

OPERATIONS RESEARCH

COURSE CODE: (MGMT 2262)

CREDIT HOUR: 3

MODULE



BY:

ATO MESFIN TESFAYE (MBA)

ATO FUAD MUSTEFA (MBA)

ARBA MINCH

OPERATIONS RESEARCH

COURSE CODE: MGMT 2262

CREDIT HOUR: 3



BY: ATO MESFIN TEFAYE (MBA)

ATO FUAD MUSTEFA (MBA)

EDITOR: ATO HABTAMU ENDRIS (MBA)

ARBA MINCH

CHAPTER ONE

Introduction to Operations Research

Objectives:

On successful completion of this chapter, you should be able to:

- Define the term operations research
- Explain the historical development of operations research
- Understand the nature and characteristics of operations research
- Explain the significance of operations research to managers
- Identify the operation research problem solving approach
- Understand the meaning of model and its different classifications

1.1. Introduction

Many people still remain in the bondage of self-incurred tutelage. Tutelage is a person's inability to make his/her own decisions. Self-incurred is this tutelage when its cause lies not in lack of reason but in lack of resolution and courage to use it without wishing to have been told what to do by something or somebody else.

The difficulty in life is the choice. Good decision-making brings about a better life. A bad decision may force you to make another one. A good decision is never an accident; it is always the result of high intention, sincere effort, intelligent direction and skillful execution; it represents the wise choice of many alternatives. One must appreciate the difference between a decision and an objective. A good decision is the process of optimally achieving a given objective.

When decision making is too complex or the interests at stake are too important, quite often we do not know or are not sure what to decide. In many instances, we resort to informal decision support techniques such as tossing a coin, asking an oracle, visiting an astrologer, etc. However formal decision support from an expert has many advantages.

Business Science focuses on the formal model-driven decision support techniques such as mathematical programs for optimization, and decision tree analysis for risky decisions. Such techniques are now part of our everyday life. For example, when a bank must decide whether a given client will obtain credit or not, a technique, called credit scoring, is often used.

Rational decisions are often made unwillingly, perhaps unconsciously. We may start the process of consideration. It is best to learn the decision-making process for complex, important and critical decisions. Critical decisions are those that cannot and must not be wrong.

The aim of this course is to make you a better decision maker by learning the decision-making process. Decision-making is a complicated process. This complication arises from the fact that your present goal (including wants, resources, and abilities) dictates your choices; however, your choices will change your goals. This influential-cycle keeps the decision-maker busy all the time. Selecting your goals and your criteria for success is a dynamic process and changes over time.

On a daily basis a manager has to make many decisions. Some of these decisions are routine and inconsequential, while others have drastic impacts on the operations of the firm for which he/she works. Some of these decisions could involve large sums of money being gained or lost, or could involve whether or not the firm accomplishes its mission and its goals.

In our increasingly complex world, the tasks of decision-makers are becoming more challenging with each passing day. The decision-maker (i.e., the responsible manager) must respond quickly to events that seem to take place at an ever-increasing pace. In addition, a decision-maker must incorporate a sometimes-bewildering array of choices and consequences into his or her decision. Routine decisions are often made quickly, perhaps unconsciously without the need for a detailed process of consideration. However, for complex, critical or important managerial decisions it is necessary to take time to decide systematically.

Making strategic decisions requires that one takes a structured approach following a formal decision making process. Otherwise, it will be difficult to be sure that one has considered all the key aspects of the decision.

1.2. Historical development of OR

It is generally agreed that OR came into existence as a discipline during world war II when there was a critical need to manage scarce resources. However, a particular model and technique of OR can be traced back to much earlier times. The scientific approach to management can be traced back to the era of Industrial Revolution and even periods before that. But operations research, as it exists today, was born during the Second World War when the British military management called upon a group of scientists to examine the strategies and tactics of various military operations with the intention of efficient allocation of scarce resources to the war effort.

The war involved strategic and tactical problems which were greatly complicated to expect adequate solution from individuals or specialists in a single discipline. Therefore groups of individuals who collectively were considered specialists in mathematics, economics, statistics and probability theory, engineering, behavioral and physical science were formed as special units within the armed forces to deal with strategic and tactical problems of various military operations.

Problems of this latter kind, often called systems problems, were studied rigorously for the first time by scientists, not in industry but in the management of operations in the 1939 - 45 war.

A number of teams of eminent scientists were employed by the British Government to apply their expertise to management and operational problems rather than technical problems.

One such team (the Radar Operational Research team - which gave OR its name) was responsible for implementing the installation and operation of radar after the technical development work was complete.

Later on another team examined the relative ineffectiveness of the Allied Forces at destroying the German U-boats, which were sinking the food convoy ships supplying Britain.

The OR team played an important part in reducing shipping losses and in the ultimate defeat of the U-boats. With no precedent in the application of Science to the management function, the scientists were still able to use their scientific approach-collecting information and developing hypothesis in order to come up with practical plans for the improvement of these wartime operations.

After the end of the war, the same approach was used with great success in the industrial and commercial field, developing most rapidly in Britain and the USA, followed by Europe and the rest of the world. This has provided the basis for the evolution of Operational Research (OR) as a separate and independent subject of Science. It is not based on any single academic engineering, social science, economics, statistics and computing but is none of these.

There are three different factors behind the rapid development in the use and development of operations research approach.

The economic and industrial boom after World War II resulted in continuous mechanization, automation, decentralization of operations and decisions of management functions. This industrialization also resulted in complex managerial problems and therefore application of operations research to managerial decision making become popular.

Many operations researchers continued their research after war. A key person in the post war development of OR was George B. Dantzing. In 1947 he developed the concept of linear programming and its solution by a method known as Simplex method.

Analytic power was made available by high-speed computers. The development of electronic digital computers, with their ability to perform arithmetic

calculations thousands or even millions of times faster than a human being can, was a tremendous boom to OR.

1.3. Meaning of operations research

Definition

Operations Research is defined by many authors in different ways .However, the following definition is proposed here.

The Operational Research Society of U.K. defines OR as:

"Operational Research is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defense. The distinctive approach is to develop a scientific model of the system, incorporating measurements of factors, such as chance and risk, with which to compare the outcome of alternative decisions, strategies and controls. The purpose is to help management determine its policies and actions scientifically."

In the USA, where it is called Operations Research, the OR Society of America says more briefly;

"OR is concerned with scientifically deciding how to best design and operate man-machine systems, usually under conditions requiring the allocation of scarce resources"

"Operations Research is a systematic analysis of a problem through scientific methods ,carried out by appropriate specialists, working together as a team ,constituted at the instance of management for the purpose of finding an optimum and the most appropriate solution ,to meet the given objective under a given set constraints."

OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing.

From the concept and definition given above, Operations Research is:

The application of scientific methods, techniques and tools to the problem to find an answer

A management tool in the hands of a manager to take a decision

A scientific approach to decision making process

An “***applied research***” aims at finding a solution for an immediate problem facing a society, industry or a business enterprise .This is not “***fundamental research***”

A decision-oriented research, using scientific methods, for providing management a quantitative basis for taking decision regarding operations under its control

Applied decision theory. It uses scientific, mathematical and logical means to take decisions.

1.4. Nature and significance of operations research

As the term implies, OR involves research on military operations. The operation research approach is particularly useful in balancing conflicting objectives where there are many alternative courses of action available to the decision maker. In a theoretical sense, the optimum decision (global optimum) must be one that is best for the organization as a whole.

Operations research attempts to resolve the conflicts of interest among various sections of the organization and seeks the optimal solution which may not be acceptable to one department but is in the interest of the organization as a whole.

Nature and characteristics of OR:

- Existence of a problem
- Intension to solve a problem
- Application of system concept and system analysis to the problem
- Scientific approach to the problem where research methods are used
- Formation of group consists of different specialists
- Assist management to make scientific decision
- Solution must meet the objective within the constraints

OR is concerned with providing the decision maker with decision aids (or rules) derived from:

- A total system orientation
- Scientific methods of investigation, and
- Models of reality, generally based on quantitative measurement and techniques

1.5. Feature of operations research

Interdisciplinary Approach

Interdisciplinary teamwork is essential because while attempting to solve a complex management problem, one person may not have the complete knowledge of all its aspects (such as economic, social, political, psychological, engineering etc).

Methodological approach

Operations research is the application of scientific methods, techniques and tools to problems involving the operation of systems so as to provide those in control of operations with optimum solutions to the problem.

Wholistic Approach

While arriving at a decision, an operation research team examines the relative importance of all conflicting and multiple objectives, and the validity of claims of various departments of the organization from the perspective of the whole organization.

Objectivistic Approach

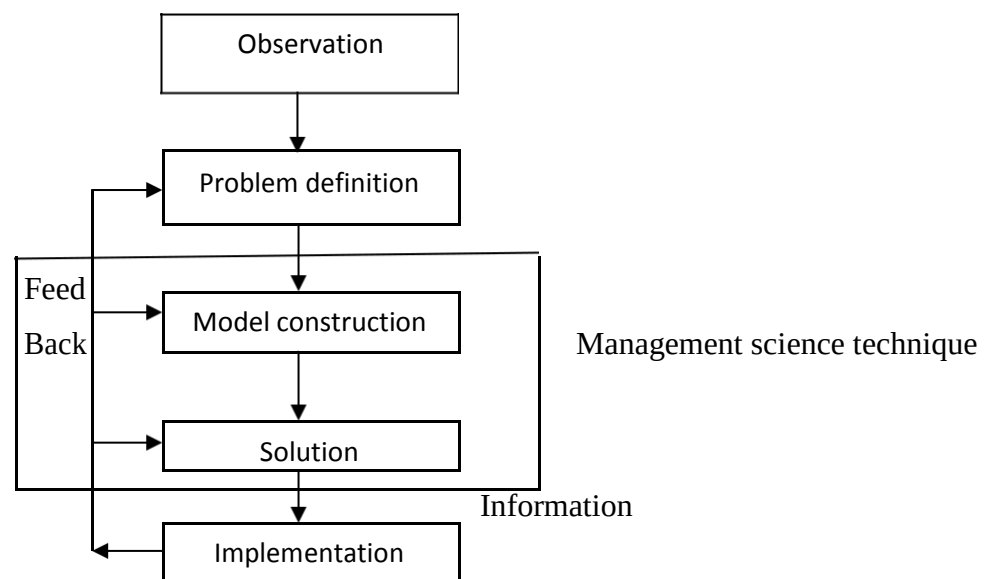
An operation research approach seeks to obtain an optimal solution to the problem under analysis. For this a measure of desirability (effectiveness) is defined, based on the objective(s) of the organization. A measure of desirability so defined is then used to compare alternative courses of action with respect to their outcomes.

1.6. The management science approach to problem solving

Management science encompasses a logical/systematic approach to problem solving, which closely parallels what is known as the scientific method for attacking problems. This approach follows a generally recognized and ordered series of steps:

- 1) Observation, 2) definition of the problem, 3) model construction, 4) model solution, implementation of solution results

Figure1.1. the management science process



Observation: - the first step in the management science process is the identification of a problem that exists in the system (organization). The system must be continuously and closely observed so that problems can be identified as soon as they occur or are anticipated. The person who normally identifies a problem is the manager because the managers work in places where problems might occur.

Definition of problem: - once it has been determined that a problem exists, the problem must be clearly and concisely defined. Improper definition of a problem can easily result in no solution or an inappropriate solution.

Model construction:- a management science model is an abstract representation of an existing problem situation. It can be in the form of a graph chart, but most frequently a management science model consists of a set of mathematical relationships.

Model solution: - once models have been constructed in management science, they are solved using the management science techniques. A management science solution technique usually applies to a specific type of model.

Implementation: - is the actual use of the model once it has been developed or the solution to the problem the models was developed to solve. It is a critical but often overlooked step in the process.

1.7. Models and modeling in operations research

Dear learners, here we will discuss models and modeling in operations research.

The approximation or abstraction, maintaining only the essential elements of the system, which may be constructed with various forms by establishing relationships among specified variables and parameters of the system, is called a model.

Models attempt to describe the essence of a solution or activity by abstracting from reality so the decision maker can study the relationship among relevant variables in isolation. Hence, models do not, and cannot represent every aspect of reality because of

the innumerable and changing characteristics of the real life problems to be represented. Instead, they are limited approximation of reality.

A model is constructed to analyze and understand the given system for the purpose of improving its performance. The reliability of the solution obtained from a model depends on the validity of the model in representing the system under study.

The key to model building lies in abstracting only the relevant variables that affect the criteria of the measures of performance of the given system and expressing the relationship in a situation form. A model should be as simple as possible so as to give the desired result.

There are many ways to classify models. Classification schemes can also provide a useful frame of reference for modelers. There are five classification schemes for models.

Self check questions

What is operations research?

What are the main features of operations research?

What are basic causes for the rapid development of operations research?

Classification based on structure

Physical models

The models provide a physical appearance of the real object under study either reduced in size or scaled up. Physical models are useful only in design problems because they are easy to observe, build and describe.

Iconic models: - retain some of the physical properties and characteristics of the system they represent. An iconic model is either in an idealized form or is a scaled version of the system. Such models represent the system as it is by scaling it up or down (i.e. enlarging or reducing the size).

Examples: blueprints of a home, maps, globes, photographs, drawings, a model airplane in a wind tunnel, trains etc.

The chief merit of an iconic model is that it is concrete and specific. It resembles visually the thing it represents and therefore, there are likely to be fewer problems in translating any “findings” from the model into the real-life situation. However, the disadvantage of such models is that they often do not lend themselves to manipulation for experimental purpose.

Analogue models: - represent a system by a set of properties different from those of the original system and does not resemble it physically. After the problem is solved, the solution is reinterpreted in terms of the original system.

Examples: organizational chart, maps in different colours may represent water, desert and other geographical feature. These models are less specific and concrete but are easier to manipulate and are more general than iconic models.

Symbolic models

Use symbols (letters, numbers) and functions to represent variables and their relationships to describe the properties of the system. These models are also used to represent relationships which can be represented in a physical form.

Verbal models: - describe a situation in written or spoken language. Example the perimeter (P) of a rectangle is equal to the sum total of two times the length (L) and two times the width of the rectangle.

