200,000 in an account that pays 12% compounded monthly. If he decides to withdraw equal monthly payments for 10 years at the end of which time the account will have a zero balance, how much should he withdraw each month?

Mrs. Almaz purchased a house for Birr 50,000. She made an amount of down paymen and pay monthly Birr 600 to retire the mortgage for 20 years at an annual interest rate of 24% compounded monthly.

Required: Find the mortgage, down payment, interest charged, and the percentage of the down payment to the selling price.

CHAPTER SUMMARY

Mathematics of finance deals with time value of money. A birr today is better than one birr tomorrow. The reason is a birr earned today earns an interest which makes it value more than a birr.

There are two types of interest. Simple interest and compound interest. Simple interest is the case where any interest earned is not added back to the principal amount invested.

Compound interest is the case where interest earned the previous interest period earns another interest for the next interest period.

Annuity is any periodical payment of a fixed amount made at a regular interval of say, one year, half year, a quarter or a month, to discharge one's obligation or to meet some amount of accumulations.

Annuity is used to create a fund to redeem a liability or to replace an asset in the future without disturbing the financial position of a person. An annuity is also applicable for someone or firm to secure a regular income for one's bread or any other purpose. Finally, an annuity is a good mechanism for determining the value of the regular installments payable or receivable under a contract or to determine the present value to be set aside under a contract.

A sinking fund is created through periodical contributions of equal amount for a certain period with a view to raising an accumulated sum inclusive of compound interest at a certain rate per annum to meet the redemption of an existing debt or replacement of an existing asset.

A sinking fund payment is funds in to which equal periodic payment are made. An account that is established for accumulating funds to meet future obligations debts is called is called a sinking Fund. It involves determination of the periodic payment (R) to accumulate a future value (S).

The present value of an ordinary annuity is the sum of the present values of all the payments, each discounted to the beginning of the term of the annuity.

Amortization means retiring a debt in a given length of time by making equal periodic payments that include compound interest. After the last payment, the obligation ceases to exist and is said to have been amortized by periodic payments.

In a typical home purchase transaction, the home buyer pays part of the cost in cash and borrows the remaining needed; usually from bank or savings and loan associations. The buyer amortizes the indebtness by periodic payments over a period of time.

CHAPTER FIVE: CALCULUS

5.1. INTRODUCTION TO CALCULUS

Calculus (Latin, *calculus*, a small stone used for counting) is a branch of mathematics that includes the study of limits, derivatives, integrals, and infinite series, and constitutes a major part of modern university education.

Historically, it was sometimes referred to as "the calculus", but that usage is seldom seen today. Calculus has widespread applications in Business, Economics, science, engineering and other fields of study and is used to solve complicated problems for which algebra alone is insufficient. Calculus builds on algebra, trigonometry, and analytic geometry and includes two major branches, **differential calculus** and **integral calculus**, that are related by the fundamental theorem of calculus. In more advanced mathematics, calculus is usually called analysis and is defined as the study of functions.

FUNCTIONAL NOTATION

The expression Y = mx + b is called the slope intercept form of the equation of the straight line. The quantities m (slope) and b (y-intercept) for any specific line are constants as in;

Y = 2x + 5, where as x and Y are variables

Thus, even though

Y = mx +b contains four letters (y, m, x, b)

X and y are variables and m and b are parameters. Since they represent constants other than variable values

Y = mx + b can be written as

F (x) = mx + b

F(x) read as the function of f of x or shortly f of x.

In the above symbol x is independent variable and f(x) or y is dependent variable.

Similarly, in C (L) = $\frac{NF}{} + \frac{iL}{2}$

L is independent variable N, F, I are said to be parameters, and C (L) is dependent variable.

In $g(x) = 3X^2 – 2x +10$;

x = independent variable

g = the name of the function

g (x) = value of the function

To find the value of the function when x = 2, written as g (2), we evaluate

$g(x) = 3x^2 – 2x + 10$

At x = 2

$g(2) = 3(2^2) - 2(2) + 10$

$3(4)-4+10$

$8+10$

18

EXAMPLE 1

If $f(x) = x^2 - 6$, Find $f(x + a)$

$F(x) = (x + a)^2 - 6$

$\underline{x^2} \ \underline{+ 2xa + a^2} \ \underline{- 6}$

EXAMPLE 2

Find $g(a) – g(x - a)$ if $g(x) = x^2 + 10$ and $g(a) = a^2 + 10$

SOLUTION

$g(x - a) = (x - a)^2 + 10$

$= x^2 – 2xa + a^2 + 10$,

Therefore, $g(a) - g(x - a)$

$a^2 + 10 – (x^2 - 2xa + a^2 + 10)$

$\underline{2ax – x^2}$

✍ SELF TEST EXERCISE 5.1

1. If h(g) = 2g – 10/g
   A) Find h(1)
   B) Find h(5)
   C) Find h(100)

2. If f(k) = k^2 - 2Kg
   A) Find f(5)
   B) Find f(2)
   C) Find f(10)

3. If f (g) = k^2 - 2Kg
   A) Find f(5)
   B) Find f(2)
   C) Find f(10)

4. If p(g) q^2 - r^2 + 5 and h(r) = q^2 – r^2 + 5
   A) Find p(2)
   B) Find h(2)

5. Find f(x + a) - f(x) if f(x) = x^2 - 3

5.2. LIMITS

Limits are the core concepts in development of calculus and analysis concerning the behavior of the function near a particular input. Informally, a function assigns an output f(x) to every input x. the function has a limit L at an input p if f(x) is close to L whenever x is close to p. in another words, f(x) becomes closer and closer to L as x moves closer and closer to P. when f is applied to each input sufficiently close to p, the result is an output that is closer L.

EXAMPLE 1

Given g(x) = $\frac{x^2}{x}$

This function is meaningless at x = 0 since the output will be undefined and we shall say g(x) is undefined when x=0.

Therefore, g(x) = $\frac{x^2}{x}$; x≠ 0

g(x) for all values of x except zero. As x approaches to zero, g(x) approaches to zero.

Symbolically, $\lim_{x \to 0} g(x) = 0$

The symbol $x \to 0$ mean as x approaches to zero or gets closer and closer to zero.

Find limit of $f(x) = \frac{x}{x^2}$ as x approaches to zero.

The above function will be undefined when x=0. Because that yields the function to be 0/0, which is undefined. But as x approaches to zero from above, f(x) approaches ∞. When x approaches to zero from below, f(x) approaches - ∞.

Therefore the limit of $f(x) = \frac{x}{x^2}$ as X approaches to zero, is non existent.

EXAMPLE 1

Find if it exists

Limit of $f(x) = \frac{x}{x-2}$ as x approaches to 2.

SOLUTION

As x approaches 2 from above, f(x) approaches ∞.

As x approaches 2 from below, f(x) approaches -∞.

Therefore limit of $f(x) = \frac{x}{x-2}$ as x approaches to 2, is non- existent.

EXAMPLE 2

Find if it exists,

Lim of $f(x) \quad \frac{x^2-4}{x-2}$ as x approaches 2.

The function is undefined if x = 2. Therefore, we need to restrict x from being to 2. Then if x is different from 2,

$$\frac{x^2-4}{x-2} = \frac{(x+2)\ (x-2)}{x-2} ;\ x \neq 2$$

= x + 2

limit of $\frac{x^2-4}{x-2}$; $x \neq 2$ as x approaches to 2 = limit of $\frac{(x+2)\ (x-2)}{x-2}$, as x approaches to 2 = limit of x + 2 as x approaches to 2.

As x approaches to 2 from above, limit of x + 2 as x approaches 2 approaches 4.

And limit of x + 2 as x approaches 2 from below approaches 4. Therefore, Limit of $\frac{x^2 - 4}{x - 2}$; x ≠ 2 as x approaches 2 = limit of $\frac{(x+2)\ (x-2)}{x-2}$ = limit x + 2 = 4.

SELF TEST EXERCISE 5.2

Find if it exists

1. Limit of $\frac{x^2 - 25}{x - 5}$ as x approaches to 5.
2. Limit of $\frac{x+1}{x-1}$ as x approaches to 1.
3. Limit of 3a + 2b as a approaches to b.
4. Limit of $\frac{a^2 - b^2}{a - b}$ as b approaches to a.
5. Limit of 2x +y as y approaches to x.
6. Limit of $\frac{x^2 - y^2}{x - y}$ as x approaches to y.

LIMIT THEOREM

If the limit of the following exists;

For any constant number k, limit of k = k.

Limit of k f(x) as x approaches to a = k limit of f(x) as a approaches to a.

EXAMPLE 1

lim $3x^2$ as x approaches to a = 3 lim x^2 as x approaches to a = $3a^2$

Limit of [f(x) ± g(x)] as x approaches to a = limit of f(x) as x approaches to a ± limit of g(x) as x approaches to a.

EXAMPLE 2

lim (x^2-2x +3) as x approaches a = limit of x^2 as x approaches to a – limit of 2x as x approaches to a + limit 3 as x approaches to a. = a^2- 2a +3

limit [f(x).g(x)] as x approaches to a = [limit of f(x) as x approaches to a] [limit of g(x) as x approaches to a]

EXAMPLE 3

Limit of (x + 3) (x - 2) as x approaches 2 = limit of [(x + 3) as x approaches 2] limit of [(x 2) as x approaches 2].

limit of $[f(x)]^n$ as x approaches to a = [limit of f(x) as x approaches to a$]^n$

EXAMPLE 4

Limit $(x-1)^5$ as x approaches to 3 = [limit of (x-1) as x approaches to 3$]^5$ = 2^5= 32

if limit of f(x) as x approaches to a = L and lim g(x) as x approaches to a = M then,

If M ≠ 0, then lim [f(x)/g(x)] as x approaches to a = L/M

If m=0 and L ≠ 0, then lim [f(x)/g(x)] as x approaches to a does not exist

If m=0 and L = 0, then lim [f(x)/g(x)] as x approaches to a have common factor and the limit can be evaluated after employing the process of cancellation.

5.3. CONTINUITY

A function is called continuous if its graph can be drawn without lifting the pencil from the paper.

Consider the following graphs

$$f(x) = \begin{cases} x + 1, & x \geq 0 \\ x - 1, & x < 0 \end{cases}$$

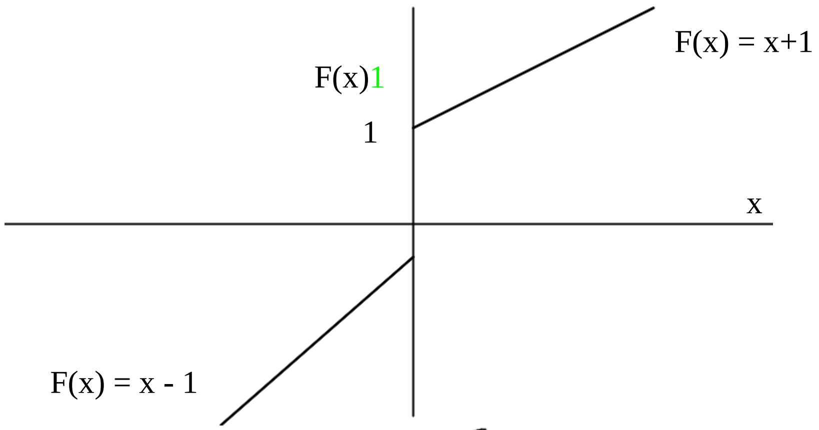


Figure 6.1 graph of the equation $f(x) = \begin{cases} x + 1, & x \geq 0 \\ x - 1, & x < 0 \end{cases}$

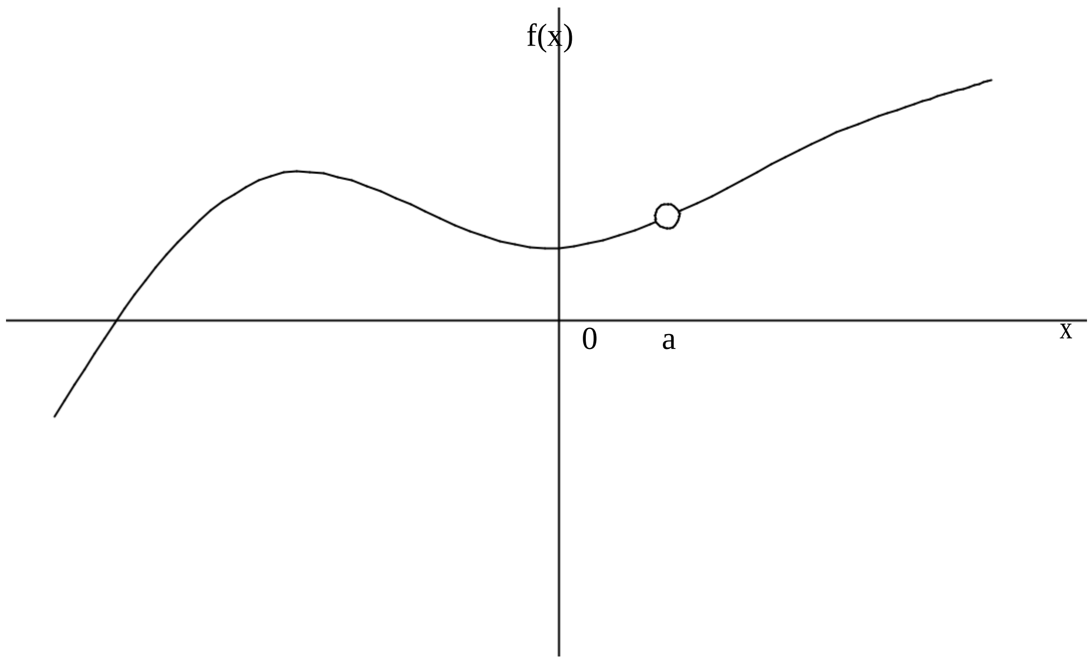


Figure 6.2

The above graph shows discontinuous function since there is a whole at x = a.

DEFINITION 5.1:

<u>A function f(x) is said to be continuous at a point a</u> if the following three conditions are satisfied.

F(x) is defined

Lim f(x) exists as x approaches a.

Lim f(x) = f(a)

If a function is not continuous at a point a we say it is discourteous or has discontinuity at a.

EXAMPLE

Discus the continuity of f(x) = $\dfrac{x+2}{X^2 - 3x + 2}$

If we factor the denominator $\dfrac{x+2}{(x-1)(X-2)}$

If we evaluate the above function using the three conditions;

The function $\frac{x+2}{(x-1)(X-2)}$ is not defined when x =1 and x= 2.

Limit of f(x) $\frac{x+2}{(x-1)(X-2)}$ as x approaches a, exists for all x except for 1 and 2.

Lim f(x) as x approaches a = f(a) for all x except for 1 and 2.

Hence, f(x) is continuous at all x except x = 2 and x =1 Certain classes of function are always continuous.

All polynomials functions are continuous

If f(x) is continuous and g(x) is continuous, then

f(x) ± g(x) is continuous

f(x).g(x) is continuous

f(x)/g(x) is continuous when g(x) ≠ 0.

SELF- TEST EXERCISE 5.3

For the following function find the discontinuities

1. $F(x) = x^3 + 2x - 4$
2. $F(x) = \frac{1}{x-1}$
3. $F(x) = \frac{x}{x^2+1}$
4. $f(x) = \frac{x-2}{x^2-4}$

$f(x) = \frac{x^2 - 4}{x-2}$

6. $f(x) = \frac{x+2}{x^2-5x+6}$
7. $f(x) = \begin{cases} x+1, & x > 1 \\ x, & x \le 1 \end{cases}$
8. $f(x) = \begin{cases} x+1, & x \ge 1 \\ 2, & x < 1 \end{cases}$
9. $f(x) = \begin{cases} x2+1, & x \ge 2 \\ x+3, & x < 2 \end{cases}$
10. $f(x) = \begin{cases} \frac{x}{2}+2, & x \ge -2 \\ x+2, & x < -2 \end{cases}$

5.4. DIFFERENTIATION

5.4.1. RULES OF DIFFERENTIATION

1. CONSTANT RULE

If f(x) = c where c is any constant;

Then f'(x) = 0

EXAMPLE 1

f(x) = -3/2

f'(x) = 0

POWER RULE

f(x) = x^n where n is natural number f'(x) = n (x^{n-1})

EXAMPLE 1

f(x) = -3/2 x^2

f'(x) = -3x

CONSTANT TIMES FUNCTION

If f(x) = c. g(x) where c is any constant function and g(x) is a differentiable function f(x) = 6 x^5

f'(x) = 6. f'(x)

6 (5) x^4

30 x^4

SUM OF DIFFERENCE OF FUNCTIONS

If f(x) = u(x) ± v (x)

F'(x) = u'(x) ± v '(x)

EXAMPLE 1

f(x) = x^2 + 4x +

9 f'(x) = 2 x + 4

5. PRODUCT RULE

If f(x) = u(x). v(x) then,

f'(x) = u'(x).V(x) + u(x). v'(x)

EXAMPLE 1

$F(x) = (x^2-3)(x-x^2)$

$F'(x) = 2x(x-x^2) + (x^2-3)(1-2x)$

$-4x^3 + 3x^2 + 6x - 3$

6. DERIVATIVE OR QUOTIENT RULE

If $f(x) = \dfrac{u(x)}{v(x)}$ then, $f'(x) = \dfrac{u'(x).v(x) - u(x).v'(x)}{[v(x)]^2}$

EXAMPLE 1

$\dfrac{3x^2+5}{x^2-1}$, then $f'(x) = \dfrac{6x(x^2-1) - 2x(3x^2+5)}{[x^2-1]^2}$

$\dfrac{-16x}{x^4 - 2x^2 + 1}$

5.5. BUSINESS APPLICATION OF DIFFERENTIAL CALCULUS

Differentiation can be used to solve different real life problems in business areas. The following example shows an application area of differential calculus.

5.5.1. MARGINAL ANALYSIS AND REVENUE APPLICATION

Dear students we have already seen linear revenue and cost functions in the first of chapter of this. In this chapter we will see non-linear revenue and cost functions. Marginal analysis deals with analysis of incremental cost.

C(X) = total cost functions C'(X) = marginal cost

R(X) = total revenue function R'(X) = marginal revenue

P(X) = R(X) – C(X) = profit P'(X) = marginal Profit

P'(X) = R'(X) – C'(X)

MARGINAL COST

Marginal cost is additional cost incurred as a result of producing and selling one more unit of a product or service. Linear cost functions assume that the variable cost per unit is constant for such functions and the marginal cost is the same at any level of output.

Non linear cost function is characterized by variable marginal costs. For the total cost functions C(X), the derivative C'(X) represents:

The instantaneous rate of change in the TC given a change in the number of units produced

A general expression for the slope of the graph of the TC function

The marginal cost, MC = C'(X)

C'(X) is used to approximate the marginal cost associated with producing the next unit. It is the rate of change in total cost per unit change in production at an output level of X unit. It is also an optimization to the actual cost of making one more unit at any production level X (non linear function)

EXAMPLE1

Suppose the total cost C(X) in thousands of dollars for manufacturing X units is given by the function

$$C(X) = 575 + 25X - \frac{x^2}{4}\ ;\ 0 \le X \le 50$$

REQUIRED

Find the MC at a production level of X units

Find the MC at production of level of 40 units and interpret the results

Find the actual cost of producing the 41st unit and compare this cost with found in question number 2.