Mathematical models: - involve the use of mathematical symbols, letters, numbers and mathematical operations (+, −, ÷, ×) to represent relationships among variables of the system to describe its properties or behaviors.

Example: $P = 2L + 2W$

Subject to:

$$x_1 + x_2 \leq 60$$

$$x_1 + 2x_2 \leq 45$$

$$x_1 \leq 15$$

$$x_1, x_2 \geq 0$$

x_1 and x_2 are decision variables

Classification based on function or purpose

Descriptive models: - simply describe some aspects of a situation, based on observation, survey, questionnaire results or other available data of a situation and do not predictive or recommend anything.

Example: - organizational chart plant layout diagram, block diagram representing an algorithm or method for solving a problem etc.

Predictive models: - these models indicate “if these occurs then what will follow”. They relate dependent and independent variables and permit trying out “what if questions’. Used to predict the outcomes due to a given set of alternatives for the problem.

Normative (optimization) models: - provide the best or optimal solution to problems subject to certain limitations on the use of resources. Are also called prescriptive models because they provide what the decision maker out to do.

Classification based on time reference

Static models: - static models represent a system at some specified time and do not account for changes over time.

Dynamic models; - time is considered as one of the variables and allows the AMU, CBE,

decisions over a period of time is made to select the optimal course of action to optimize the given objective.

4. Classification based on degree of certainty

a) Deterministic models

If all the parameters, constants and functional relationships are assumed to be known with certainty when the decision is made, the model is said to be deterministic. Thus, in such case, the outcome associated with a particular course of action is known. That is for a specific set of input values, there is a uniquely determined output which represents the solution of the model under considerations of certainty. The results of the models assume single value.

b) Probabilistic (stochastic) models

Models in which at least one parameter or decision variable is a random variable are called probabilistic (or stochastic) models. Since at least one decision variable is random, therefore, an independent variable which is the function of dependent variable(s) will also be random. This means consequence or payoff due to certain changes in the independent variable cannot be predicted with certainty. However, it is possible to predict a pattern of values of both the variables by their probability distribution.

Example: insurance against risk of fire, accidents, sickness, where the pattern of events is studied in the form of probability distribution.

5. Classification based on method of solution or quantification:

- a. Heuristic models: These models employ some set of rules which, though perhaps not optimal, do facilitate solution of problems when applied in a consistent manner.
- b. Analytical models: it has specific analytical structure and thus can be solved by known analytical or mathematical techniques. Any optimization model

(which requires maximization or minimization of an objective function) is an analytical model.

Simulation model: it has a mathematical structure but is not solved by applying mathematical techniques to get a solution. Instead it is computer assisted experimentation on a mathematical structure of a real life problem. It is more flexible than a mathematical model and helps to represent complex systems which otherwise can't be represented by mathematically.

Self check questions

Briefly explain the difference between deterministic and probabilistic models?
Explain the nature of OR?

Chapter Summary

In our increasing complex world, the tasks of decision-makers are becoming more challenging with each passing day. The decision-maker is expected to respond quickly to events that seem to take place at an ever increasing pace.

Operations research attempts to resolve the conflicts of interest among various sections of the organization and seeks the optimal solution which may not be acceptable to one department but is in the interest of the organization as a whole.

It is generally agreed that OR came into existence as a discipline during the second world war when there was a critical need to manage scarce resources.

OR has interdisciplinary, methodological, wholistic and objectivistic approach for problem solving.

OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing.

The operation research problem solving approach involves five major steps: observation, AMU,

The approximation or abstraction, maintaining only the essential elements of the system, which may be constructed with various forms by establishing relationships among specified variables and parameters of the system, is called a model.

A model is constructed to analyze and understand the given system for the purpose of improving its performance. The reliability of the solution obtained from a model depends on the validity of the model in representing the system under study. There are different classifications of models which are constructed to represent and resolve real world business problem.

Exercises

Operations research analysts do not

a) Predict future operation	c) Collect relevant data
b) Build more than one model	d) Recommend decision and accept

Decision variables are

a) Controllable	c) Parameters
b) Uncontrollable	d) None of the above

Ä	Ä □ Ä □	Ä □	Ä □
model is			

a) An essence of reality	c) An idealization
b) An approximation	d) All of the above

A physical model is example of

a) An iconic model	c) A verbal model
b) An analogue model	d) A mathematical model

Operations research approach is

Multi-disciplinary

Scientific

Intuitive

All of the above

CHAPTER TWO

Linear Programming

Learning objectives:

On successful completion of this chapter, you should be able to:

Understand what linear programming is and explain the component and assumptions of Linear Programming?

Formulate Linear Programming model and solve it by using graphic and simplex solution techniques.

Demonstrate special cases of graphic solution method.

Solve maximization problem that have mixed constraint and interpret the solution.

Solve minimization problems and interpret the solution.

Study and demonstrate special cases of simplex solution method.

Formulate the dual of a primal

Read and interpret the solution to a dual problem, and relate the dual solution to the primal solution.

Explain the economic interpretation of dual variables and the dual solution.

Use sensitivity analysis to evaluate change in RHS of a constraint and change in the value of an objective function coefficient.

2.1. Introduction to Linear Programming

Linear Programming is a branch of mathematical programming which is designed to solve *optimization problems* where all the constraints as well as the objectives are expressed as *linear function*. It was developed by George B. Dantzig in 1940. Its earlier application was solely related to the activities of the second World War. However soon its importance was recognized and it came to occupy a prominent place in the industry and trade.

Linear Programming is a technique for making decisions under *certainty* i.e.; when all the courses of options available to an organization are known & the objective of the firm along with its constraints are quantified. That course of action is chosen out of all possible alternatives which yield the optimal results. Linear Programming can also be used as a verification and checking mechanism to ascertain the accuracy and the reliability of the decisions which are taken solely on the basis of manager's experience-without the aid of a mathematical model.

Some of the definitions of Linear Programming are as follows:

"Linear Programming is a method of planning and operation involved in the construction of a model of a real-life situation having the following elements:

Variables which denote the available choices and

The related mathematical expressions which relate the variables to the controlling conditions, reflect clearly the criteria to be employed for measuring the benefits following out of each course of action and providing an accurate measurement of the organization's objective. The method may be so devised' as to ensure the selection of the best alternative out of a large number of alternative available to the organization.

Linear Programming is the analysis of problems in which a linear function of a number of variables is to be optimized (maximized or minimized) when whose variables are subject to a number of constraints in the mathematical inequalities.

From the above definitions, it is clear that:

Linear Programming is an optimization technique, where the underlying objective is either to *maximize the profits* or to *minimize the Cost*.

It deals with the problem of allocation of *finite limited resources* amongst different competing activities in the most optimal manner.

- iii. It generates solutions based on the feature and characteristics of the actual problem or situation.

Linear Programming has been highly successful in solving the following Types of problems :

- Product-mix problems.
- Investment planning problems.
- Blending strategy formulations and
- Marketing & Distribution management.

Even though Linear Programming has wide & diverse' applications, yet all LP problems have the following properties in common:

- The objective is always the same (i.e.; profit Maximization or cost minimization).
- Presence of constraints which limit the extent to which the objective can be achieved.
- Availability of alternatives i.e.; different courses of action to choose from,
- The objectives and constraints can be expressed in the form of linear relation.

Regardless of the size or complexity, all LP problems take the same form i.e. allocating scarce resources among various competing alternatives. Irrespective of the manner in which one defines Linear Programming, a problem must have certain basic characteristics before this technique can be utilized to find the optimal values.

2.2. The characteristics or the basic assumptions of linear programming

1. **Decision or Activity Variables & Their Inter-Relationship.** The decision or variables refer to any activities which are in competition with other variables for limited resources. Examples of such activity variables are: services, projects, products etc. These variables are most often inter-related in terms of utilization of the scarce resources and need simultaneous solutions. It is important to ensure that the relationship between these variables be linear.

2. **Finite Objective Functions.** A Linear Programming problem requires a clearly defined, unambiguous objective function which is to be optimized. It should be capable of being

expressed as a linear function of the decision variables. The single-objective optimization is one of the most important prerequisites of linear programming. Examples of such objectives can be: cost-minimization, sales, profits or revenue Maximization & the idle-time minimization etc

Limited Factors/Constraints. These are the different kinds of limitations on the available resources e.g. important resources like availability of machines, number of man hours available, production capacity and number of available markets or consumers for finished goods are often limited even for a big organization. Hence, it is rightly said that each and every organization function within overall constraints both internal and external. These limiting factors must be capable of being expressed as linear equations or in equations in terms of decision variables.

Presence of Different Alternatives. Different courses of action or alternatives should be available to a decision maker, who is required to make the decision which is the most effective or the optimal. For example, many grades of raw material may be available, the same raw material can be purchased from different supplier, the finished goods can be sold to various markets.

Non-Negativity Restrictions. Since the negative values of (any) physical quantity has no meaning, therefore all the variables must assume non-negative values. If some of the variables are unrestricted in sign, the non-negativity restriction can be enforced by the help of certain mathematical tools - without altering the original information contained in the problem.

Linearity Criterion. The relationship among the various decision variables must be directly proportional. Both the objective and the constraint, must be expressed in terms of linear equations or inequalities. For example. If one of the factor inputs (resources like material, labor, plant capacity etc.) increases, then it should result in a proportionate manner in the final output. These linear equations can graphically be presented as a straight line.

resource utilized would be exactly equal to the sum of the respective individual amounts. Thus the function or the activities must be additive - and the interaction among the activities of the resources does not exist.

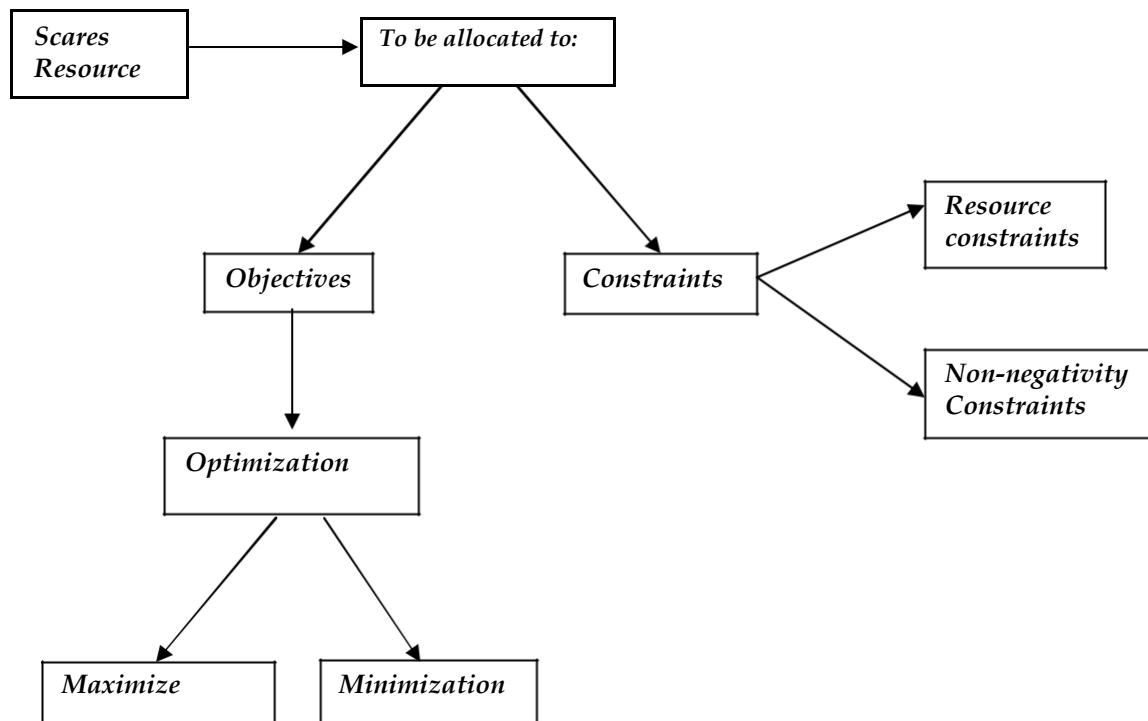
Mutually Exclusive Criterion. All decision parameters and the variables are assumed to be mutually exclusive. In other words, the occurrence of anyone variable rules out the simultaneous occurrence of other such variables.

Divisibility. Variables may be assigned fractional values. i.e.; they need not necessarily always be in whole number. If a fraction of a product cannot be produced, an integer programming problem exists. Thus, the continuous values of the decision variables and resources must be permissible in obtaining an optimal solution.

Certainty. It is assumed that conditions of certainty exist i.e.; all the relevant parameters or coefficients in the Linear Programming model are fully and completely known and that they don't change during the period. However, such an assumption may not hold good at all times.

Finiteness. LP assumes the presence of a finite number of activities and constraints without which it is not possible to obtain the best or the optimal solution.

Diagrammatically,



2.3. Advantages & Limitations of Linear Programming

Advantages of LP; The following are some of the advantages of LP approach:

- 1. Scientific Approach to Problem Solving.** Linear Programming is the application of scientific approach to problem solving. Hence it results in a better and true picture of the problems-which can then be minutely analyzed and solutions ascertained.
- 2. Evaluation of All Possible Alternatives.** Most of the problems faced by the present organizations are highly complicated - which cannot be solved by the traditional approach to decision making. The technique of Linear Programming ensures that all possible solutions are generated - out of which the optimal solution can be selected.
- 3. Helps in Re-Evaluation.** Linear Programming can also be used in re- evaluation of a basic plan for changing conditions. Should the conditions change while the plan is carried out only partially, these conditions can be accurately determined with the help of Linear

☐ ☒ ☐

those situations where a given problem can clearly be represented in the form of linear relationship between different decision variables. Hence it is based on the implicit assumption that the objective as well as all the constraints or the limiting factors can be stated in term of linear expressions - which may not always hold good in real life situations. In practical business problems, many objective function & constraints cannot be expressed linearly. Most of the business problems can be expressed quite easily in the form of a quadratic equation (having a power 2) rather than in the terms of linear equation.

- 2. Constant Value of objective & Constraint Equations.** Before a LP technique could be applied to a given situation, the values or the coefficients of the objective function as well as the constraint equations must be completely known. Further, Linear Programming assumes these values to be constant over a period of time. However, in real life practical situations often it is not possible to determine the coefficients of objective function and the constraints equations with absolute certainty. These variables in fact may, lie on a probability distribution curve and hence at best, only the likely hood of their occurrence can be predicted. Moreover, often the value's change due to external as well as internal factors during the period of study. Due to this, the actual applicability of LP tools may be restricted.

No Scope for Fractional Value Solutions. There is absolutely no certainty that the solution to a LP problem can always be quantified as an integer quite often, Linear Programming may give fractional-varied answers, which are rounded off to the next integer. Hence, the solution wouldn't be the optimal one. For example, in finding out the number of men and machines required to perform a particular job, a fractional Larson-integer solution would be meaningless.

Degree of Complexity. Many large-scale real life practical problems can't be solved by employing Linear Programming techniques even with the help of a computer due to highly complex and Lengthy calculations. Assumptions and approximations are required to be made so that the given problem can be broken down

into several smaller problems and then, solve separately.

Multiplicity of Goals. The long-term objectives of an organization are not confined to a single goal. An organization, at any point of time in its operations has a multiplicity of goals or the goals hierarchy - all of which must be attained on a priority wise basis for its long term growth. Some of the common goals can be Profit Maximize or cost minimization, retaining market share, maintaining leadership position and providing quality service to the consumer. In cases where the management has conflicting, multiple goals, Linear Programming model fails to provide an optimal solution. The reason being that under Linear Programming techniques, there is only one goal which can be expressed in the objective function.

2.4. Main Application Areas of Linear Programming

In the last few decades since 1960s, no other mathematical tool or technique has had such a profound impact on the management's decision making criterion as Linear Programming well and truly it is one of the most important decision making tools of the last century which has transformed the way in which decisions are made and businesses are conducted.

Some of the important application areas of Linear Programming are the following:

Military Applications. Paradoxically the most appropriate example of an organization is the military and worldwide, Second World War is considered to be one of the best managed or organized events in the history of the mankind. Linear Programming is extensively used in military operations. Such applications include the problem of selecting an air weapon system against the enemy so as to keep them pinned down and at the same time minimize the amount of fuel used. Other examples are dropping of bombs on pre-identified targets from aircraft and military attacks against localized terrorist outfits.

Agriculture. Agriculture applications fall into two broad categories, farm economics and farm management. The former deals with the agricultural

economy of a nation or a region, while the latter is concerned with the problems of the individual form. Linear Programming can be gainfully utilized for agricultural planning e.g. allocating scarce limited resources such as capital, factors of production like labor, raw material etc. in such a way so as to maximize the net revenue.

Environmental Protection. Linear programming is used to evaluate the various possible alternative for handling wastes and hazardous materials so as to satisfy the stringent provisions laid down by the countries for environmental protection. This technique also finds applications in the analysis of alternative sources of energy paper recycling and air cleaner designs.

Facilities Location. Facilities location refers to the location nonpublic health care facilities (hospitals, primary health centers) and public recreational facilities (parks, community halls) and other important facilities pertaining to infrastructure such as telecommunication booths etc. The analysis of facilities location can easily be done with the help of Linear Programming. Apart from these applications, LP can also be used to plan for public expenditure and drug control.

Product-Mix. The product-mix of a company is the existence of various products that the company can produce and sell. However, each product in the mix requires finite amount of limited resources. Hence it is vital to determine accurately the quantity of each product to be produced knowing their profit margins and the inputs required for producing them. The primary objective is to maximize the profits of the firm subject to the limiting factors within which it has to operate.

Production. A manufacturing company is quite often faced with the situation where it can manufacture several products (in different quantities) with the use of several different machines. The problem in such a situation is to decide which course of action will maximize output and minimize the costs.

Another application area of Linear Programming in production is the assembly by-line balancing, where a component or an item can be manufactured by assembling different parts. In such situations, the objective of a Linear Programming model is to set the

assembly process in the optimal (best possible) sequence so that the total elapsed time could be minimized.

Mixing or Blending. Such problems arise when the same product can be produced with the help of a different variety of available raw-materials each having a fixed composition and cost. Here the objective is to determine the minimum cost blend or mix (i.e. the cost minimizations) and the various constraints that operate are the availability of raw materials and restrictions on some of the product constituents.

Transportation & Trans-Shipments. Linear Programming models are employed to determine the optimal distribution system i.e. the best possible channels of distribution available to an organization for its finished product at minimum total cost of transportation or shipping from company's go down to the respective markets. Sometimes the products are not transported as finished products but are required to be manufactured at various sources. In such a situation, Linear Programming helps in ascertaining the minimum cost of producing or manufacturing at the source and shipping it from there.

Portfolio Selection. Selection of desired and specific investments out of a large number of investment options available for the managers (in the form of financial institutions such as banks, non-financial institutions such as mutual funds, insurance companies and investment services etc.) it is a very difficult task, since it requires careful evaluation of all the existing options before arriving at decision. The objective of Linear Programming, in such cases, is to find out the allocation/which maximizes the total expected return or minimizes the total risk under different situations.

Profit Planning & Control. Linear Programming is also quite useful in profit planning and control. The objective is to maximize the profit margin from investment in the plant facilities and machinery.

Traveling Salesmen Problem. Traveling salesmen problem refers to the problem of a salesman to find the shortest route originating from a particular city, visiting each of the specified cities and then returning back to the originating point of departure. The restriction being that no city must be visited more than once during a particular tour. Such types of problems can quite easily be solved with the help of Linear Programming.

Media Selection/Evaluation. Media selection means the selection of the optimal media-mix so as to maximize the effective exposure. The various constraints in this case are: Budget limitation, different rates for different media (i.e.; print media, electronic media like radio and T.V. etc.) and the minimum number of repeated advertisements in the various media. The use of Linear Programming facilities like the decision making process.

Staffing. Staffing or the man-power costs are substantial for a typical organization which make its products or services very costly. Linear Programming techniques help in allocating the optimum employees (man-power or man-hours) to the job at hand. The overall objective is to minimize the total man-power or overtime costs.

Job Analysis. Linear Programming is frequently used for evaluation of jobs in an organization and also for matching the right job with the right worker.

Wages and Salary Administration. Determination of equitable salaries and various incentives and perks becomes easier with the application of LP. LP tools can also be utilized to provide optimal solutions in other areas of personnel management such as training and development and recruitment etc.

2.5. Linear Programming problem Formation

Linear programming is one of the most useful techniques for effective decision making. It is an optimization approach with an emphasis on providing the optimal solution for resource allocation. How best to allocate the scarce organizational or national resources among different competing and conflicting needs (or uses) forms the core of its working. The scope for application of linear programming is very wide and it occupies a central place in many diversified decisional problems. The effective use and application of linear programming requires the formulation of a realistic model which represents accurately the objectives of the decision making subject to the constraints in which it is required to be made.

The basic steps in formulating a linear programming model are presented as follows:

Step I Identification of the decision variables. The decision variables (parameters) having a bearing on the decision at hand shall first be identified, and then expressed or determined in the form of linear algebraic functions or in equations.

Express each constraint in words. For this first see whether the constraint is of the form $\geq$ (at least, as large as), or of the form $\leq$ (no larger than) or $=$ (exactly equal)

Then express the objective function verbally.

Step (a) and (b) should then allow you to verbally identify the decision variables.

Step II Identification of the constraints. All the constraints in the given problem which restrict the operation of a firm at a given point of time must be identified in this stage. Further these constraints should be broken down as linear functions in terms of the pre-defined decision variables.

Step III Formulate the objective function. In the last stage, the objective which is required to be optimized (i.e., maximized or memorized) must be clearly identified and expressed in terms of the pre-defined decision variables.

Example 1

3F furniture Ltd. manufactures two products, tables & chair. Both the products have to be processed through two machines M_1 & M_2 the total machine-hours available are: 200 hours of M_1 and 400 hours of M_2 respectively. Time in hours required for producing a chair and a table on both the machines is as follows:

Time in Hours:

Machine	Table	Chair
M_1	7	4
M_2	5	5

Profit from the Sale of table is Birr 40 and that of a chair is Birr 30.

Required: formulate the LP Problem (LPP)?

Solution:

Step I Identification of the decision variables. Let x_1 = Number of tables produced and

x_2 = Number of Chairs produced

Step II List down all the constraints.

a) Total time on machine M_1 cannot exceed 200 hour

$$\therefore 7x_1 + 4x_2 \leq 200$$

(Since it takes 7 hours to produce a table & 4 hours to produce a chair on machine M_1)

b) Total time on machine M_2 cannot exceed 400 hour

$$\therefore 5x_1 + 5x_2 \leq 400$$

(Since it takes 5 hours to produce both a table & a chair on machine M_2)

Step III. The objective function for maximizing the profit is given by

$$\text{Maximize } Z = 40x_1 + 30x_2 \text{ (objective function)}$$

(Since profit per unit from a table and a chair is Birr 40 & Birr. 30 respectively).

Presenting the problem as LPP, the given problem can now be formulated as a linear programming model as follows:

$$\text{Maximize } Z = 40x_1 + 30x_2$$

$$\text{Subject to: } 7x_1 + 4x_2 \leq 200$$

$$5x_1 + 5x_2 \leq 400$$

$$\text{Further; } x_1 \& x_2 \geq 0$$

(Since if $x_1 \& x_2 < 0$ it means that negative quantities of products are being manufactured which has no meaning).

Example 2

Alpha Limited produces & sells 2 different products under the brand name black & white. The profit per unit on these products is Birr 50 & Birr 40 respectively. Both black & white employ the same manufacturing process which has a fixed total capacity of 50,000 man-hours. As per the estimates of the marketing research department of Alpha Limited, there is a market demand for maximum 8,000 units of Black & 10,000 units of white. Subject to the overall demand, the products can be sold in any possible combination. If it takes 3 hours to produce one unit of black & 2 hours to produce one unit of white. Formulate the about as a linear programming model?

Solution:

Step I Identification of the decision variables. Let x_1 & x_2 denote the number of units produced, black & white brand respectively.

Step II: List down all the constraints.

a) Capacity or man-hours constraint:

$$3x_1 + 2x_2 \leq 50,000$$

(Since it takes 3 hours to produce 1 unit of x_1 & 2 hours to produce 1 unit of x_2 & the total available man – hours are 50,000).

b) Marketing constraints:

$$x_1 \leq 8,000$$

(Since maximum 8,000 units of x_1 can be sold)

$$x_2 \leq 10,000$$

(Since maximum 10,000 units of x_2 can be sold).

Step III. The objective function for maximizing the profit is given by

$$\text{Maximize } Z = 50x_1 + 40x_2 \text{ (objective function)}$$

Presenting the problem as LPP Now, the given problem can be written as a linear programming model in the follows:

$$\text{Maximize } Z = 50x_1 + 40x_2$$

$$\text{Subject: } 3x_1 + 2x_2 \leq 50,000$$

$$x_1 \leq 8,000$$

$$x_2 \leq 10,000$$

$$\text{Further; } x_1 \& x_2 \geq 0$$

(Since if $x_1, x_2 < 0$, it means that negative Quantities of products are being manufactured – which has no meaning)

Exercise

The management of ABC Chemicals is considering the optimal mix of two possible processes. The values of input & output for both these process are given as follows:

Process	Unit-input		Unit-output	
	I ₁	I ₂	O ₁	O ₂
X	2	6	3	7
Y	4	8	5	9

Maximum 500 units of Input I₁ and 300 units of I₂ are available to ABC Chemicals in the local market. The forecasted demand for outputs O₁ & O₂ are at least 5,000 units & 7,000 units respectively. The respective profits from process X & Y are Birr 1,000 & Birr 2,000 per production run.

Required: formulate the above problem as a linear programming model.

2.6. Method for solving LPP

There are two types of finding a solution for Linear programming problems.

Graphic solution and
Simplex method

2.7. Graphical Method of solving Linear Programming Problems

2.7.1. Introduction

Dear learners, during the previous section, you have seen how to formulate a given problem as a Linear Programming model. And you are also familiar with the method of solving LP models by using graphics method (see module of mathematics for management mgt221). So in this module we try to remind you some points of graphic solution method.

The next step, after the formulation of LP model is to devise effective methods to solve the model and ascertain the optimal solution.

Dear learners, we start with the graphical method of solving LPP and now let we move on to simplex algorithm for solving the Linear Programming model.

The graphic solution procedure is one of the methods of solving two variable linear programming problems.

2.7.2. Steps in graphic solution method:-

Step I *Defining the problem.* Formulate the problem mathematically. Express it in terms of several mathematical constraints & an objective function. The objective function relates to the optimization aspect, i.e. Maximization or minimization Criterion.

Step II *Plot the constraints Graphically.* Each inequality in the constraint equation has to be treated as an equation. An arbitrary value is assigned to one variable & the value of the other variable is obtained by solving the equation. In the similar manner, a different

arbitrary value is again assigned to the variable & the corresponding value of other variable is easily obtained. These 2 sets of values are now plotted on a graph and connected by a straight line. The same procedure has to be repeated for all the constraints. Hence, the total straight lines would be equal to the total no of equations, each straight line representing one constraint equation.

Step III *Locate the solution space.* Solution space or the feasible region is the graphical area which satisfies all the constraints at the same time. Such a solution point (x, y) always occurs at the corner Points of the feasible Region the feasible region is determined as follows:

For "greater than" & "greater than or equal to" constraints, the feasible region or the solution space is the area that lays above the constraint lines.

For "Less Than" & "Less than or equal to" constraint, the feasible region or the solution space is the area that lays below the constraint lines.

Step IV Selecting the graphic solution technique. Select the appropriate graphic technique to be used for generating the solution. There are two graphic techniques to find solution;

Corner Point Method and

Iso-profit (or Iso-cost) method may be used; however, it is easier to generate solution by using the corner point method and we use this method to find solution.

2.7.3. Corner Point Method to find Solution for Graphics Method

Since the solution point (x, y) always occurs at the corner point of the feasible or solution space, identify each of the extreme points or corner points of the feasible region by the method of simultaneous equations.