SOLUTION

MC= C'(X) = 25- X/2

C'(X)= C'(40) = 25-40/2

5

The actual cost of producing the 41st unit = C(41) – C(40)

$C(41) = 575 + 25X - \frac{x^2}{4} = 575 + 25(41) - 41^2/4 = 1179.75$

$C(40) = 575 + 25X - \frac{x^2}{4} = 575 + 25(41) - 41^2/4 = 1175$

C(41)-C(40) = 1179.75 – 1750= 4.75 birr which is approximately 5 birr.

Hence, at a production level of 40 units the rate of change of TC relative to production is birr 5.

MARGINAL REVENUE

Marginal revenue is the additional revenue derived from selling one more unit of a product or service. If each unit of a product sells at the same price, the MR is always equal to the price. However, for non-linear total revenue function is not constant. For total revenue function, R(X), the derivative R'(X) represents the instantaneous rate of change in the total revenue given a change in the number of units sold.

EXAMPLE 1

Given the following revenue function:

$R(X) = 500x - 0.005x^2$

Calculate the marginal revenue.

SOLUTION

Marginal revenue = $R'(X) = 500 - 0.01x$

MARGINAL PROFIT

Marginal profit is concerned with the effect of one additional unit of a product on profit. As a decision maker as long as the additional revenue is greater than the additional cost of producing a unit, the firm has to continue production; however, if the marginal revue of producing the nth unit is less than the marginal cost of producing the **n**$^{\mathbf{th}}$ unit, the firm has to stop production.

IF MR > MC PRODUCE THE NEXT UNIT

IF MR < MC, DO NOT PRODUCE THE NEXT

N. B If the MR = MC for the last unit produced and sold, total profit will be maximized;

P(X) = R(X) – C(X) = 0

P'(X) = R'(X) – C'(X) = 0

⇨P'(X) = 0

⇨R'(X) = C'(X)

EXAMPLE 1

The market research department of a company recommends that the company to manufacture and market a new transistor radio after suitable test. The marketing department also presents the following demand equation.

x = 10,000 – 1000p   i.e P = 10 - x/1000

sFurthermore, the financial department provides the following cost equation:

C(x) = 7,000 + 2x

FIND:

The production level that leads to maximum revenue and the maximum revenue.

The production level that leads to the maximum profit and the maximum profit

The production level that leads to minimum cost and minimum cost.

1. TR= P(x) = 10x – x^2/1,000

   Marginal revenue = R'(X) = 10- x/500

   10- x/500 = 0

   x/500 = 10

   x= 5,000 units

The maximum revenue = 10(5000) – 5000^2/1000

= 25,000 birr

2. Profit = TR- TC

   10x – x^2/1,000 – (7,000 +

   2X) Profit = 8x – x^2/1000 -7000

   P'(X) =8- x/500

Profit maximization level of output is the level where MC=MR. Therefore, at profit maximization level of output,

$10- x/500 = 2$

$8 = x/500$

$x = 8(500)$

4,000 units

Maximum profit = $8(4{,}000) – 4{,}000^2/1000 – 7{,}000$

9,000 birr

Marginal cost = C'(X) = 2

Cost is minimum when production level is zero. The minimum cost is also zero. For non-linear cost functions the cost minimizing level of output is usually different from zero. So is the minimum cost.

EXAMPLE

The demand for the tyres produced by Addis Tyre manufacturing company varies with the price the firm charges. The firm has determined that annual total revenue is the function of price p stated in birr as;

$R = f(p) = -15p^2 + 4500p$

Determine the price that should be charged in order to maximize total revenue.

What is the maximum value of annual total revenue?

SOLUTION

$R = f(p) = -15p^2 +$

$4500p$ $f'(x) = -30p +$

$4{,}500$ $30p = 4500$

$p = 4500/30$

150 birr

b. $f(p) = -15p^2 + 4500p$

$-15(150)^2 + 4500(150)$

337, 500 birr

5.6. INTEGRAL CALCULUS

Calculus has two main Branches: 1) ***Differential Calculus*** - which deals with the problems of finding the rate at which a known but varying, quantity changes: and 2) ***Integral Calculus*** which deals with the problem of finding a quantity, given that we know the rate at which it is changing.

Integral calculus is the reciprocal of the differential calculus. Given the rate of change f '(x), by integral calculus we can find the original function f(x).

EXAMPLE: f(x) = 5

f(x) = 5x +c

f'(x) = 5

Because the derivative of a constant is zero, once an anti derivative of a function is found, another anti derivative of the same function can be formulated simply by adding a non-zero constant to the first derivative. Thus, f(x), by itself is not unique; it is only one member of a family of anti derivatives for some specified function, each of these anti derivatives for some specified function, each of these anti-derivatives differing from the others only by an arbitrary constant C. Thus, if f(x) is an anti derivative of f(x), the general expression f(x) + C is called the **anti derivative** of f(x), or the ***indefinite integral***, and is denoted by the symbol ∫f(x) dx.

Given f(x, an anti derivative of f(x), the indefinite integral of f(x) is defined to be f(x)+c = ∫f(x)dx read as the indefinite integral of function f(x) with respect to the variable x.

- Integral sign

indicates that the operation of anti differentiation (integration) is to be performed.

f(x) – Integrand

dx - indicates integration is to be performed with respect to the variable

x C - Constant of integration

The process for finding ∫f(x) dx is called indefinite integration.

EXAMPLE: Find the indefinite integral for the following

1. $f(x) = 10x$ 2. $f(x) = 3x - 10$

$\int f(x)dx = 5x^2+c$ $\int f(x)dx = 3/2x^2-10x+c$

Find the function that has a slope of $10x^4$ at each point and that passes through the point (2, 74).

$f(x) = 10x^4$

$\int f(x)dx = \int 10x^4dx = 2x^5+c$ Therefore, $f(x) = 2x5 + 10$ $y = 2(2^5)$

$+ C$

$74 = (2 \times 32) + C$

$C=10$

5.6.1. BASIC RULES FOR INTEGRATION

The determination of anti-derivatives by trial and error is cumbersome. So many rules for integration have been developed from the corresponding theorems concerning derivatives. The most common are:

1. CONSTANT RULE

$f(x) = K$	$f(x) = 3$	$f(x) = 0$
$\int f(x)dx = Kx+C$	$\int 3dx = 3x+C$	$\int 0dx = 0+c = C$

2. POWER RULE

$f(x) = x^n$	$f(x) = x$	$f(x) = x^2$
$\int f(x)dx = \frac{x^{n+1}}{n+1} + C$; $n \neq 1$	$\int f(x)dx = \frac{x^2}{2} + C$	$\int f(x)dx = \frac{x^3}{3} + C$

If $n = -1$, $\ln|x| + C$

It is because logarithmic functions are not defined for negative values of x.

3. CONSTANT MULTIPLIER RULE

f(x) = kg(x)   f(x) = 5x

if ∫f(x) exists, $\int f(x)dx = 5\int \frac{x}{2}^2 + C$

$\int f(x)dx = K\int g(x)dx + c$ $= 5\left(\frac{x^2}{2}\right) + C$

$f(x) = \frac{x}{2}^2$ $= \frac{5x}{2}^2 + C$

$\int f(x)dx = \frac{1}{2}\int x^2 dx + C$

$\frac{1}{2}\left(\frac{x^3}{3}\right) + C$

$\left(\frac{x^3}{6}\right) + C$

SUM OR DIFFERENCE RULE

f(x) = h(x) ± g(x) , if ∫h(x) and ∫g(x) exist

∫f(x) = ∫h(x)dx ± ∫g(x)dx

EXAMPLE: f(x) = 3x+6

∫f(x) = ∫3xdx + ∫6dx

$\left(\frac{3}{2}x^2 + C_1\right) + (6x + C_2)$

$= \frac{3}{2}x^2 + 6x + C$ where $C = C_1 + C_2$

5. PRODUCT RULE

2

$= x^2+4x+4$

$\int f(x)dx = \frac{x^3}{3} + \frac{4x^2}{2} + 4x + C$

$\frac{x^3}{3} + 2x^2 + 4x + C$

QUOTIENT RULE

$f(x) = \frac{3x^3 + 2}{5x^2}$

$= \frac{3x^3}{5x^2} + \frac{2}{5x^2} + C$

$\frac{3}{5}x + 2/5x^{-2} + C$

$\int f(x)dx = 3/5x + 2/5x^{\frac{-2+1}{-2+1}} + C$

$3/5x - 2/5x + C$

5.6.2. BUSINESS APPLICATIONS OF INTEGRAL CALCULUS

Finding Total Functions

EXAMPLE 1: The marginal revenue function for a company product is given by the function f(x) = 50,000 - x; where x is the number of units produced and sold. Moreover, TR = 0 when no units are produced and sold. Find the company's TR function.

f(x) - marginal revenue

TR = the integration of MR

$\int F(x)dx = \int(50,000 - x)dx$

$\int 50,00dx - \int xdx$

$50,000x - x^2/2 + C$

$0 = 50,000(0) - (0/2) + C$

$0 = C$

R(x) = $\underline{50,000x - x^2/2}$

EXAMPLE 2: The function describing the MC of producing a product is f(x) = x+100, where x = the number of units produced. It is also known that TC = 40,000 Birr when x is 100 units. Determine the total cost function.

SOLUTION.