By putting the value of the corner point's co-ordinates [e.g. (2, 3)] into the objective function, calculate the profit (or the cost) at each of the corner points.

In a Maximize problem, the optimal solution occurs at that corner point which gives the highest profit. In a minimization problem, the optimal solution occurs at that corner point which gives the lowest profit.

Dear learners, let us now turn our attention to the important theorems which are used in solving a linear programming problem.

Important Theorems in solving LPP

On the process of determining an optimum feasible solution to the linear programming problem, the following four important theorems are used:-

Theorem I the feasible solution space constitute a convex set.

Theorem II within the feasible solution space, feasible solution correspond to the extreme (or Corner) points of the feasible solution space.

Theorem III There is a finite number of basic feasible solutions, with the feasible solution space.

Theorem IV The optimum feasible solution, if it exists, will occur at one, or more, of the extreme points that are basic feasible solutions.

Note. Convex set is a polygon "Convex" implies that if any two points of the polygon are selected arbitrarily then straight line segment joining these two points lays completely within the polygon. The extreme points of the convex set are the basic solution to the linear programming problem.

Important Terms

Some of the important terms commonly used in linear programming are disclosed as follows:

Solution: Values of the decision variable $x_i (i=1,2,3,...)$ satisfying the constraints of a general linear programming model is known as the solution to that linear programming model.

Feasible solution: Out of the total available solution a solution that also satisfies the non-negativity restrictions of the linear programming problem is

Basic solution: For a set of simultaneous equations in Q unknowns (P, Q) a solution obtained by setting ($P - Q$) of the variables equal to zero & solving the remaining P equation in P unknowns is known as a basic solution. The variables which take zero values at any solution are detained as non-basic variables & remaining are known as basic variables, often called basic solution.

Basic feasible solution: A feasible solution to a general linear programming problem which is also basic solution is called a basic feasible solution.

Optimal feasible solution: Any basic feasible solution which optimizes (i.e. maximize or minimizes) the objective function of a LP models is known as the optimal feasible solution to that linear programming model.

Degenerate Solution: A basic solution to the system of equations is termed as degenerate if one or more of the basic variables become equal to zero.

Example 1

XYZ Ltd. Co. Wishes to purchase a maximum of 3600 units of two types of product, A & B are available in the market. Product A occupies a space of 3 cubic feet & cost Birr 9 whereas B occupies a space of 1 cubic feet & cost Birr 13 per unit. The budgetary constraints of the company do not allow spending more than Birr 39,000. The total availability of space in the company's god own is 6000 cubic feet. Profit margin of both the product A & B is Birr 3 & Birr 4 respectively. Formulate the above problem as a linear programming model and solve it by using graphical method. You are required to ascertain the best possible combination of purchase of A & B so that the total profits are maximized.

Solution

(a) The problem can be formulated as a linear programming model as follows:-

Let x_1 = Number of units of product A &

x_2 = Number of units of product B

Objective function, $\text{Maximize } Z = 3x_1 + 4x_2$

Subject to: -

$$x_1 + x_2 \leq 3600 \text{ (Maximum unites constraint)}$$

$$3x_1 + x_2 \leq 6000 \text{ (Storage area constraint)}$$

$$9x_1 + 13x_2 \leq 39000 \text{ (Budgetary constraint)}$$

$$x_1 \& x_2 \geq 0$$

Find the solution by using graphics method Step I

Treating all the constraints as equality,

The 1st constraint is: $x_1 + x_2 \leq 3600$ change this in to $x_1 + x_2 = 3600$ and find the X and Y intercept as follows.

$$\text{Put } x_1 = 0 \Rightarrow x_2 = 3600 \quad \therefore \text{the point is } (0, 3600)$$

$$\text{Put } x_2 = 0 \Rightarrow x_1 = 3600 \therefore \text{the point is } (3600, 0)$$

Draw the graph with x_1 on x-axis & x_2 on y-axis as shown in the following figure.

—
The 2nd constraint is: $3x_1 + x_2 \leq 6000$ change this in to $3x_1 + x_2 = 6000$ and find X and Y intercept.

$$\text{Put } x_1 = 0 \Rightarrow x_2 = 6000 \quad \therefore \text{the point is } (0, 6000)$$

$$\text{Put } x_2 = 0 \Rightarrow x_1 = 2000 \therefore \text{the point is } (2000, 0)$$

Draw the graph with x_1 on x-axis & x_2 on y-axis as shown in the following figure.

The 3rd constraint is: $9x_1 + 13x_2 \leq 39,000$ change this in to and $9x_1 + 13x_2 = 39,000$
find the X and Y intercept as follows.

Put $x_1 = 0 \Rightarrow x_2 = 3000$ $\therefore$ the point is (0, 3000)

Put $x_2 = 0 \Rightarrow x_1 = 4333.3$ the point is (4333.3, 0)

Draw the graph with x_1 on x-axis & x_2 on y-axis as shown in the following figure.

Step II Determine the set of the points which satisfy the constraint:

$$x_1 + x_2 \leq 3600$$

This can easily be done by verifying whether the origin (0, 0) satisfies the constraint.
Here, $0 + 0 \leq 3600$ hence the solution is true; all the points below the line will satisfy the constraint.

Like it's in the above iteration, determine the set of points which satisfy the constraint $3x_1 + x_2 \leq 6000$. At origin;

$0 + 0 \leq 6000$. Hence, all the points below the line will satisfy the constraint.

Like it's in the above iteration, determine the set of points which satisfy the constraint.
 $9x_1 + 13x_2 \leq 39,000$. At origin;

$0 + 0 \leq 39,000$. Hence, all the points below the line will satisfy the constraint.

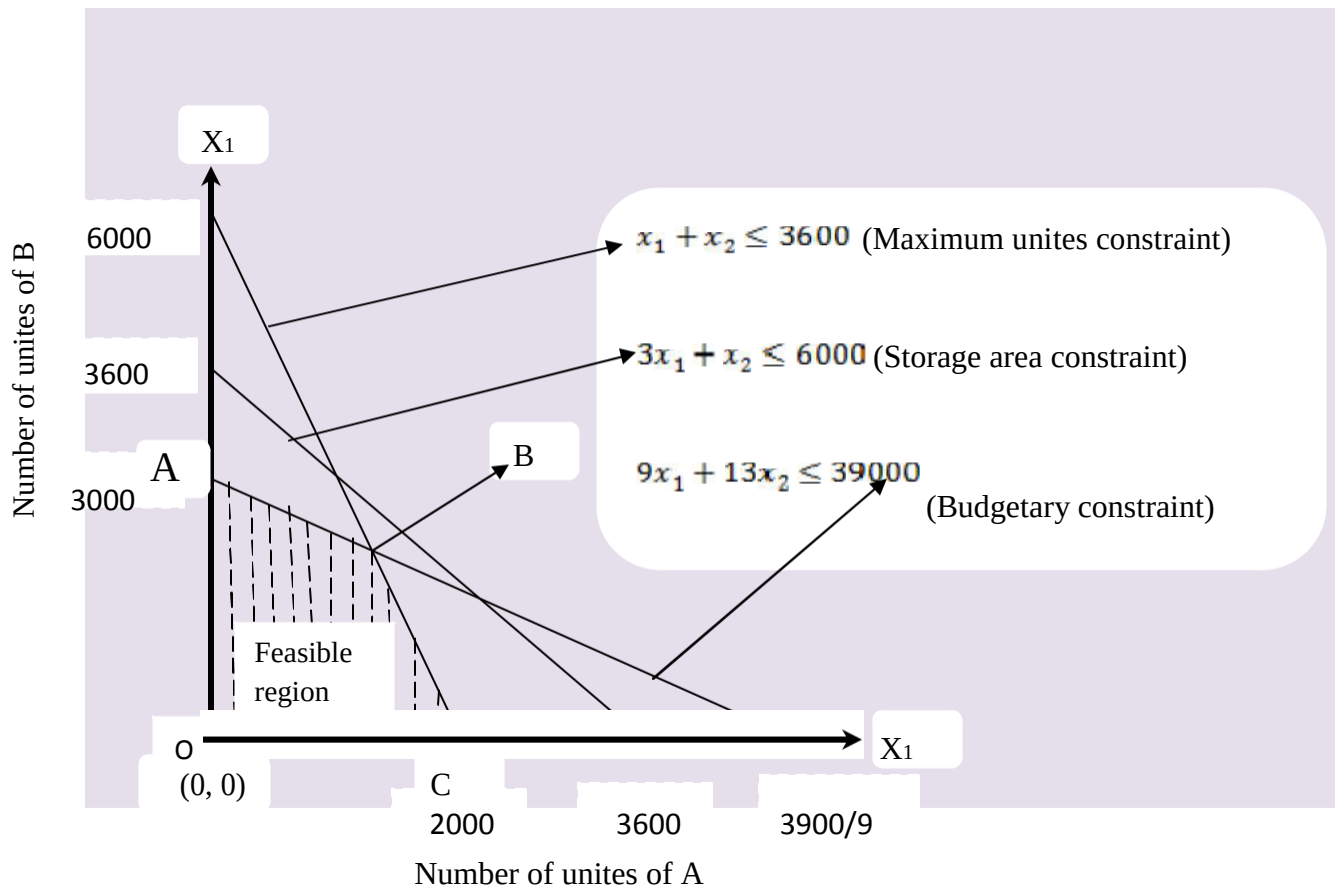
The intersection of the above graphic denotes the feasible region for the given problem.

Step III Finding feasible region by plotting the graph

Always keep in mind two things: -

For $\geq$ constraint the feasible region will be the area, which lays above the constraint lines, and for $\leq$ constraints, it will lay below the constraint lines. This would be useful in identifying the feasible region.

According to a theorem on linear programming, an optimal solution to a problem (if it exists) is found at a corner point of the solution space.



Step IV. Find the optimal solution by the corner point method.

At corner points (O, A, B, C), find the profit value from the objective function. Those points which maximize the profit are the optimal point.

	Corner point	Coordinates	Objective function	Value
solution	O	(0, 0)	$Maximize Z = 3x_1 + 4x_2$	
	A	(0, 3000)		
	B			
	C	(2000, 0)		

	O	(0,0)	$Z=0+0$	0
	A	(0,3000)	$Z=0+4 \times 3000$	12,000
→	B	(1300,2100)	$Z=3 \times 1300 + 4 \times 2100$	12,300
	C	(2000,0)	$3 \times 2000 + 0$	6000

In order to get the value of point B apply simultaneous equation by taking the two intersection lines. Solve the point.

$3x_1 + x_2 = 6000$ (Storage area constraint)

$9x_1 + 13x_2 = 39000$ (Budgetary constraint)
 ($\therefore$ A+B these two lines are intersecting)

$$3x_1 + x_2 = 6000 \quad \dots 1$$

$$9x_1 + 13x_2 = 39000 \quad \dots 2$$

Multiply equation (1) by -3 on both sides:

$$-9x_1 - 3x_2 = -18000 \quad \dots 3$$

$$9x_1 + 13x_2 = 39000 \quad \dots 4$$

$$10x_2 = 21,000 \quad \therefore x_2 = 2100$$

Put the Value of x_2 in first equation:
 $\Rightarrow x_1 = 1300$

At point (1300, 2100) $Maximize Z = 3x_1 + 4x_2$

$$Z = 3 \times 1300 + 4 \times 2100$$

= 12,300 which is the maximum value

Result

The optimal solution is:

No of units of product A=1300

No of units of product B=2100

Total profit, = 12, 300 which is the maximum

Example 2:

Suppose that a machine shop has two different types of machines; machine 1 and machine 2, which can be used to make a single product. These machines vary in the amount of product produced per hour, in the amount of labor used and in the cost of operation.

Assume that at least a certain amount of product must be produced and that we would like to utilize at least the regular labor force. How much should we utilize each machine in order to utilize total costs and still meet the requirement? The resource used, the cost and the required hour is given in the following table.

	Resource used		
	Machine 1 (X_1)	Machine 2 (X_2)	Minimum required hours
Product produced/hr	20	15	100
Labor/hr	2	3	15
Operation Cost	Birr25	Birr30	

1. Formulation of the linear programming model

$$\begin{array}{l}
 \text{Min. } Z = 25 X_1 + 30 X_2 \\
 \text{Subject to :} \\
 20 X_1 + 15 X_2 \geq 100 \\
 2 X_1 + 3 X_2 \geq 15 \\
 X_1, X_2 \geq 0
 \end{array}
 \left. \vphantom{\begin{array}{l} \text{Min. } Z = 25 X_1 + 30 X_2 \\ \text{Subject to :} \\ 20 X_1 + 15 X_2 \geq 100 \\ 2 X_1 + 3 X_2 \geq 15 \\ X_1, X_2 \geq 0 \end{array}} \right\} \text{LPP Model}$$

The 1st constraint is: $20x_1 + 15x_2 \geq 100$ change this in to $20x_1 + 15x_2 = 100$ and find X and Y intercept.

Put $x_1 = 0 \Rightarrow x_2 = \frac{20}{3} \therefore$ the point is $(0, 20/3)$

Put $x_2 = 0 \Rightarrow x_1 = 5 \therefore$ the point is $(5, 0)$

Draw the graph with x_1 on x-axis & x_2 on y-axis as shown in the following figure.

The 2nd constraint is $2X_1 + 3X_2 \geq 15$ change this in to $2X_1 + 3X_2 = 15$ and find X and Y intercept.

Put $x_1 = 0 \Rightarrow x_2 = 5 \therefore$ the point is $(0, 5)$

Put $x_2=0 \Rightarrow x_1=7.5$. the point is (7.5,0)

Draw the graph with x_1 on x-axis & x_2 on y-axis as shown in the following figure.

Determine the set of the points which satisfy the constraint:

$$20x_1 + 15x_2 \geq 100$$

This can easily be done by verifying whether the origin (0, 0) satisfies the constraint.

Here,

$0 + 0 > 100$ Hence the solution is false; all the points above the line will satisfy the constraint.

Like it's in the above iteration, determine the set of points which satisfy the constraint

$$2X_1 + 3X_2 \geq 15 \text{ at origin;}$$

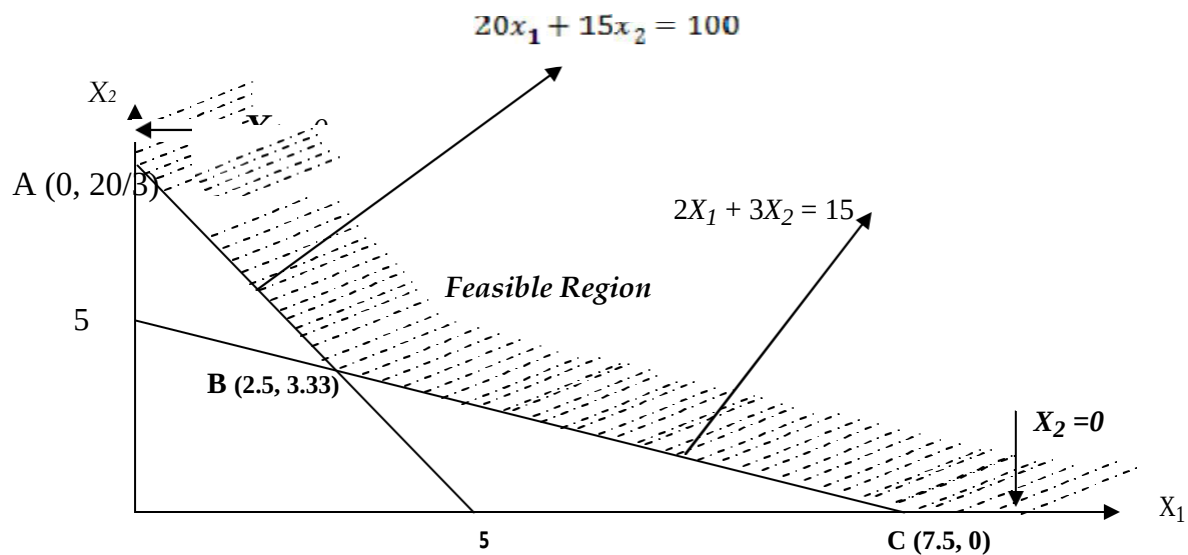
$0+0>1$. Hence, all the points above the line will satisfy the constraint.

Finding feasible region by plotting the graph

Always keep in mind two things: -

For $\geq$ constraint the feasible region will be the area, which lays above the constraint lines, and for $\leq$ constraints, it will lay below the constraint lines. This would be useful in identifying the feasible region.

According to a theorem on linear programming, an optimal solution to a problem (if it exists) is found at a corner point of the solution space.



Corner point	Coordinates	Objective function	Value
<i>Minimize</i> . $Z = 25 X_1 + 30 X_2$			
A	(0,20/3)	$Z=0+20/3 \times 30$	200
B	(2.5, 3.33)	$Z=2.5 \times 25 + 3.33 \times 30$	162.5
C	(7.5, 0)	$Z=7.5 \times 25 + 0$	187.5

Feasible solution

Use the same stapes to find the value of B by using the simultaneous equation as presented previously.

Result

The optimal solution is:

$X_1 = 2.5$, $X_2 = 3.33$ and Minimum cost $Z = 162.5$

2.7.3. Special Cases in Graphics Methods

The following are the major special cases in graphics solution

Alternative Optima

Infeasible Solution

Unbounded solution

Alternative Optimal solution

When the objective function is parallel to a binding constraint; (a constraint that is satisfied in the equality sense by the optimal solution), the objective function will assume the same optimal value at more than one solution point, for this reason they are called **alternative optima**. Example 1 shows that normally there is infinity of such solutions. The example also demonstrates the practical significance of encountering alternative optima.

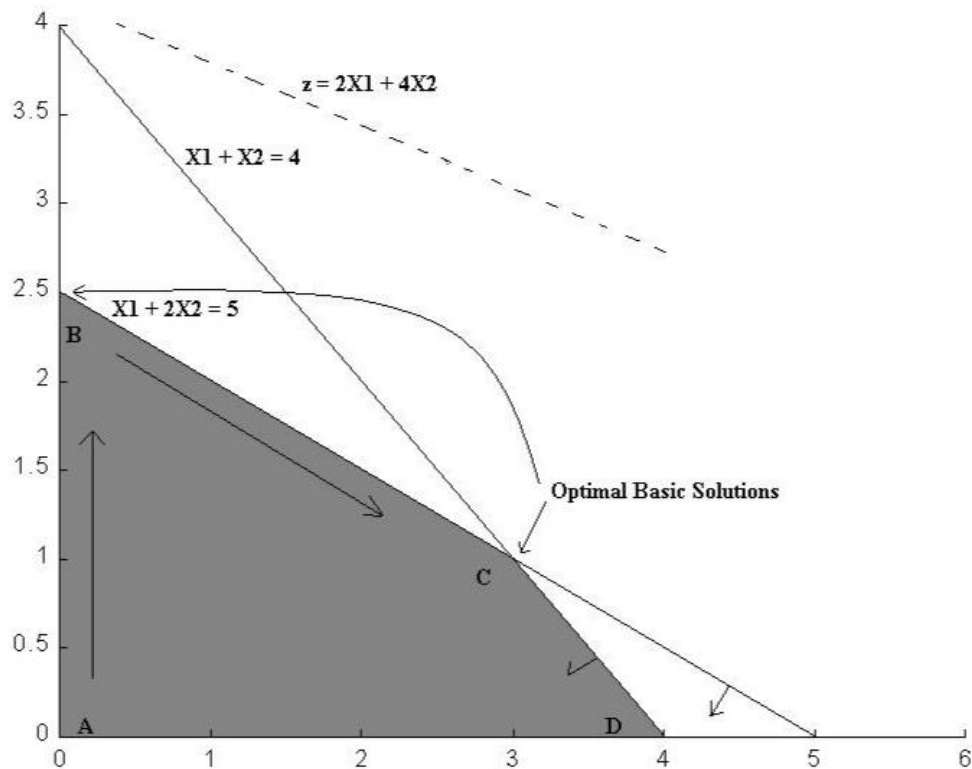
Example1

$$\text{Maximize } z = 2x_1 + 4x_2$$

Subject to:

$$x_1 + 2x_2 \leq 5$$

$$x_1 + x_2 \leq 4 \text{ and } x_1, x_2 \geq 0$$



The above Figure demonstrates how alternative optima can arise in LP model when the **objective function is parallel to a binding constraint**. Any point on the line segment BC represents an alternative optimum with the same objective value $z = 10$. Mathematically, we can determine all the points (x_1, x_2) on the line segment BC as a nonnegative weighted average of the points B and C. That is, given $0 \leq \alpha \leq 1$ and

$$x_1 = 0, x_2 = 5/2$$

$$x_1 = 3, x_2 = 1$$

Then all points on the line segment BC are given by

$$x_1 = \alpha(0) + 3(1-\alpha) = 3 - 3\alpha$$

$$x_2 = \alpha(5/2) + 1(1-\alpha) = 1 + 3\alpha/2$$

Observe that when $\alpha=0$, $(x_1, x_2) = (3, 1)$, which is point C. When $\alpha=1$, $(x_1, x_2) = (0, 5/2)$, which is point B. For values of α between 0 and 1 (x_1, x_2) lays between B and C

Example 2:

The information given below is for the products A and B.

<i>Department</i>	<i>Machine hours per week</i>		<i>Maximum available</i>
	<i>Product A</i>	<i>Product B</i>	<i>per week</i>
Cutting	3	6	900
Assembly	1	1	200
<i>Profit per unit</i>	<u><u><i>Birr 8</i></u></u>	<u><u><i>Birr 16</i></u></u>	

Assume that the company has a marketing constraint on selling products B and therefore it can sale a maximum of 125 units of this product.

Required:

- Formulate the LPP of this problem?
- Find the optimal solution?

Solution: Let X_1 =The Number of units' f product A produced per week

X_2 = The number of units f product B produced per week

$$\text{Max. } Z = 8X_1 + 16X_2$$

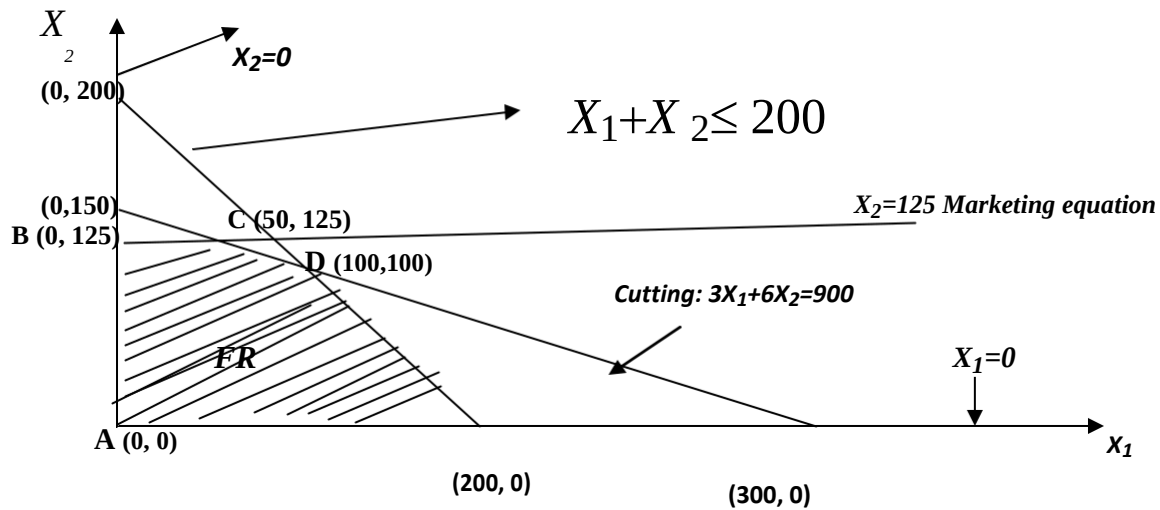
Subject to :

$$3X_1 + 6X_2 \leq 900$$

$$X_1 + X_2 \leq 200$$

$$X_2 \leq 125$$

$$X_1, X_2 \geq 0$$



<u>Corners</u>	<u>Coordinates</u>	<u>Maximize $Z = 8X_1 + 16X_2$</u>
A	(0, 0)	0
B	(0, 125)	2000
C	(50, 125)	2400
D	(100, 100)	2400
E	(200, 100)	1600

Interpretation:

Both **C** and **D** are optimal solutions. Any point on the line segment **CD** will also lead to the same optimal solution.

==>Multiple optimal solutions provide more choices for management to reach their objectives.

2. Infeasible Solution

A solution is called feasible if it satisfies all the constraints and the constraints and non-negativity condition. However, it is sometimes possible that the constraints may be inconsistent so that there is no feasible solution to the problem. Such a situation is called **infeasibility**.

Example:

$$\text{Maximize } Z = 20X_1 + 30X_2$$

Subject to:

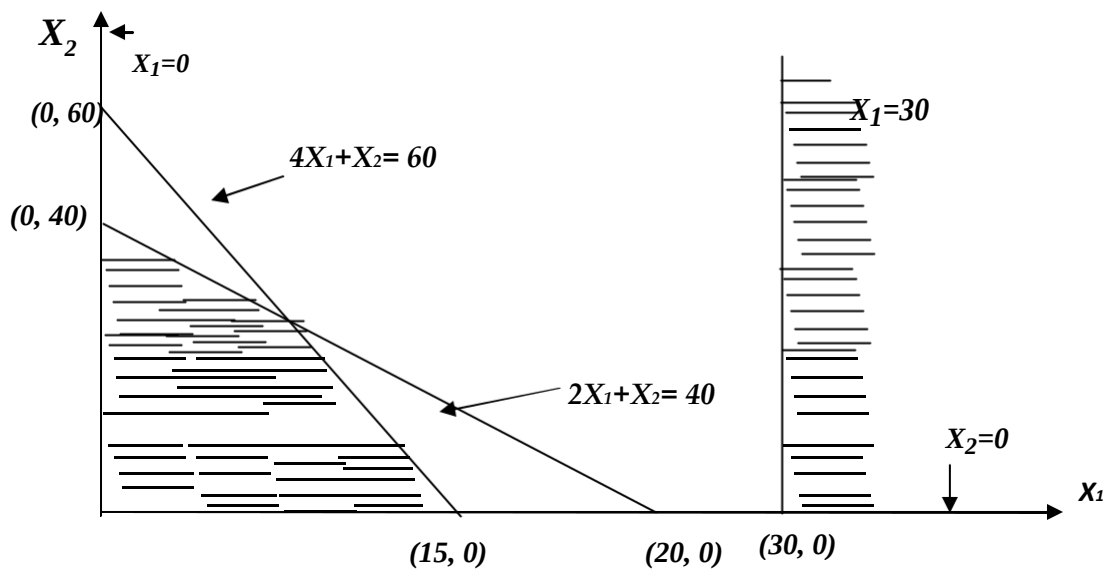
$$2X_1 + X_2 \leq 40$$

$$4X_1 + X_2 \leq 60$$

$$X_1 \geq 30$$

$$\text{and } X_1, X_2 \geq 0$$

Solution:



Note: -In the above graph, there is no common point in the shaded area.

-All constraints cannot be satisfied simultaneously and there is no feasible solution to the problem.

3. Unbounded Solution

When the value of decision variables in LP is permitted to increase infinitely without violating the feasibility condition, then the solution is said to be unbounded. Here, the objective function value can also be increased infinitely. However, an unbounded feasible region may yield some definite value of the objective function.