MC = x + 100

TC = ∫MC

$\int (X+100)\,dx$

$\int x\,dx + \int 100\,dx$

$x^2/2 + 100x + C$

$40{,}000 = (100^2/2) + 100(100) + C$

C = 25,000

C(x) = $\underline{x^2/2 + 100x + 25{,}000}$

5.6.3. DEFINITE INTEGRAL

Consider a firm producing some commodity. At a production level of x units, the marginal cost is: C'(x) = 6x + 8. The anti derivative of the marginal cost is: $C(x) = \int (6x + 8)\,dx$

$= 3x^2 + 8x + C$

If we are told that the fixed cost is Birr 600, then we determine the arbitrary constant to be C=600.Hence, the cost function is: $C(x) = 3x^2 + 8x + 600$.

Suppose we wish to determine the total net change in cost if production rises from x=10 to x=15. This total net change in cost is determined by evaluating

C (15) - C (10)

Since

$C(15) = 3(15^2) + 8(15) + 600$ = Birr 1,395

and

$C(10) = 3(10^2) + 8(10) + 600$ = Birr 980 the total net change in cost is C (15) - C (10) = Birr 1,395 - Birr 980 = Birr 415.

Thus, as production changes from x=10 to x=15, the total cost increases by Birr 415.

DEFINITION

If f is a bounded function on the interval [a, b] we define the definite integral of f as:

$$\int_a^b f(x)dx = f(x)\Big|_a^b = f(b) - f(a)$$

WHERE

The numbers a and b are called limits of integration.

F (b) - the value of the anti derivative at x=b.

F (a) - the value of the anti derivative at x=a.

a - lower limit

b - upper limit

$\int_a^b f(x)dx$ – the definite integral of f between a lower limit x=a and an upper limit x=b.

According to this definition the calculation of the definite integral proceeds as follows:

Determine the anti derivative, F(x), of f(x)

Evaluate the anti derivative at x=b and at x= a

Subtract F(a) from F(b)

There are three basic assumptions in computing a definite integral. These are:

continuity over an interval

non-negativity

anti derivative exists

EXAMPLE 1

The marginal revenue function is given as f(x) = 10.

Find the change in total revenue associated with increasing sales from 1500 to 1800 units [1500, 1800]

Find the incremental revenue from selling 1,500 units.

SOLUTION

A)

$$\int_{1500}^{1800} f(x)dx = 10x + C \Big|_{1500}^{1800}$$

f (1800) - f (1500)

10(1800) + C - [10(1500) + C

10(1800) - 10(1500)

Birr <u>3,000</u>

B)

$$\int_{0}^{1500} f(x)dx = 10x \Big|_{0}^{1500}$$

f (1500) - f (0)

Birr <u>15,000</u>

5.6.4. AREA PROBLEMS

The fundamental theorem of calculus allows us to compute the area under the graph of a continuous non-negative function f(x) over the interval a ≤ x ≤ b by

$$\int_a^b f(x)dx = f(x) \Big|_a^b = f(b) - f(a)$$, where f(x) is an anti derivative of f(x).

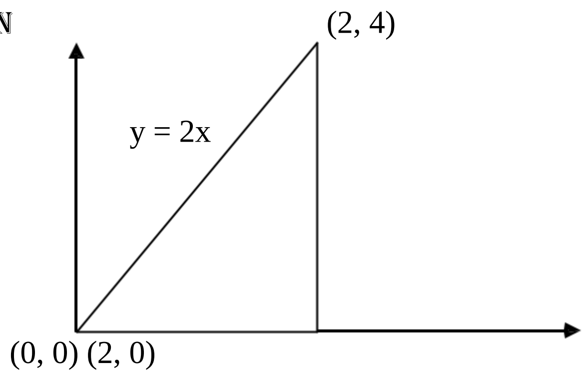
To find areas under a curve we employ the following steps.

- Graph the function
- Find any x-intercepts
- Shade the area to be found
- Note whether the shaded area contains any regions that lie below the x-axis as well as above the x-axis.
- Evaluate the appropriate definite integrals

EXAMPLE 1

Find the area of the region bounded by the lines f(x) = 2x and x = 2, and the x axis.

SOLUTION



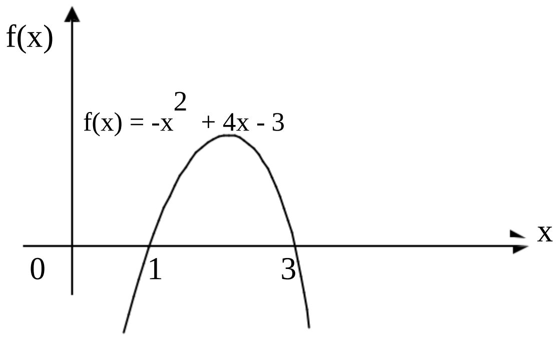
To compute this area using calculus, apply the integral formula with f(x) = 2x. Take b = 2 since the region is bounded on the right by the line x=2, and take a=0 since, on the left, the boundary consists of the single point (0, 0), which is part of the vertical line x=0. You will find, as expected that

$$\text{Area} = \int_0^2 2x\,dx = x^2 \Big|_0^2 = f(2) - f(0) = 4$$

EXAMPLE 2: Find the area of the region bounded by the curve $f(x) = -x^2 + 4x - 3$ and the x axis.

SOLUTION

From the factored form of the polynomial $f(x) = -x^2 + 4x - 3 = -(x-3)(x-)$, we see that the x-intercepts of the curve are (1, 0) and (3, 0). From the corresponding graph we see that the region in question is below the curve $f(x) = -x^2 + 4x - 3$ and extends from x=1 to x=3. Hence,



The region is bounded by $f(x) = -x^2 + 4x - 3$ and the x-axis.

$$\text{Area} = \int_1^3 (-x^2 + 4x - 3)dx$$

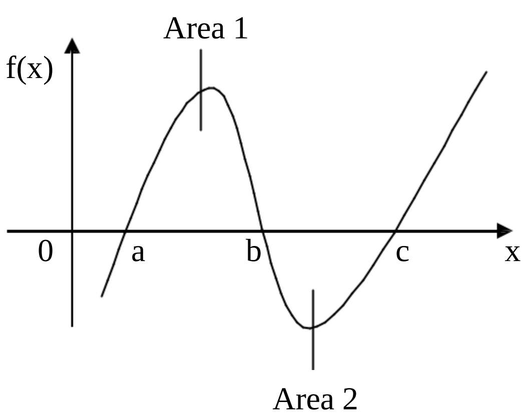
$$= \left(-\frac{1}{3}x^3 + 2x^2 - 3x\right)\Big|_1^3$$

$$f(3) - f(1)$$

$$\frac{4}{3}$$

Area below the X-Axis

If $f(x) < 0$ over some interval $b < x < c$, the corresponding area between the graph of f(x) and the x-axis lies below the x-axis. The definite integral $\int_b^c f(x)dx$ results in a value that is the negative of the area, so the absolute value of such a definite integral gives the area.



Area = Area 1 + Area 2. WHERE;

Area 1 = $\int_a^b f(x)dx$ and Area 2 $\left|\int_b^c f(x)dx\right|$ or $-\int_b^c f(x)dx$

SELF CHECK EXERCISES 5.4

The function describing the marginal profit from producing and selling a product is $f(x) = 6x+750$ where x = the number of units produced and sold. Moreover, when 100 units are produced and sold, total profit equals Birr 25,000. Determine the total profit function.

The marginal propensity to save, S'(x), is a function of a nation's income, x (in billions of Birr), as defined by the equation: $S'(x) = .5 - .12x^{-1/2}$ (x>0). If S = 0 when x = 81, then

Find the equation defining total savings, S(x).

Find the total savings at a national income of Birr 144 billion.

A truck carrying natural gas gets stuck at a low underpass and leaks natural gas at the rate of $L'(t) = 10t + 20$ cubic feet per minute, where t denotes time (in minutes) elapsed since the gas first began leaking.

A) Find the total amount of natural gas that has leaked during the first five minutes.

B) Find the total amount of natural gas that has leaked during the first 10 minutes.

C) Find the total amount of gas that has leaked during the fifth minute.

D) Find the total amount of gas that has leaked during the sixth minute

E) If the amount of natural gas that the truck was carrying was 5,100 cubic feet, how many minutes will be elapsed before the truck is empty?

The marginal profit for producing and selling x items is given by $f(x) = 600- 5x$. What is the change in total profit if the level of production changes from 40 to 60 units?

Find the area between the x-axis and the curve $f(x) = x^2-9$ from x=1to x = 4.

CHAPTER SUMMARY

Calculus is a branch of mathematics that includes the study of limits, derivatives, integrals, and infinite series. Calculus has widespread applications in business, economics, science, engineering and other fields of study and is used to solve complicated problems for which algebra alone is insufficient.

The two major branches of calculus are called differential calculus and integral calculus that are related by the fundamental theorem of calculus.

Limits are the core concepts in development of calculus and analysis concerning the behavior of the function near a particular input. The function has a limit L at an input p if f(x) is close to L whenever x is close to p.

Limit theorem:

For any constant number k, limit of k = k.

Limit of k f(x) as x approaches a = k limit of f(x) as a approaches a.

Limit of (f(x) $\pm$ g(x)) as x approaches a = limit of f(x) as x approaches a $\pm$ limit of g(x) as x approaches a.

Limit of (f(x) . g(x)) as x approaches a = (limit of f(x) as x approaches a) (limit of g(x) as x approaches a)

Limit of $(f(x))^n$ as x approaches a = (limit of f(x) as x approaches a)^n

If limit of f(x) as x approaches a = L and limit of g(x) as x approaches a = M then,

If M $\neq$ 0, then limit of (f(x)/g(x)) as x approaches a = L/M

If m = 0 and L $\neq$ 0, then limit (f(x)/g(x)) as x approaches a doesn't exist.

If m = 0 and L = 0, then limit (f(x)/g(x)) as x approaches a have common factor and the limit can be evaluated after employing the process of cancelation.