Example:

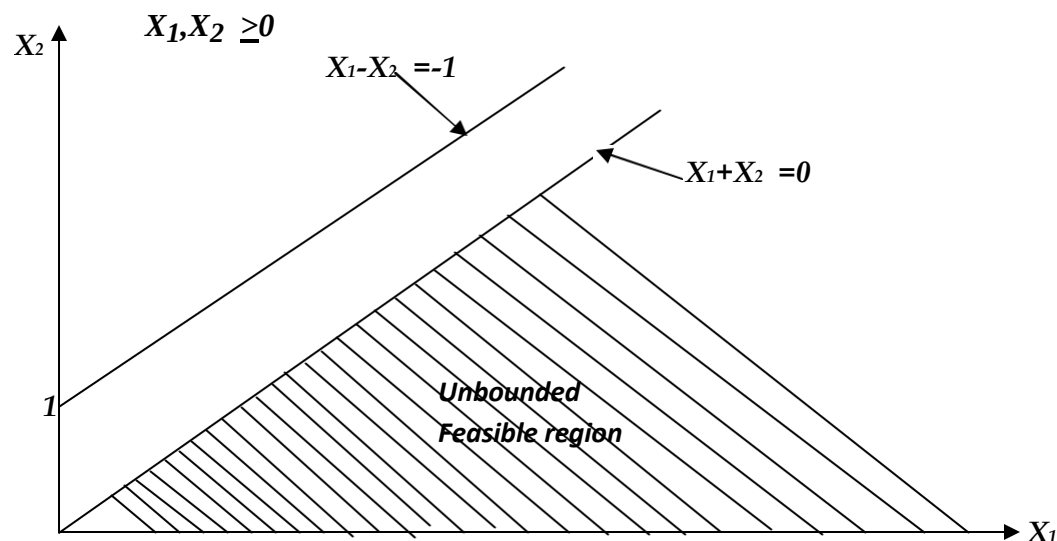
Use the graphical method to solve the following LPP.

$$\text{Maximize } Z = 3X_1 + 4X_2$$

Subject to:

$$X_1 - X_2 \leq -1 \Rightarrow -X_1 + X_2 \geq 1 \text{ since the quantity solution is positive}$$

$$-X_1 + X_2 \leq 0$$



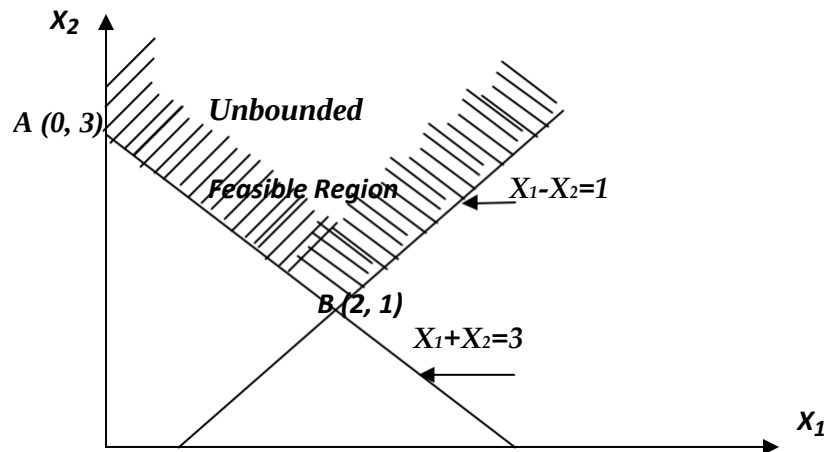
2. Maximize $Z=3X_1+2X_2$

Subject to:

$$X_1 - X_2 \leq 1$$

$$X_1 + X_2 \geq 3$$

$$X_1, X_2 \geq 0$$



Note: here that the two corners of the region are A(0,3) and B(2,1). The value of Maximize $Z(A)=6$ and Maximize $Z(B)=8$. But there exist number of points in the shaded region for which the value of the objective function is more than 8. For example, the point (10, 12) lies in the region and the function value at this point is 70 which is more than 8.

Remark: An unbounded solution does not mean that there is no solution to the given LPP, but implies that there exists an infinite number of solutions.

Exercise 1:

Use graphical method to solve the following LPP.

1. Maximize $Z=7/4X_1+3/2X_2$

2. Maximize $Z=X_1+X_2$

Subject to:

Subject to:

$$8X_1 + 5X_2 \leq 320$$

$$X_1 + X_2 \leq 1$$

$$4X_1 + 5X_2 \leq 20$$

$$-3X_1 + X_2 \geq 3$$

$$X_1 \geq 15$$

$$X_1, X_2 \geq 0$$

$$X_2 \geq 10$$

$$X_1, X_2 \geq 0$$

Answer: No feasible solution

Answer: Unbounded solution

3. Maximize $Z = 3X_1 + 2X_2$

4. Maximize $Z = 6X_1 - 4X_2$

Subject to:

Subject to:

$$X_1 - X_2 \leq 1$$

$$2X_1 + 4X_2 \leq 4$$

$$X_1 + X_2 \geq 3$$

$$4X_1 + 8X_2 \geq 16$$

$$X_1, X_2 \geq 2$$

$$X_1, X_2 \geq 0$$

Answer: Unbounded solution

Answer: Infeasible solution

Exercise2.

I. Solve the following LP problems using the graphical method.

1. Maximize $Z = 15X_1 - 10X_2$

2. Maximize $Z = 2X_1 + X_2$

Subject to:

Subject to:

$$4X_1 + 6X_2 \leq 360$$

$$X_1 + 2X_2 \leq 10$$

$$3X_1 + 0X_2 \leq 180$$

$$X_1 + X_2 \leq 6$$

$$0X_1 + 5X_2 \leq 280$$

$$X_1 - X_2 \leq 2$$

$$X_1, X_2 \geq 0$$

$$X_1 - 2X_2 \leq 1$$

Answer: $X_1 = 60, X_2 = 20$

$X_1, X_2 \geq 0$

and Maximize $Z = 1,100$

Answer: $X_1 = 4, X_2 = 2$

and Maximize $Z = 10$

3. Maximize .Z=10X₁+15X₂

Subject to:

$$2X_1 + X_2 \leq 26$$

$$2X_1 + 4X_2 \leq 56$$

$$-X_1 + X_2 \leq 5$$

$$X_1, X_2 \geq 0$$

Answer: X₁=4, X₂=2

Maximize Z=230

4.Min.Z=3X₁+2X₂

Subject to:

$$5X_1 + X_2 \geq 10$$

$$X_1 + X_2 \geq 6$$

$$X_1 + 4X_2 \geq 12$$

$$X_1, X_2 \geq 0$$

Answer: X₁=1, X₂=5

and Minimize Z=13

and

II. A manufacturer produces two different models; **X** and **Y**, of the same product. The raw materials **r₁** and **r₂** are required for production. At least 18 Kg of **r₁** and 12 Kg of **r₂** must be used daily. Almost at most 34 hours of labor are to be utilized. 2Kg of **r₁** are needed for each model **X** and 1Kg of **r₁** for each model **Y**. For each model of **X** and **Y**, 1Kg of **r₂** is required. It takes 3 hours to manufacture a model **X** and 2 hours to manufacture a model **Y**. The profit realized is \$50 per unit from model **X** and \$30 per unit from model **Y**. How many units of each model should be produced to maximize the profit?

Answer: 10 units of model X, 2 units of model Y and the maximum profit is \$ 560.

III. A manufacturing firm produces two machine parts **P₁** and **P₂** using milling and grinding machines. The different machining times required for each part, the machining times available on different machines and the profit on each machine part are as given below:

Machine	Manufacturing time		Maximum time	
	Required (min)		available per week (minimum)	
	P₁	P₂		

<i>Lathe</i>	<i>10</i>	<i>5</i>	<i>25,000</i>
<i>Milling Machine</i>	<i>4</i>	<i>10</i>	<i>2000</i>
<i>Grinding Machine</i>	<i><u>1</u></i>	<i><u>1.5</u></i>	<i>450</i>
<i>Profit per unit (\$)</i>	<i><u>\$50</u></i>	<i><u>\$100</u></i>	

Determine the number of pieces of P_1 and P_2 to be manufactured per week to maximize profit.

Answer: $X_1=187.5$, $X_2 =125$ and Maximize $Z=21,875$

2.8. Simplex Method

Dear learners, all of you have by now understand the graphical method of solving a linear programming model and the various special cases of graphic solution of linear programming model.

Let us see what the limitations of graphic solution are, and how can these be deal with?

Limitations of the Graphical Method

The applicability of the graphical method is very limited in scope. This is due to the fact that it is quite simple to identity all the corner points & then tests them for optimality-in the case of a two-variable problem. As a result, the graphical method cannot be always employed to solve the real-life practical Linear programming models which involve more than two decision-variables.

The above limitation of the graphical method is tackled by what is known as the simplex method. Developed in 1940 by George B-Dantizg, it remains a widely applicable method for solving complex LP problems. It can be applied to any LP problem which can be expressed

in terms of a Linear Objective function subject to a set of Linear Constraints. As such, no theoretical restrictions are placed on the number of decision variables or constraints contained in a LPP.

The Simplex method is an **iterative or “step by step”** method or repetitive algebraic approach that moves automatically from one basic feasible solution to another basic feasible solution improving the situation each time until the optimal solution is reached at.

Note: The Simplex method starts with a corner that is in the solution space or feasible region and moves to another corner, the solution space improving the value of the objective function each time until optimal solution is reached at the optimal corner.

2.8.1. Standard forms of Linear Programming (LP) problems

The use of the Simplex method to solve an LP problem requires that the problem be converted into its standard form. The standard form of the LP problem should have the following characteristics:

i. All the constraints should be expressed as equations by adding slack or surplus and/or artificial variables.

The right hand side of each constraint should be made non-negative; if it is not, this should be done by multiplying both side of the resulting constraint by -1.

The objective function should be of the maximization type.

The standard form of LP problem is expressed as follow;

Optimize (Maximize or Minimize) $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n + 0s_1 + 0s_2 + \dots + 0s_m$

Subject to the linear constraint:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + s_2 = b_2$$

$$\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + s_m = b_m$$

And $x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_m \geq 0$

Where $c = (c_1, c_2, \dots, c_n)$ the coefficient of the objective function; $x = (x_1, x_2, \dots, x_n)$ decision variables; $b = (b_1, b_2, \dots, b_n)$ the limited resource available and $s = (s_1, s_2, \dots, s_n)$ slack variables.

In matrix notation the standard form is expressed as:

Optimize (Maximize or Minimize) $Z = cx + 0s$

Subject to the linear constraint:

$$Ax + s = b$$

$$x, s \geq 0$$

Where $c = (c_1, c_2, \dots, c_n)$ is the row vector, $x = (x_1, x_2, \dots, x_n)_T$, $b = (b_1, b_2, \dots, b_n)_T$, and $s = (s_1, s_2, \dots, s_n)$ are column vector.

Hence, this method is used, which can solve LP problems with any number of variable or constraints it is geared towards solving optimization problems which have constraints of less than or equal to type.

And

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The above matrix is the $m \times n$ matrix of coefficient of variables $x_1, x_2, \dots, x_n$ in the constraints.

Remarks; three types of additional variables, namely

Slack variable (s)

Surplus variable (-s), and

Artificial variable (A)

Are added in the given LP problem to convert it in to the standard form for the following reasons:

These variables allow as converting inequalities in to equalities, there by converting the LP problem in to a form that is amenable to algebraic solution.

These variables permit as to make a more comprehensive economic interpretation of a final solution.

Help us to get an initial feasible solution presented by the column of the identity matrix.

Types of constraint	Extra variable needed	Coefficient of extra variables In the objective function		Presence of extra variable in initial solution mix
		Max Z	Min Z	
Less than or equal to $\leq$	A slack variable is added	0	0	Yes
Greater than or	A surplus variable is	0	0	No

Operations Research

equal to ($=$)	artificial variable is added	-M	+M	Yes
Equal to ($=$)	Only an artificial variable is added	-M	+M	Yes

The summary of the extra variable to be added in the given LP problem to convert it in to standard form is given in the following table.

Remark: a *slack variable* represents unused resource, either in the form of time on a machine, labor hour, money, warehouse space or any number of such resources in various business problems. Since variable yields no profit, therefore such variable are added to the original objective function with zero coefficients.

A *surplus variable* represents amount by which solution values exceed a resource. These variables are also called *negative slack variables*. Surplus variables carry a zero coefficient in the objective function.

2.8.2. The major steps and activities to solve LP model by using Simplex method

This method utilizes the property of a LP problem of having optimal solution only at the corner point of the feasible solution space. It systematically generates corner point solutions & evaluates them for optimality. The method stops when an optimal solution is found. Hence, it is an iterative (repetitive) technique.

If we get more variables & less equation, we can set extra variables equal to zero, to obtain a system of equal variables & equal equations. Such solution is called ***basic solution***.

The variables having positive values in a basic feasible solution are called basic variable while the variables which are set equal to zero, so as to define a corner point are called ***non-basic variables***.

Slack variables are the fictitious variables which indicate how much of a particular resource remains unused in any solution. These variables can't be assigned negative values. A zero value indicates that all the resources are fully used up in the production process.

C_j Column denotes the unit contribution margin.

C_j Row is simply a statement of the projective function.

Z_j Row denotes the contribution margin lost if one unit is brought into the solution.

Hence, it represents the opportunity cost. (Opportunity cost is the cost of sacrifice i.e., the opportunity foregone by selecting a particular course of action out of a number of different available alternatives).

$C_j - Z_j$ Row denotes the Net Potential contribution or the Net unit margin potential, per unit.

The rules used under simplex method, for solving a linear programming problem are as follows:-

Convert the LP to the following form:

Convert the given problem into Standard maximization Problem i.e. minimization problem into a maximization one (by multiplying the objective function by -1). All variables must be non-negative. All right hand side values must be non-negative (multiply both sides by -1, if needed). All constraints must be in $\leq$ form (except the non-negativity conditions). No strictly equality or $\geq$ constraints are allowed.

Convert all $\leq$ constraints to equalities by adding a different slack variable for each one of them.

Construct the initial simplex tableau with all slack variables in the Basic Variables.

The last row in the table contains the coefficient of the objective function (row C_j).

Determine whether the current tableau is optimal. That is: If all the right hand side values are non-negative (called the feasibility condition). If all elements of the last row, that is C_j rows are non-positive (called, the optimality condition)

If the answers to both of these two questions are yes, then stop. The current tableau contains an optimal solution. Otherwise, go to the next step.

If the current *BASIC VARIABLES* is not optimal, determine, which non basic variable should become a basic variable and, which basic variable should become a non basic variable. To find the new *BASIC VARIABLES* with the better objective function value, perform the following tasks:

- ➡ **Identify the entering variable:** The entering variable is the one with the largest positive C_j value (In case of a tie, select the variable that corresponds to the left most of the columns).
- ➡ **Identify the outgoing variable:** The outgoing variable is the one with smallest non-negative column ratio (to find the column ratios, divide the RHS column by the entering variable column, wherever possible). In case of a tie select the variable that corresponds to the upper most of the tied rows.
- ➡ **Generate the new tableau**
 - a. Select the largest value of $C_j - Z_j$ row. The column, under which this value falls, is the pivot-column.
 - b. Pivot-row selection rule. Find the ratio of quantity to the corresponding pivot-column coefficient. The pivot-row selected is the variable having the least ratio.

Remark: Rows having negative or zero coefficients in the pivot-column are to be neglected.

- c. The coefficient, which is in both, the pivot-row & the pivot column, is called the pivot-element or pivot-number.

Up-dating Pivot-row. Pivot-row, also called replaced rows, are updated as under all elements of old-row divided by Pivot-element. Now, in the basic activities column, write the pivot-column variable in place of the pivot-row variable. i.e.; the pivot-row variable is to be replaced by the pivot-column variable.

Up-Dating all other rows. Update all other rows by updating the formulae.

$$(\text{Old} - \text{row element}) - (\text{Corresponding pivot column element} \times \text{updated})$$

$$\text{Corresponding pivot row element} = (\text{New element})$$

Up-dating Z_j & $C_j - Z_j$ rows. Each Z_j is obtained as the sum of the products of the C_j column coefficients multiplied by the corresponding coefficient in the j^{th} column. (i.e.) the Quantity column). It is then subtracted from $C_j - Z_j$ row values to get values. $C_j - Z_j$. This pivoting is to be repeated till no positive coefficients exist in the row, the $C_j - Z_j$ optimal solution is known.

2.8.3. Simplex algorithm (Maximization Case)

What is a Standard Maximization Problem?

A Standard Maximization Problem is the one that satisfies the following four conditions

The Objective function is to be maximized.

All the inequalities are of $\leq$ type.

All right hand constants are non-negative.

All variables are non-negative.

Dear learners let us consider some examples to test our understanding of the solution algorithm that has been discussed so far.

Example 1

Smart Limited manufactures two types of cement which are sold under the brand name quick and Tuff. Each product consumes the same raw materials but in varying proportions. The following table depicts the amount of raw materials along with their respective cost.

Raw material type	Price (Birr/tonne)	Quick	Tuff
N	600	350	200
A	400	50	100
P	400	50	100
I	200	550	600
		1000kg	1000kg

Quick can be blended @ 1000 kg /hour where as the blending rate for Tuff is 1250 kg/hour. Their respective selling prices are Birr. 1010 & Birr 845 You may assume the variable costs to be Birr 500 per hour of plant production time. The maximum availability of raw materials is:

Raw Material Types	Max available (kg)
N	1000
A	300
P	250

I

1800

Formulate as a linear programming model and find out the optimal units of quick & tuff to be produced so as to maximize the profits.

Solution

Step I List the objective & constraint equations.

Step II Introduce the slack variables

Step III Arrange in the form of initial table.

Step IV Find out the profit-margins from given sales price.

Step V Generate solutions.

The detailed solutions are as under:

Step I List the objective & constraint equations.

Maximize Cont-margin = $150x_1 + 125x_2$

Subject to: $350x_1 + 200x_2 \leq 1000$

$50x_1 + 100x_2 \leq 250$

$550x_1 + 600x_2 \leq 1800$

$X_j \geq 0$ for all j

How to calculate the Profit Margin?

First find the time taken to manufacture 1000 kg of both types. It is required for allocating variable production costs to finished product.

∴ 1250 kg Tuff is made in one hour.

∴ 100 kg of Tuff = 0.8 hr.

∴ Variable production cost for Tuff $0.8 \times 500 = \text{Rs. } 400$

∴ Variable production cost for Quick Rs. 500 (∴ in 1 hr. 1000 kg is made)

Particulars Revenue/sales (given) (-) variable costs: Direct Mat	Quick (Rs.) 1,010	Tuff (Rs.) 845
N –	310	120
A –	20	40
P –	110	120
I –	20	40
Sub-total	360	320
Direct paid cost	500	400
Total var. cost	860	720
Contribution margin	150	125

Remarks

1. Given cost of 1000 kg of Birr 600

Given cost of 350 kg of = Birr 210 (for Quick)

Given cost of 200 kg of Birr 120 (for Tuff) etc.

Contribution margin represents the profit which remains if after deducting variable costs from sales i.e.; It covers fixed costs & net profit i.e. it fixed cost = Birr 200 X five – 100 kg of quick are sold then the net profit = $5 \times 150 - 200 = \text{Rs. } 550$

(Fixed cost do not vary with the output)

All calculation has been done to obtain the contribution margin (i.e., profit)

Let x_1 = thousands of kg of quick to be produced.

x_2 = thousands of kg of tuff to be produced.

Now, we have to find those values of x_1 & x_2 for which the contribution is maximum. Here, the constraint is the availability of raw material. Hence, the problem is formulated as;

$$\text{Maximize } Z = 150x_1 + 125x_2$$

$$\text{Subject to: } 350x_1 + 200x_2 \leq 1000$$

$$50x_1 + 100x_2 \leq 250$$

$$550x_1 + 600x_2 \leq 1800$$

$$x_1, x_2 \geq 0$$

Hence, the problem has been formulated

Step II Introduce the slack variables:

Slack Variables:

A slack variable (s) is added to the left hand side of a $\leq$ constraint to convert the constraint inequality into equality. The value of the slack variable shows unused resource. A slack variable emerges when the LPP (linear programming problem) is a maximization problem.

Slack variables represent unused resource or idle capacity. Thus, they don't produce any product and their contribution to profit is **zero**.

Note:

In general, whenever there are n variables and m constraints (excluding the non-negativity), where m is less than n ($m < n$), $n-m$ variables must be **set equal to zero** before the solution can be solved algebraically.

a. Basic variables are variables with non-zero solution values.

Or: basic variables are variables that are in the basic solution. Basic variables have **0** values in the $C_j - Z_j$ row.

b. Non-basic variables are variables with zero solution values.

Or: non-basic variables are variables that are out of the solution.

$\Rightarrow n=5$ variables (x_1 , x_2 , s_1 , s_2 , and s_3) and $m=3$ constraints (Labor, machine and marketing constraints), excluding non-negativity.

Therefore, $n-m=5-3=2$ variables (x_1 and x_2) are set equal to zero in the 1st simplex tableau. These are non-basic variables. 3 Variables (s_1 , s_2 , and s_3) are basic variables (in the 1st simplex tableau) because they have non-zero solution values.

Converting inequalities into equalities by using slack-variable

$$\text{Maximize Cont-margin} = 150x_1 + 125x_2$$

$$\text{Subject to: } 350x_1 + 200x_2 + s_1 = 1000$$

$$50x_1 + 100x_2 + s_2 = 250$$

$$550x_1 + 600x_2 + s_3 = 1800$$

$$s_i, x_j \geq 0 \text{ for all } i \text{ and } j$$

Note. We have dropped: $50x_1 + 100x_2 < 300$ as it is already contained in $50x_1 + 100x_2 \leq 250$

Slack can't be larger than the constant. On right hand side (RHS) (i.e. $s_1=1000, s_2=250$ etc) It is to be so, and then some other variable must be negative for equality to exist, which is not possible.

(2) Rewriting as:

$$\text{Maximize Cont-margin} = 150x_1 + 125x_2 + 0s_1 + 0s_2 + 0s_3$$

$$\text{Subject to: } 350x_1 + 200x_2 + s_1 + 0s_2 + 0s_3 = 1000$$

$$50x_1 + 100x_2 + 0s_1 + s_2 + 0s_3 = 250$$

$$550x_1 + 600x_2 + 0s_1 + 0s_2 + s_3 = 1800$$

$$s_i \text{ and } x_j \geq 0 \text{ for all } i \text{ and } j$$

$C_j(\text{Birr})$	Basic	150	125	0	0	0	(Birr.)	Min ratio
	Activities	x_1	x_2	s_1	s_2	s_3	Qty. RHS	RHS/ x_1
0	S1	350	200	1	0	0	1000	$1000/350 = 2.857$
0	S2	50	100	0	1	0	250	$250/50 = 5$
0	S3	550	600	0	0	1	1800	$1800/550 = 3.273$
	$C_j - Z_j$	150	125	0	0	0		

Entering variable (Pivot Column)
 Leaving variable (Pivot Row)
 Pivot point (Intersection of Pivot Column and Pivot Row)

Living variable (Pivot row)

Pivot Column = x_1 (Largest value of $C_j - Z_j$)

Pivot row the lowest value in the min ratio column that is 2.875
 S_1 is the pivot- row

Pivot element = 350, the intersection of the pivot row and pivot column

Up-dating Pivot-row

$$\frac{1000}{350} = 2.857, \quad \frac{350}{350} = 1, \quad \frac{200}{350} = 0.571 \text{ and } \frac{1}{350} = 0.0029$$

Make the pivot-row of 2nd tableau:

C _j Birr	Basic Act.	Birr Qty.	150 x ₁	125 x ₂	0 S1	0 S2	0 S3
150	x ₁	2.8571	1	0.5714	0.0029	0	0
0	S2						
0	S3						

(Pivot column variable in place of Pivot row variable)

S₂ row is up-dated as follows:

$$\begin{aligned} \text{Old row element} - \text{Corresponding Pivot column element} \\ \times \text{Updated corresponding Pivot row element} \\ = \text{New } S_2 \text{ row element} \end{aligned}$$

Old Row	(Corresponding Pivot	Updated corresponding	= New S ₂ row
Element -	column element x	Pivot row element	element
250 -	(50 x	2.8571)	= 107.145
50 -	(50 x	1)	= 0
100 -	(50 x	0.5714	= 71.43

$$0 - (50 \times 0.0029) = -0.145$$

$$1 - (50 \times 0) = 1$$

$$0 - (50 \times 0) = 0$$

S_3 row is similarly updated:

Old Row (Corresponding Pivot Updated corresponding = New S_2 row

Element - column element $\times$ Pivot row element element

$$1800 - (550 \times 2.8571) = 228.595$$

$$550 - (550 \times 1) = 0$$

$$600 - (550 \times 0.5714) = 285.73$$

$$0 - (550 \times 0.0029) = -1.595$$

$$0 - (550 \times 0) = 0$$

$$1 - (550 \times 0) = 1$$

Complete 2nd tableau

C _j Birr	Basic Act.	Birr Qty.	150 x_1	125 x_2	0 S1	0 S2	0 S3
150	x_1	2.857	1	0.5714	0.0029	0	0
0	S2	107.145	0	71.43	-0.145	1	0
0	S3	228.595	0	285.73	1.595	0	1
-	Z_j (Birr)	428.55	150	85.71	0.435	0	0

-	$C_j - Z_j$ (Birr)	--	0	39.29	-0.435	0	0

Z_j is calculated as under:

$$Z_j \text{ value for QTY Column} = 150 \times 2.857 + 0 \times 107.145 + 0 \times 228.595 = 428.55$$

$$\text{For } x_1 = 150 \times 1 + 0 \times 0 + 0 \times 0 = 150$$

$$x_2 = 150 \times 0.5714 + 0 + 0 = 85.71$$

$$S_1 = 150 \times 0.0029 + 0 + 0 = 0.435$$

$$S_2 = 0$$

$$S_3 = 0$$

Now perform $C_i - Z_i$ to get $C_i - Z_i$ row values.

The positive value of 39.29 in $C_j - Z_j \Rightarrow$ it is not the optimal solution hence, once again pivoting is required.

Developing 3rd tableau

Now, pivot column x_2 (Largest value of $C_j - Z_j$), Pivot row = S_3

Pivot element = 285.73

Updating Pivot row (S_3 ; Old row element $\div$ Pivot number)

C_i Birr	Basic Activities	Birr Qty.	150 x_1	125 x_2	0 S_1	0 S_2	0 S_3	
150	x_1	2.857	1	0.5714	0.0029	0	0	
0	S_2	107.145	0	71.43	-0.145	1	0	
0	S_3	228.595	0	285.73	1.595	0	1	→ Leaving variable

-	Z_j (Birr)	428.55	150	85.71	0.435	0	0
-	$C_j - Z_j$ (Birr)	--	0	39.29	-0.435	0	0

Entering
rd variable

Completed 3 Tableau

C_j Birr	Basic Activities	Birr Qty.	150 x_1	125 x_2	0 S1	0 S2	0 S3
150	x_1	2.4	1	0	0.006	0	-0.002
0	S_2	50	0	0	0.225	1	-0.25
125	x_2	0.8	0	1	-0.0056	0	0.0035
-	Z_j (Birr)	460	150	125	0.02	0	0.1375
-	$C_j - Z_j$ (Birr)	--	0	0	-0.02	0	-0.1375

Note

(1) x_1 row is up-dated as:

$$2.857 - (0.5714 \times 0.8) = 2.4$$

$$1 - (0.5714 \times 0) = 1$$

$$0.5714 - (0.5714 \times 1) = 0$$

$$0.0029 - (0.5714 \times -0.0056) = 0.006$$

$$0 - (0.5714 \times 0) = 0$$

$$0 - (0.5714 \times 0.0035) = -0.002$$

(2) S_2 row is updated as:

$$107.145 - (71.43 \times 0.8) = 50$$

$$0 - (71.43 \times 0) = 0$$

$$71.43 - (71.43 \times 1) = 0$$

$$-0.145 - (71.43 \times -0.0056) = 0.255$$

$$1 - (71.43 \times 0) = 1$$

$$0 - (71.43 \times 0.0035) = -0.25$$

Since no positive co-efficient exists in the $C_1 - z_1$ row, this is the optimal solution.

$$X_1 = 2.4 \qquad X_2 = 0.8$$

$$Z_1 = 460.$$

Example 2

X Ltd. Produces two products p_1, p_2 having profit of Rs. 4 \$ Rs. 3 each p_1, p_2 require 4 hrs. & 2 hours of machining respectively, the total available machining time is 10 hours. P_1, p_2 consume 2 units & $8/3$ units of raw material respectively subject to a total of maximum 8 units. Any no. of p_2 j can be produced & sold but the no. of p_1 must not be more than 6 Formulate as a (P model & solve by the simplex method.)