A function is called continuous if its graph can be drawn without lifting the pencil from the paper. A function f(x) is said to be continuous at a point a if the following three conditions are satisfied.

f(x) is defined

limit of f(x) exists as x approaches a.

limit of f(x) = f(a)

If a function is not continuous at a point a we say it is discourteous or has discontinuity at

Certain classes of function are always continuous.

All polynomial functions are continuous

If f(x) is continuous and g(x) is continuous, then,

f(x) $\pm$ g(x) is continuous

f(x) . g(x) is continuous

f(x)/g(x) is continuous when g(x) $\neq$ 0.

If y = f(x) be any function of x and △x be any increment in the value of x (in the dependent variable) and △y be the corresponding increment in the value of y(the dependent variable), then, △y/△x is called the average rate of change or just the rate of y with respect to x. the limiting value △y/△x as △x approaches 0, if it exists indefinitely, is called the derivative (or the different coefficient or instantaneous rate of change) of y with respect to x and is denoted by dy/dx.

Therefore, limit of $\frac{y}{x}$ = limit of $\frac{f\left(x + \quad x\right) - f(x)}{\Delta x}$

One can use the following rules to find the differentiation of a function.

If f(x) = c where c is any constant, then f'(x) = 0

If f(x) = x^n where n is natural number, f'(x) = n(x^{n-1})

If f(x) = c.g(x) where c is any constant function and g(x) is a differentiable function, then f'(x) = (c) g'(x).

If f(x) = u(x) $\pm$ v(x), then f'(x) = u'(x) $\pm$ v'(x)

If f(x) = u(x).v(x) then, f'(x) = u'(x).v(x) + u(x).v'(x).

If f(x) = u(x) then, f'(x) = u'(x).v(x) - u(x).v'(x)

$x\,)$ 2v(x)v

KEY FOR CHAPTER ONE SELF CHECK EXERCISES

EXERCISE 1.1

y = 26/5 – 3/5x, negative relationship

y = 2x + 1, positive relationship

y + 8, no relationship between x and y.

x = 2, undefined relationship

EXERCISE 1.2

1.

y = 0.2x + 22

For every additional rent per day, total cost increased by 0.2 cents.

With no rent of car, the agency incurred a cost of Br. 22 (which is equal to the fixed cost).

C. When x = 100

y = 0.2(100) + 22

y = 42

When x = 200

y = 0.2(200) + 22

y = 62

2.

(100, 4500) and (2000, 7500)

The equation of the line is deduced by substituting this slope, and either point into equation

$$Slope = \frac{y2 - y1}{x2 - x1} = \frac{7500 - 4500}{2000 - 1000} = \frac{3000}{1000} = 3$$

$y - y_1 = m(x - x_1)$.

$y - 4500 = 3(x - 1000)$

$y - 4500 = 3x - 3000$

$\underline{y = 3x + 1500}$

$y = mx + b$

$40 = 3(10) + b$

$b\ (FC) = 10$

Therefore, $y = 3x + 10$

EXERCISE 1.3

1.

FC = 10,000

VC =5

Selling price = Birr

10 R = pq

R = 10q

TC = vc/unit + FC

B. $Q = \dfrac{FC}{P-V} = \dfrac{10{,}000}{10-5} = \dfrac{10{,}000}{5}$ = **2,000 units (Break even Quantity)**

Sales volume at BEP = PQ

10 (2,000)

Birr 20,000

The producer should continue production, since the loss (Birr 500) is less than the total fixed cost (10,000). In this situation if production is going on the loss will be minimized. In our case, the loss is Birr 500 whereas the fixed cost is Birr 10,000 (Loss < TFC).

2.

A. $x_e = \dfrac{FC}{1 - m}$

Variable cost per birr of sales = (1- 0.52) + 0.07 = 0.55

$$\frac{36,000}{(1-0.55)}$$

$$\frac{36,000}{0.45} = 80,000$$

Y = 0.55x+36,000

π=R-C

75,000-055(75,000) +36,000

75,000-77250

Loss = (2,250)

KEY FOR CHAPTER TWO SELF CHECK EXERCISES

EXERCISE 2.1

The product **AB** = **C** is

$$\mathbf{C} = \mathbf{AB} = [1 \quad 2 \quad 3] \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = [(1 \text{ x } 4) + (2 \text{ x } 5) + (3 \text{ x } 6)]$$

=[4+10+18]=[32]

2. i. $\begin{pmatrix} 3 & -4 \\ 2 & 3 \end{pmatrix}$ ii. $\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$ iii. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ iv. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

i. not conformable

ii. $\begin{pmatrix} 64 & -2 & 96 \\ 12 & 14 & 74 \\ 16 & 2 & 8 \end{pmatrix}$

iii. $\begin{pmatrix} 32 & -1 & 48 \\ 6 & 7 & -37 \\ 8 & 1 & 4 \end{pmatrix}$

4.

$$\mathbf{C} = \mathbf{PQ} = [5\ 2.25\ 0.90\ 1.90\ 1.75] \begin{bmatrix} 6 \\ 18 \\ 12 \\ 3 \\ 4 \end{bmatrix} = \text{Br. } 94.00$$

Given

$$\mathbf{A} = [1 \quad 2 \quad 3] \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 4 & 5 \\ 6 & 7 \\ 8 & 9 \end{bmatrix}$$

The product **C** = **AB** is computed

$$\mathbf{C} = \mathbf{AB} = \left[[1 \quad 2 \quad 3] \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} \qquad [1 \quad 2 \quad 3] \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \right]$$

$$= [(1 \text{ x } 4 + 2 \text{ x } 6 + 3 \text{ x } 8) \qquad (1 \text{ x } 5 + 2 \text{ x } 7 + 3 \text{ x } 9)]$$

[40 46]

Costs at the various four locations are

First place birr 17.6

Second place- birr 18.8

Third place birr 17.75

➔ Fourth place birr 17.5

Therefore, Abebe had better use the fourth place.

5. $\begin{pmatrix} 56 & 53 & 97 & 90 \\ 41 & 55 & 91 & 106 \\ 104 & 110 & 83 & 61 \end{pmatrix}$

6. i. $A = \begin{pmatrix} 16 \\ 40 \\ 23 \end{pmatrix}$ $\quad B = \begin{pmatrix} 20 \\ 35 \\ 0 \end{pmatrix}$

ii. $\begin{pmatrix} 325 \\ 110 \end{pmatrix}$

7. $\begin{pmatrix} 482 & 444 & 222 \end{pmatrix}$

EXERCISE 2.2

The total material, total labor, and total overhead cost associated with filling this order can be computed as follows:

$$\textbf{C}=\textbf{VQ}= \begin{bmatrix} 50 & 20 & 15 & 25 \\ 30 & 15 & 12 & 15 \\ 30 & 15 & 8 & 20 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 4 \\ 12 \end{bmatrix} = \begin{array}{l} \begin{bmatrix} 730 \\ 468 \\ 512 \end{bmatrix} \end{array} \begin{array}{l} \text{Material} \\ \text{Labor} \\ \text{Overhead} \end{array}$$

2.

Note that the computations made to determine that total material cost is $730 were as follows:

Br.50	material per desk	x 5 desks	= Br.250
Br.20	material per chair	x 6 chairs	= 120
Br.15	material per table	x 4 tables	= 60
Br.25	material per cabinet	x 12 cabinet	= 300
			Br.730

Similar computations yielded the total labor cost of Br.468 and the total overhead cost of Br.512.

The inverse is the matrix

$$A^{-1} = \begin{vmatrix} 1 & -3 \\ -2 & 7 \end{vmatrix}$$

We compute the inverse of a matrix of order 3, as shown below:

$$\left|\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 5 & 6 & 4 & 0 & 0 & 1 \end{array}\right|$$

Divided the first row above by 2 to obtain

$$\left|\begin{array}{ccc|ccc} 1 & 3/2 & ½ & ½ & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 5 & 6 & 4 & 0 & 0 & 1 \end{array}\right|$$

Multiply the first row above by -1 and add to the second row to obtain

$$\left|\begin{array}{ccc|ccc} 1 & 3/2 & ½ & ½ & 0 & 0 \\ 0 & 5/2 & 3/2 & -1/2 & 1 & 0 \\ 5 & 6 & 4 & 0 & 0 & 1 \end{array}\right|$$

Multiply the first row above by -5 and add to the third row to obtain

$$\left|\begin{array}{ccc|ccc} 1 & 3/2 & ½ & ½ & 0 & 0 \\ 0 & 5/2 & 3/2 & -1/2 & 1 & 0 \\ 0 & -3/2 & 3/2 & -5/2 & 0 & 1 \end{array}\right|$$

The tactics we have followed are these: first, get a 1 in the first column, first row, and then use combinations of this row with each of the other rows to get zeros in the first columns of these rows. We now repeat these tactics, starting by getting a 1 in the second column of the second row, then using this row to get zeros in the second columns of the other rows.

Divide the second row by 5/2 to obtain the new second row

$$0 \quad 1 \quad 3/5 \quad -1/5 \quad 2/5 \quad 0$$

If we multiply this new second row by -3/2 and add to the first row, then multiply the new second row by 3/2 and add to the third row, we obtain the following new matrices.

$$\left|\begin{array}{ccc|ccc} 1 & 0 & -2/5 & 4/5 & -3/5 & 0 \\ 0 & 1 & 3/5 & -1/5 & 2/5 & 0 \\ 0 & 0 & 12/5 & -14/5 & 3/5 & 1 \end{array}\right|$$

Finally, we repeat the tactics by getting a 1 in the third row of the third column and then using this row to get zeros in the third column of the other rows. The new third row is

$$\left|\begin{array}{ccc|ccc} 0 & 0 & 1 & -7/6 & 1/4 & 5/12 \end{array}\right|$$

If we multiply this new third row by 2/5 and add to the first row, then multiply the new third row by -3/5 and add to the second row, we find

$$\left|\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -1/2 & 1/6 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{4} & -1/4 \\ 0 & 0 & 1 & -7/6 & \frac{1}{4} & 5/12 \end{array}\right|$$

which has the matrix I to the left and the desired inverse matrix to the right.

X -12/5 Z =9

Y+4/5Z=7

4 units of type A car, 3 units of Type B car and 4 units of type C car

The grocery should purchase 300 bottles of Guder, 200 bottles of Axumite, 100 bottles of Crystal.

i) 25/4 tons of Alloy A, and 5 tons of Alloy B.

5/2 tons of Alloy A and 2 Tons of Alloy B

Using the available capacity, the company can manufacture,

X + Z= 45/2

Y–Z=15/2

This shows that there is multiple optimal solutions for the problem. Therefore, how any units of small calculator to produce depend on that of the large and medium.

12 units of A and 10 units of B

EXERCISE 2.3

1. Apply elementary row operation to the augmented matrix of the system,

$$\left[\begin{array}{cc|c} 4 & 6 & 8 \\ 6 & 9 & 12 \end{array}\right]$$

$$1/4R_1 \rightarrow R_1 = \left[\begin{array}{cc|c} 1 & 3/2 & 2 \\ 6 & 9 & 12 \end{array}\right]$$

$$R_2 - 6R_1 \rightarrow R2 \sim \left[\begin{array}{cc|c} 1 & 3/2 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

At this point we cannot continue the process since all entries in the last row are 0. The system of equations corresponding to this final augmented matrix is,

$$X + 3/2y = 2$$

$$0 = 0$$

Which means we really have a single equation, x + 3/2 y = 2. The graphs of the two original equations coincide and there are unlimited or infinitely many solutions, if we let value of x = k, then using the equation x + 3/2y = 2 we get y = $\frac{4 - 2K}{3}$

$\left(K, \; 4-2K/3\right)$ is a solution for any real number K. Replacing value of K with a real number produces a particular solution to the system. For instance if K = -1 then one particular solution is (- 1, 2)

1.

Apply elementary row operations to the augmented matrix of the system

$$\left[\begin{array}{cc|cc} 1 & 1 & -2 & 8 \\ 2 & -1 & 2 & 7 \end{array}\right]$$

$R2-2R1 \rightarrow R2=$ $\left[\begin{array}{cc|cc} 1 & 1 & -2 & 8 \\ 0 & -3 & -6 & -9 \end{array}\right]$

$-\frac{1}{3} R_2 \rightarrow R2 =$ $\left[\begin{array}{cc|cc} 1 & 1 & -2 & 8 \\ 0 & 1 & -2 & 3 \end{array}\right]$

$R_1-R_2 \rightarrow R_1=$ $\left[\begin{array}{cc|cc} 1 & 0 & 0 & 5 \\ 0 & 1 & -2 & 3 \end{array}\right]$

At this point we stop the process. Why? The last matrix corresponds to the

system X =5

Y – 2z = 3

Thus we have infinitely many solutions. Solving the first equation in terms of z we get y= 3+2z. Let z = K. then for any real number K,

X=5,

Y = 3 + 2k,

Z=k

The solution given by (5, 3+2k, K) is called a general solution. We can have particular solutions for the system by assuming particular values for K. for instance, if K = 4, then

X= 5

Y = 3+2(4) =11

Z= 4

That is (5, 11, 4) is one particular solution for the system. Other particular solutions can be obtained in a similar manner. For instance if k= -2, we will have another particular solution (5, -1, -2).

2.

Since we are required to find the number of products that can be produced there are only two unknown variables.

Let x = the number of product A to be produced

Y = the number of product B to be produced

We have the following 3 by 2 system of linear equation representing the

problem, $X + y = 8$

$X + 2y = 10$

$3x + y = 20$

Apply elementary row operation to the augmented matrix of the system

$$\left[\begin{array}{cc|c} 1 & 1 & 8 \\ 1 & 2 & 10 \\ 3 & 1 & 20 \end{array}\right]$$

$R_2 - R_1 \rightarrow R_2$ $\left[\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{array}\right]$

R3- 3R1 → R3

$R_1 - R_2 \rightarrow R_1 =$ $\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$

$R_3 + 2R_2 \rightarrow R_3$

We have a unique solution $x = 6$ and $y = 2$, hence with the available hours on each machine, 6 units of product A and 2 units of product B can be produced per day.

EXERCISE 2.4

1. (0.5648 0.2731 0.2159)

2. (.0453 0.9547)

Let. U - Stands for users, and N- stands for nonuser

Initial state $V_{UN}^{(0)} = 0.2 \quad 0.8$

To the next month

From one month	Users (U)	Non Users (N)
Users (U)	.95	.05
Non Users (N)	.10	.90

i) $V^{(1)}_{UN} = V^{(0)}_{UN} \times P$

$$= (0.2 \quad 0.8)\begin{pmatrix} .95 & .05 \\ .10 & .90 \end{pmatrix}$$

$$= \underline{\underline{(0.27 \quad 0.73)}}$$

ii) $V^{(2)}_{UN} = V^{(1)}_{UN} \times P$

$$= (0.27 \quad 0.73)\begin{pmatrix} .95 & .05 \\ .10 & .90 \end{pmatrix}$$

$$= \underline{\underline{(.3295 \quad 0.6705)}}$$

iii) $V_U \; V_N = (? \quad ?)$

$$V_U = \frac{switchtoU}{Switchtou + SwitchtoN} = \frac{.1}{.15} = 0.65 \qquad V_U = \frac{switchtoN}{Switchtou + SwitchtoU} = \frac{.05}{.15} = 0.33$$

$V_U \; V_N = 0.67 \quad 0.33$

$V_{UN}^{(n)} = \underline{\underline{0.67 \quad 0.33}}$

KEY FOR CHAPTER THREE SELF CHECK EXERCISES

EXERCISE 3.1

Linear programming is an optimization model that deals with how to allocate scarce resources such as materials, capital, and time and how to make such an allocation in a best possible way subject to more than one limiting conditions expressed in the form of inequalities and or equations.

It is impossible to list all the application areas of Linear programming models. However, it is possible to mention some common application areas which include:

- Selection of product mix
- Selection of different blends of raw materials in feed mills to produce finished feed combinations at minimum cost.
- Determination of distribution system to for minimizing total shipping costs.
- Allocation of limited advertising budget among various advertising media.
- Selection of an investment portfolio from variety of stocks and bonds.

Linear programming model has got pros and cons which include among other things the following:

Advantage

- Linear programming model helps in attaining optimum use of productive factors
- It also indicates how a decision maker can employ his productive factors effectively by selecting and distributing these elements
- Linear programming technique improve the quality of decisions
- Highlighting of bottlenecks in the production process is one of the most significant advantages of this technique
- Linear programming provides possible and practical solutions since there might be other constraints operating outside of the problem which might be taken in to account.

Drawbacks

In reality objective functions and constraints can not be expressed in linear form

In LP problem fractional values are permitted for the decision variables. However, many decision problems require that the decision variables should be non fractional.

The technique can not be used when the problem consists of multiple objectives.

It does not take in to consideration the effect of time and Uncertainty.

Parameters appearing in the LP Model are assumed to be constant but in real life they are neither known nor constant.

In case of large, complex and constrained problems, computational problems are enormous.

Basic requirements of linear programming models include: Decision variables and their relationship, Well defined objective function, Presence of constraints, Alternative courses of action, Non-negativity restriction, Linearity

The basic assumptions of linear programming Model include: proportionality, additivity, divisibility, certainty, finiteness, and optimality.

EXERCISE 3.2

1. maximize $\pi = 500x_1 + 600x_2 + 1200x_3$

   Subject to: $2x_1 + 4x_2 + 6x_3 \leq 160$

   $3x_1 + 2x_2 + 4x_3 \leq 120$

   $x_1, x_2, x_3 \geq 0$

2. minimize $\pi = 5x + 8y$

   Subject to $2x + y \geq 8$

   $x + 2y \geq 10$

   $x, y \geq 0$

3. Maximize $\pi = 30x + 40x2 + 20x3$

   Subject to: $100x_1 + 120x_2 + 70x_3 \leq 100{,}000$

   $7x_1 + 10x_2 + 8x_3 \leq 80$

   $x_1, x_2, x_3 \geq 0$

4. Maximize $\pi = 4x + 3y$

   subject to: $2x + y \leq 72$

   $x + 2y \leq 48$

   $x, y \geq 0$

5. Maximize $Z = 20x + 30y$

   subject to: $3x + 3y \leq 36$

   $5x + 2y \leq 50$

   $2x + 6y \leq 60$

   $x1, y \geq 0$

Exercise 3.3

X_1=3

X_2=3

Objective function Value = 30

X_1=40

X_2 = 100

Objective function Value = 3,200

X_1= 2

X_2= 3

Objective function Value= 230

X_1=5

X_2=3

Objective function Value = 35

X_1 = 1.33

X_2 = 3.33

Objective function Value = 36

$X_1=30$

$X_2=0$

Objective function Value 300

EXERCISE 3.4

I.

1. No feasible solution
2. Unbounded solution
3. Unbounded solution
4. Unbounded solution
5. Infeasible solution
6. Infeasible solution

II.

1. Answer: $X_1 = 60$, $X_2 = 2$ and Max. Z = 1,100
2. Answer: $X_1 = 4$, $X_2 = 2$ and Max. Z = 10
3. Answer: $X_1 = 4$, $X_2 = 2$ and Max. Z = 230
4. Answer: $X_1 = 1$, $X_2 = 5$ and Min. Z = 13
5. Answer: $X_1 = 2$, $X_2 = 0$ and Min. Z = 240
6. Answer: $X_1 = 6$, $X_2 = 12$ and Min. Z = -2

III.

Answer: 10 units of model X, 2 units of model Y and the maximum profit is $ 560.

Answer: $X_1 = 375$, $X_2 = 50$ and Max. Z = 23750

Answer: 1 Unit of liquid, 5 units of dry product and Min. Z = $13

KEY FOR CHAPTER FOUR SELF CHECK EXERCISES

EXERCISE 4.1

Let the principal be Birr 1,000 and hence, double of this amount yields Birr 2,000. So, within the given period (i.e., 5 years), the interest is calculated as follows.

$$A = P(1 + rt)$$

$$\frac{2000}{1000} = \frac{1000\,(1 + 5r)}{1000}$$

$$2 = 1 + 5r$$

$$2 - 1 = 5r$$

$$\frac{1}{5} = \frac{5r}{5}$$

$$r = \frac{1}{} = 20\%$$

Let the principal be Birr 1,000 and hence, a triple of this amount yields Birr 3,000. So, at the given rate of simple interest (i.e., 5%), the period is calculated as follows.

$$A = P(1 + rt)$$

$$\frac{3000}{1000} = \frac{1000\ (1 + 0.05t)}{1000}$$

$$3 = 1 + 0.05t$$

$$3 - 1 = 0.05t$$

$$\frac{}{0.05} = \frac{0.05t}{0.05}$$

$$t = 40 \text{ years}$$

3. We know $A = P(1 + rt) \Rightarrow P = A\left(\frac{1}{1 + rt}\right)$

We are given; A = Birr 20,400

$i = 0.08$

$t = 3/12 = 0.25$ year

$$P = 20{,}400 \left(\frac{1}{1 + (0.08 \times 0.25)} \right) = \text{Birr } 20{,}400 \times \frac{1}{1.02}$$

P = Birr 20,400 (0.980392) = 19,999.99 ≈ Birr 20,000

4. Birr 9,120

5. 0.12 0r 12%

6. I = Prt

150,000 x 0.12 x 6

Birr 288,000

A=P+I

= 250,000 + 108,000

= Birr 258000 or

A = P (1 + rt)

150,000 (1 + 0.12 x 6)

150,000 (1.72) = Birr 258,000

EXERCISE 4.2

Birr 16,713

0.14 year

Birr 151, 634.12

4.

P=500

A=650

r = 10%

i=2.5%

m = 4

t=? n/m = $\frac{10.625}{} = 2\frac{2}{3}$ *years*

n =? 10.625 quarters

A = p(1+i)^{n} =

650 = 500 (1.025)^{n}

1.3 = (1.025)^{n}

log$^{1.3}$= log$^{(1.025)n}$

$\frac{\log 1.3}{\log 1.3} = n\frac{\log 1.025}{\log 1.025}$

n = $\frac{\log 1.3}{\log 1.025}$

= 10.625 quarters

5.

P = Birr 2,000

A = Birr 2,166

=?

i=r/12

t = 1 m

= 12

A = p(1+i)^{n}

2166 = 2000 (1+i)^{12}

1.083 = (1+i)^{12}

log$^{1.083}$= log$^{(1+i)12}$

=12log^{1+i}

$\frac{\log 1.083}{12} = \log 1 + i$

0.0028857 = log$^{1+i}_{.0028857}$

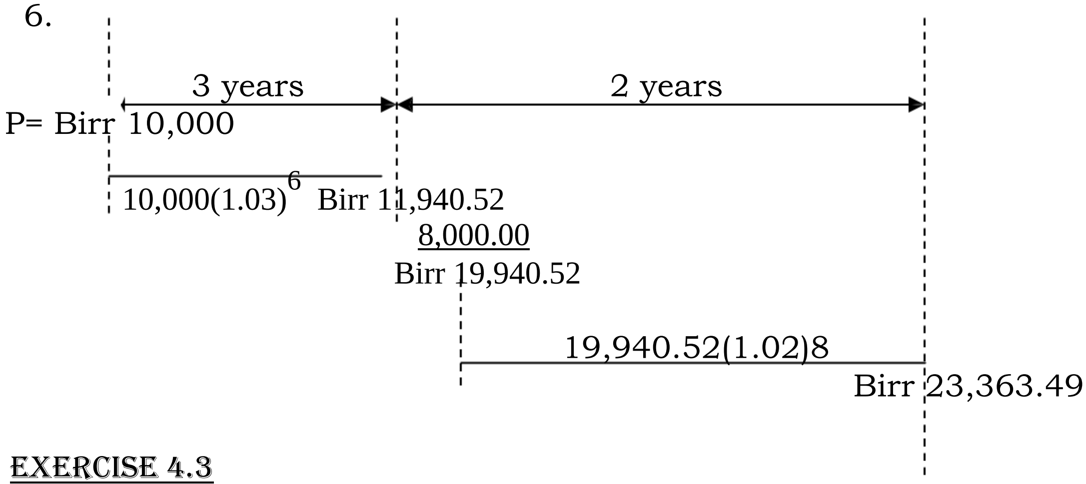
anti log = 1+i

.0066667 = i

.006667 x 12 = r = i x m

= 8% = r

6.



EXERCISE 4.3

A = 20,000 Birr
t = 4 years

m = 12

r = 6%

P =?

$= A(1+i)^{-n}$

$20{,}000\ (1.005)^{-48}$

Birr 15,742

We can solve this problem in two ways:

By finding the PV of 600 and compare it with 500

By finding the FV of 500 and compare it with 600.

A=600 t = 18 months = 1.5years m = 2 r = 14% p =? $P = A(1+i)^{-n}$ $600\ (1.07)^{-3}$ **Birr 489.78** Since 489.78 < 500, it is better to pay the debt after 18 months.	P=500 t = 1.5 years m = 2 r = 14% A =? $A = P(1+i)^{n}$ $500\ (1.07)^{3}$ = **Birr 612.52** Since 612.52 > 600, it is better to pay the debt after 18 months.

$P = A\ (1+i)^{-n}$

3. A = Birr 8,000

t = 9 years

n = 36

m = 4

i=2%

r = 8%

P =?

$= 8,000(1.02)^{-36}$

= 3,921.79 Birr

Present value with continuous compounding

$$\frac{}{erterte} = \frac{Pe^{rt}}{} \Rightarrow P = \frac{A}{rt}$$

$$Ae^{-rt}$$

EXERCISE 4.4

1. $re = e^{r} - 1$

$.10 = e^{r} - 1$

$1.1 = e^{r}$

$\ln^{1.1} = \ln^{er}$

$\ln^{1.1} = r \ln e$

$\ln^{1.1} = r(1)$

9.531% = r

2. A) 9.2025%
B) 9.3080%
C) 9.38%

3. Birr 20,000

EXERCISE 4.5

1.

R = Birr 1000

t = 3 years.

m = 12

n = 36

r = 12%

i=1%

A =?

$$A = \frac{[(1+i)^{n} - 1]}{i}$$

$$\frac{1000\,[(1.01)\,36 - 1]}{0.01}$$

= Birr 43,076.88

Compound interest = A - R(n)

= 43,076.88 - 36,000

= 7,076,88 Birr

2.

For the first 10 years:

R = Birr 500

t = 10years

m = 1

r = 6%

A_{10} =?

$$A_{10} = 500\,\frac{[(1.06)^{10} - 1]}{.06}$$

$= 500\ (13.180795)$

$= \text{Birr}\ \underline{\underline{6590.40}}$

For the next 8 years, Birr 6590.40 is taken as single deposit (Principal) in an account which pays 8% compounded quarterly.

$P_8 = A_{10}$ = Birr 6950.40

t = 8years

m = 4

r = 8%

A_{18} =?

$A_{18} = A_{10}\ (1 + i)^{n}$

$= 6950.40\ (1.02)^{32}$

$= 6950.40\ (1.88454)$

A_{18} = **Birr 12,419.87**

The balance after 18 years is Birr 12,419.87 out of which Birr 7,419.87 (Birr 12,419.87 – Birr 5000) is interest earned.

Birr 954.06

Birr 1337.8

Birr 124.44

A) Birr 572.73

Interest = Birr 172.73

1669.89

EXERCISE 4.6

1. Cash Value = down payment + PV of an ordinary annuity

$$200 + 82\left[\frac{1 - (1.01)^{-18}}{.01}\right]$$

$200 + 82(16.39827)$

$200 + 1,344.658$

Birr 1,544.658

2. R = Birr 1000

t = 3years

m = 4

r = 6%

PV =?

$$PV = 1000\left[\frac{1 - (1.015)^{-12}}{.015}\right]$$

$= 1000(10.9075)$

$=$ **Birr 10, 907.50**

Since we have quarterly payment periods and monthly interest periods, we must change the interest rate to coincide with the quarterly payment periods. Specifically, we must find the equivalent interest rate compounded quarterly corresponding to 12% compounded monthly and use it as r (i). Using the equivalent rate formula, we have

$(1 + r/4)^4 = (1 + .12/12)^{12}$

$(1 + r/4)^4 = (1.01)^{12}$

$r/4 = (1.03)^3 - 1 \Rightarrow i = .030301$

R = Birr 1000

i = .030301

n = 7

P =?

$$P = R\left[\frac{1 - (1 + i)^{-n}}{i}\right] = 1,000\left[\frac{1 - (1.030301)^{-7}}{.030301}\right] = \textbf{Birr 6,223.22}$$

4. PV = Birr 200,000

t = 10years

m = 12

r = 12%

=?

$$R = 200,000 \left[\frac{.01}{1 - (1.01)^{-120}} \right]$$

= 200,000(0.014347)

= **Birr 2,869.42**

Selling price = Birr 50,000

Down payment =?

Mortgage (A) =?

R = Birr 600

r = 24%

m = 12

i = 2%

t = 20

Mortgage

$$A = \frac{R\left[1 - (1+i)^{-n}\right]}{i}$$

$$A = \frac{600\left[1 - (1.02)^{-240}\right]}{0.02}$$

Birr <u>29,741.13</u>

Down payment = Selling price – mortgage

50,000 – 29,741.13

Birr <u>20,258.87</u>

Interest charged = actual payment- mortgage

600 x 240 - 29,741.13

144,000 - 29741.13

Birr <u>114,258.87</u>

Percentage of down payment = $\frac{Down\ payment}{Selling\ Price} x100$

$$\frac{20,258.87}{50,000} x100$$

<u>40.52%</u>

KEY FOR CHAPTER FIVE SELF CHECK EXERCISES

EXERCISE 5.1

1.

A) -8

B) 8

C) 199.9

2.

A) 25 – 10g

B) 4 – 4g

C) 10,000 – 200g

3.

A) $k^2 - 10k$

B) $k^2 - 4k$

C) $k^2 - 200k$

4.

A) $9 - r^2$

B) $q^2 + 1$

C) $a^2 + 2a$

5. $a^2 + 2a$

EXERCISE 5.2

Does not exist

5b

2a

3x

2y

EXERCISE 5.3

None

X = 2

None

X=+2

X = 2

X = 2, x = 3

X = 1

None

None

None

EXERCISE 5.4

1.

$MP = -6x+750$

$TP = \int MP\,dx$

$= \int(-6x + 750)dx$

$= \int -6x dx + \int 750 dx + C$

$= -6x/2 + 750x + C$

$= -3x^2+750x+c$

$25,000 = -3(100^2) + 750(100) + C$

$25,000 = 45,000 + C$

$C = -20,000$

$P(x) = \underline{\underline{-3x^2+750x-20,000}}$

2.

$S(x) = \int S'(x)\,dx$

$= \int(.5 - .12x^{-1/2})\,dx$

$= \int .5dx - \int .12x^{-1/2}dx$

$= .5x - .12x^{(-1/2+1)}/1/2 + C$

$= .5x - .24x^{1/2} + C$

$0 = .5(81) - .24(81^{1/2}) + C$

$40.5 - 2.16 + C$

$38.34 + C$

$S(x) = .5x - .24x^{1/2} - 38.34$

$S(144) = .5(144) - .24(144^{1/2}) - 38.34$

$= 72 - .24(12) - 38.34$

$= 72 - 41.22$

= **Birr 30.78 billion**

3.

A) 225 cubic feet.

B) 700 cubic feet

C) 65 cubic feet

D) 75 cubic feet

E) 30 minutes

4.

$$\int_{40}^{60} f(x)dx = 600x - 2.5x^2 \Big|_{40}^{60}$$

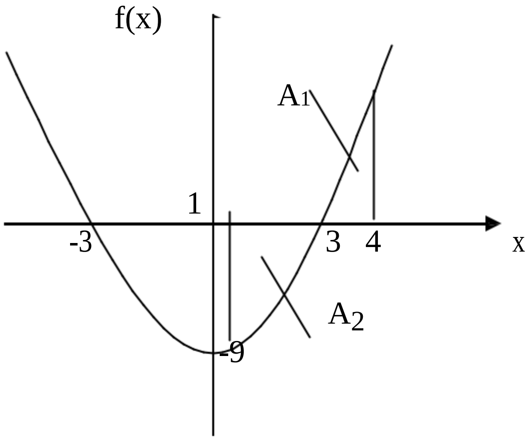
$= f(60) - f(40)$

$= [600(60) - 2.5(60^2)] - [600(40) - 2.5(40^2)]$

= Birr $\underline{7,000}$

5.

Sketch the graph and identify the region needed.



Find A_1 and A_2.

$$A_2 = \int_3^4 (x^2-9)dx = \left. \frac{1}{3}x^3 - 9x \right|_3^4 = f(4)-f(3) = 3\frac{1}{3} \text{ square units}$$

$$A_1 = \int_1^3 (x^2-9)dx = \left. \frac{1}{3}x^3 - 9x \right|_1^3 = -[f(3)-F(1)] = -(-9\frac{1}{3}) = 9\frac{1}{3} \text{ square units}$$

Total area = A_1 + A_2

$$9\frac{1}{3} + 3\frac{1}{3} = 12\frac{1}{3} \text{ square units}$$

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ARBAMINCH UNIVERSITY

COLLEGE OF BUSINESS & ECONOMICS

DEPARTEMENT OF MANAGEMENT

MATHEMATICS FOR MANAGEMENT ASSIGNMENT FOR DISTANCE STUDENTS

Part I: Multiple Choice Questions (5 x 1 = 5 Points)

The marketing department estimates that if the selling price of the new product is set at Br.40 per unit, sales will be 400 units per week. If the selling price is Br.20 per unit, sales will be 800 units per week. The production department estimates that variable costs will be Br.7.50 per unit and fixed costs Br.10, 000 per week.

1. The cost equation is
   - A. Br.10, 000 + Br.7.5x
   - B. Br.10, 000 - Br.7.5x
   - C. Br.10, 000 + Br.40
   - D. Br.10, 000 + Br.75x
2. The sales revenue equation is
   - A. 400–9
   - B. 1,200–9
   - C. $60x - x^2/20$
   - D. $60x - x^2$
3. The profit equation is
   - A. $-x^2/20 + 52.5x - 10,000$
   - B. $-x + 52.5 - 10,000$
   - C. $x^2 - 52.5x + 10,000$
   - D. $X^2 - 52.5x - 10,000$

How much would need to be invested today at 6% per annum to provide an annuity of Br.5, 000 per annum for ten years commencing in five years' time?

- A. Br.5,000
- B. Br.19,000
- C. Br.29,150
- D. Br.39,420

A computer company plans to produce 30,000 computers next year. They will sell for Br. 700 each. The fixed cost of operation care Br. 5 million and total variable costs are Br. Br. 6 million. What is the break-even point?

- A. 30,000
- B. 20,000
- C. 10,000
- D. 40,000
- E. 15,00

Part I: Work Out the following Questions

A company can use two types of machine *A* and *B*, in a manufacturing plant. The number of operators required and the running cost per day are given as

	Cost per day	Available operators	Floor area (m^2)	Profit per machine
Machine A	6	2	2	20
Machine B	3	4	2	30
AMaximum.W available r	360	280	160	

ite down the inequality constraints and the profit function. **(1 point)**

Graph the inequalities. From the graph determine the number of machines A and B which should be used to maximize profits. **(1 point)**

Confirm your answer in (B) algebraically. **(1 point)**

A gardener requires fertilizer to have a minimum of 80 units of nitrogen, 15 units of potassium and 10 units of iron. Two fertilizer mixes are available. Brand X contains 20, 2 and 1 units of the minerals while Brand Y contains 4, 1, 2 units of the minerals respectively. Brand X costs Br.18 per kg while brand Y costs Br.6 per kg.

Write down the inequality constraints and the equation of the cost function. **(1 point)**

Graph the inequality constraints. Hence determine the number of kg of each brand which provide the minimum mineral requirements at minimum cost. **(1 point)**

Confirm your answer in (B) algebraically. **(1 point)**

A distributor records the weekly sales of personal computers (PCs) in three retail outlets in different parts of the country. The cost price of each model is:

Table 1: Number of computers sold in each shop

	Pentium (basic)	Pentium (extra)	Pentium (latest)
Shop A	150	320	180
Shop B	170	420	190
Shop C	201	63	58

Table 2: Selling price of computers in each shop

	Pentium (basic)	Pentium (extra)	Pentium (latest)
Shop A	560	750	1580

Shop B	520	690	1390
Shop C	590	720	1780

Calculate the total weekly cost of computers to each shop. **(1 point)**

Calculate the total weekly revenue for each model for each shop. **(1 point)**

Calculate the total weekly profit for each shop. **(1 point)**

Which shop makes the greatest overall profit? **(1 point)**

The variable cost of the electric kettle manufactured by Fair Oak Domestic Appliances Ltd is Br. 4. The company, which sells its kettles direct to retailers for Br. 10, expects its net profit for the year just ended to be Br. 270,000 after allowing for fixed costs of Br. 90,000. Annual sales are 60,000 kettles per year. The productive capacity of the company is under-utilised and the marketing manager suggests that a 10 per cent reduction in selling price will bring about a 25 per cent increase in sales.

What level of sales is necessary to break even? **(1 point)**

Calculate the sales revenue at this volume of sales. **(1 point)**

Calculate the sales revenue resulting from implementing the marketing manager's proposal. **(1 point)**

As a result of the increased volume, by how much has revenue changed? **(1 point)**

As a result of the increased volume, by how much have costs changed? **(1 point)**

Should Fair Oak reduce their selling price by 10 per cent? **(1 point)**

A firm produces x tonnes of output at a total cost

$C(X) = 1/10\ x^3 - 4\ x^2 + 20\ x + 5$

Find;

Average cost **(1 point)**

Average Variable Cost **(1 point)**

Average Fixed Cost **(1 point)**

Marginal Cost **(1 point)**

Marginal Average Cost. **(1 point)**

11. A marketing research team has just completed a survey of consumer buying habits with respect to three brands of coffee and has produced the following brand switch matrix for customer change in brands:

Brand next purchased

$$\text{Brand last purchased:}\quad \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{pmatrix} \end{array}$$

Draw the state diagram for this Markov chain model. **(1 point)**

What is the probability that a customer who bought brand B this time will buy brand A two time periods from now? **(1 point)**

What percent of the customers who buy brand A this week will buy brand A three time periods from now? **(1 point)**

What percent of the customers who buy brand A this week will buy brand B three time periods from now? **(1 point)**