Solution

$$\text{Maximise } = 4x_1 + 3x_2 + 0s_2 + 0s_3$$

Subject to :

$$\left. \begin{array}{l} 4x_1 + 2x_2 + s_1 + 0s_2 + 0s_3 = 10 \\ 2x_1 + \frac{8}{3}x_2 + 0s_1 + s_2 + 0s_3 = 8x_1; \\ x_1 + 0x_2 + 0s_1 + 0s_2 + s_3 = 6 \end{array} \right\} x_1, x_2 \geq 0$$



C_j (Birr)	Basic activities	Qty.	4	3	0	0	0	Min ratio
	Birr			x_2	s_1	s_2	s_3	Qty./ x_1
0	S_1	10	4	2	1	0	0	10/4= 2.5
0	S_2	8	2	8/3	0	1	0	8/2= 4
0	S_3	6	1	0	0	0	1	6/1= 6
	Z_j (Birr)	0	0	0	0	0	0	
	$C_j - Z_j$ (Birr)		4	3	0	0	0	

Pivot Column = x_1 because in the $C_j - Z_j$ row x_1 have the highest value.

Pivot Row = S_1 it have the lowest contribution (2.5)

Pivot Element = 4 the intersection of pivot row and pivot column

Up-dating S_2 row

$$8 - \left(2 \times \frac{5}{2}\right) = 3$$

$$2 - (2 \times 1) = 0$$

$$\frac{8}{3} - \left(2 \times \frac{1}{2}\right) = \frac{5}{3}$$

Updating s_1 row

$$8 - \left(2 \times \frac{5}{2}\right) = \frac{7}{2}$$

$$1 - (1 \times 1) = 0$$

$$0 - \left(2 \times \frac{1}{2}\right) = -1$$

Updating Z_j row

$$\text{Qty. } 4 \times \frac{5}{2} = 10$$

$$X_1 = 4$$

$$X_2 = 4 \times \frac{1}{2} = 2$$

$$0 - (-1) \times 1 = \frac{-1}{2} \quad 0 - (1) \times 1 = -\frac{1}{4} \quad S_1 = 4 \times \frac{1}{4} = 1$$

$$1 - (2 \times 0) = 1 \quad 0 - (1 \times 0) = 0 \quad S_2 = 0$$

$$0 - (2 \times 0) = 0 \quad 1 - (1 \times 0) = 1 \quad S_3 = 0$$

Second tableau:

C_j (Birr)	Basic activities	Qty.	4	3	0	0	0	Min ratio
			x_1	x_2	s_1	s_2	s_3	Qty./ x_1
4	x_1	5/2	1	1/2	-1/2	0	0	5/2 ÷ 1/2 = 5
0	s_2	3	0	5/3	1/4	1	0	3 ÷ 5/3 = 9/5 →
0	s_3	7/2	0	-1/2		1	1	7/2 ÷ -1/2 = -7
	Z_j (Birr)	10	4	2	-1/4	1	0	
	$C_j - Z_j$ (Birr)		0	1	1	1	0	

Pivot row = s_2 the smallest value in the min ratio column (9/5). Don't take negative and zero value.

Updating pivot row:

$$3 \div 5/3 = 9/5, 0, 1, -1/2 \div 5/3 = -3/10, 1/5 \div 3 = 3/5, 0$$

Up-dating x_1 row

Updating s_3 row

Updating Z_j row

$$\frac{5}{2} - \left(\frac{1}{2} \times \frac{9}{5}\right) = \frac{8}{5} \quad \frac{7}{2} - \left(\frac{-1}{2} \times \frac{9}{5}\right) = \frac{22}{5} \quad \text{Qty. } 4 \times \frac{8}{5} + \frac{3 \times 9}{5} = 11.6$$

$$1 - \left(\frac{1}{2} \times 0\right) = 1 \quad 0 - \left(0 \times -\frac{1}{2}\right) = 0$$

$$\frac{1}{2} - \left(\frac{1}{2} \times 1\right) = 0 \quad \frac{-1}{2} - \left(\frac{-1}{2} \times 1\right)$$

$$\frac{1}{4} - \left(\frac{1}{2} \times \frac{1}{4}\right) = \frac{2}{5} \quad \frac{-1}{4} - \left(\frac{-1}{2} \times \frac{-3}{4}\right) = \frac{-2}{5}$$

$$0 - \left(\frac{1}{2} \times \frac{3}{5}\right) = \frac{-3}{10} \quad 0 - \left(\frac{-1}{2} \times \frac{3}{5}\right) = \frac{3}{10}$$

$$0 - \left(\frac{1}{2} \times 0\right) = 0 \quad 1 - \left(\frac{-1}{2} \times 0\right) = 1$$

3rd Tableau

C_j (Birr)	Basic activities	Qty.	4	3	0	0	0
		Birr	x_1	x_2	s_1	s_2	s_3
0	x_1	8/5	1	0	-	2/5	-3/10
					1/2		
3	x_2	9/5	0	1	1/4	-	3/5
					3/10		
4	s_3	22/5	0	0		-2/5	3/10
	Z_j (Birr)	59/5	4	3	-	7/10	3/5
					1/4		
	$C_j - Z_j$ (Birr)		0	0	1	-	-3/5
						7/10	

There are no positive values in $C_j - Z_j$ row, optimal solution is reached.

Hence $X_1 = 8/5$, $X_2 = 9/5$ & max. = $59/5$ is the answer.

Example 3:

Solve the problem using the simplex approach

$$\text{Maximize } Z = 300x_1 + 250x_2$$

Subject to:

$$2x_1 + x_2 \leq 40 \text{ (Labor)}$$

$$x_1 + 3x_2 \leq 45 \text{ (Machine)}$$

$$x_1 \leq 12 \text{ (Marketing)}$$

$$x_1, x_2 \geq 0$$

Solution

Step 1 Formulate LPP Model

Step 2 Standardize the problem

i.e. Convert constraint inequality into equality form by introducing a variable called **Slack variable**.

Slack Variables:

A slack variable(s) is added to the left hand side of a $\leq$ constraint to convert the constraint inequality into equality. The value of the slack variable shows unused resource.

A slack variable emerges when the LPP (linear programming problem) is a maximization problem.

Slack variables represent unused resource or idle capacity. Thus, they don't produce any product and their contribution to profit is **zero**.

Slack variables are added to the objective function with **zero coefficients**.

Let that, s_2 and s_3 be unused labor, machine and marketing hours, respectively.

Let that s_1 , s_2 , and s_3 are unused labor, machine and marketing hours respectively.

$$\text{Maximize } Z = 300x_1 + 250x_2 + 0s_1 + 0s_2 + 0s_3$$

Subject to:

$$\left. \begin{aligned} 2x_1 + x_2 + s_1 + 0s_2 + 0s_3 &= 40 \\ x_1 + 3x_2 + 0s_1 + s_2 + 0s_3 &= 45 \\ x_1 + 0s_1 + 0s_2 + s_3 &= 12 \\ x_1, x_2, s_1, s_2, s_3 &\geq 0 \end{aligned} \right\} \text{Standard form}$$

Step 3 Obtain the initial simplex tableau

To represent the data, the simplex method uses a table called the **simplex table** or the **simplex matrix**.

==> In constructing the initial simplex tableau, the search for of the optimal solution begins at the origin. Indicating that nothing can be produced;

Thus, first assumption, No production implies that $x_1=0$ and $x_2=0$

$$\Rightarrow 2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 40 \quad \Rightarrow x_1 + 3x_2 + 0s_1 + s_2 + 0s_3 = 45$$

$$2(0) + 0 + s_1 + 0s_2 + 0s_3 = 40 \quad 0 + 3(0) + 0s_1 + s_2 + 0s_3 = 45$$

$s_1 = 40$ – Unused labor hours.

$s_2 = 45$ – Unused machine hrs.

$$\Rightarrow x_1 + 0s_1 + 0s_2 + s_3 = 12$$

$$0 + 0s_1 + 0s_2 + s_3 = 12$$

$s_3 = 12$ – Unused Marketing hours.

Note:

In general, whenever there are n variables and m constraints (excluding the non-negativity), where m is less than n ($m < n$), $n-m$ variables must be **set equal to zero** before the solution can be solved algebraically.

c. Basic variables are variables with non-zero solution values.

Or: basic variables are variables that are in the basic solution. Basic variables have **0** values in the **$C_j - Z_j$** row.

Non-basic variables are variables with zero solution values. Or:

non-basic variables are variables that are out of the solution.

$\Rightarrow n=5$ variables (x_1 , x_2 , s_1 , s_2 , and s_3) and $m=3$ constraints (Labor, machine and marketing constraints), excluding non-negativity.

Therefore, $n-m=5-3=2$ variables (x_1 and x_2) are set equal to zero in the 1st simplex tableau. These are non-basic variables. 3 Variables (s_1 , s_2 , and s_3) are basic variables (in the 1st simplex tableau) because they have non-zero solution values.

Step 3 Construct the initial simplex tableau. Dear learners alternatively you may solve LP simplex method by using the following method (you may solve the problem by using the following table arrangement).

Initial simplex tableau

C_j							
		300	250	0	0	0	
SV	X_1	X_2	S_1	S_2	S_3		Q
0	S_1	2	1	1	0	0	40
0	S_2	1	3	0	1	0	45
0	S_3	1	0	0	0	1	12
Z_j		0	0	0	0	0	0
$C_j - Z_j$		300	250	0	0	0	

Profit per unit column
Basic or Solution variable column
Real or decision variables column
Slack variables column
quantity Solution column

Profit per unit
Row
Constraint equation
Gross Profit row
Net Profit row /Indicator row/

R_1
 R_2
 R_3

Step 4: Choose the “*incoming*” or “*entering*” variables

Note:

The entering variable is the variable that has **the most positive value** in the $C_j - Z_j$ row also called as **indicator row**. Or the entering variable is the variable that has **the highest contribution to profit per unit**.

X_1 in our case is the entering variable

The column associated with the entering variable is called **key** or **pivot column** (X_1 column in our case)

Step 5: Choose the “*leaving*” or “*outgoing*” variable

==> In this step, we determine the variable that will leave the solution for X_1 (or entering variable)

Note: The row with **the minimum** or **lowest positive (non-negative) replacement ratio** shows the variable to leave the solution.

$$\text{Replacement Ratio (RR) or min ratio} = \frac{\text{Solution Quantity (Q)}}{\text{Corresponding values in pivot column}}$$

Note: $RR > 0$

The variable leaving the solution is called **leaving variable** or **outgoing variable**.

The row associated with the leaving variable is **called key or pivot row** (s_3 column in our case)

The element that lies at the intersection of the pivot column and pivot row is called pivot element ($N^o 1$ in our case)

Step 6: Repeat step 3-5 till optimum basic feasible solution is obtained.

Or: repeat step 3-5 till no positive value occurs in the $C_j - Z_j$ row.

Note:

Divide each element of the pivot row by the pivot element to find new values in the key or pivot row.

Perform row operations to make **all other entries** for **the pivot column equal to zero**.

2nd simplex tableau

C_j	300	250	0	0	0	
SV	X_1	X_2	S_1	S_2	S_3	Q

0	S_1	0	1	1	0	-2	16	$R'_1 = R_1 - 2R_2$
0	S_2	0	3	0	1	-1	33	$R'_2 = R_2 - R_3$
300	X_1	1	0	0	0	1	12	$R'_3 = R_3$
	Z_j	300	0	0	0	300	3600	
	$C_j - Z_j$	0	250	0	0	-300		

3rd simplex tableau

C_j		300	250	0	0	0		
	SV	X_1	X_2	S_1	S_2	S_3	Q	
0	S_1	0	0	1	-1/3	-5/3	5	$R''_1 = R'_1 - R'_2$
250	X_2	0	1	0	1/3	-1/3	11	$R''_2 = R'_2/3$
300	X_1	1	0	0	0	1	12	$R''_3 = R'_3$
	Z_j	300	250	0	250/3	650/3	6350	
	$C_j - Z_j$	0	0	0	-250/3	-650/3		

Since all the $C_j - Z_j \leq 0$ optimal solution is reached at.

Therefore, $X_1=12$, $X_2=11$, $S_1=5$ and $Max Z=6350$

2.8.4. Minimization Problems

Minimize Z with inequalities of constraints in " $\geq$ " form

There are two methods to solve minimization LP problems:

Direct method/Big M-method/

Using artificial variables

Conversion method

Minimization by maximizing the dual

Surplus Variable (-s):

A variable inserted in a greater than or equal to constraint to create equality. It represents the amount of resource usage above the minimum required usage.

Surplus variable is subtracted from a $\geq$ constraint in the process of converting the constraint to standard form.

Neither the slack nor the surplus is negative value. They must be positive or zero.

Example:

1. $2x_1 + x_2 \leq 40$ ==> is a constraint inequality

$$x_1 = 12 \text{ and } x_2 = 11 \Rightarrow 2x_1 + x_2 + s = 40 \Rightarrow 2(12) + 11 + s = 40$$

$$\Rightarrow \underline{\underline{s=5}} \text{ unused resource}$$

2. $5x_1 + 3x_2 \leq 45$

$$x_1 = 12 \text{ and } x_2 = 11 \Rightarrow 5x_1 + 3x_2 + s = 45 \Rightarrow 5(12) + 3(11) + s = 45$$

$$\Rightarrow \underline{\underline{s=0}} \text{ unused resource (No idle resource)}$$

3. $5x_1 + 2x_2 \geq 20$

$$x_1 = 4.5 \text{ and } x_2 = 2 \Rightarrow 5x_1 + 2x_2 - s = 20 \Rightarrow 5(4.5) + 2(2) - s = 20$$

$$\Rightarrow \underline{\underline{s=6}} \text{ unused resource}$$

4. $2x_1 + x_2 \geq 40$

$$x_1 = 0 \text{ and } x_2 = 0 (\text{No production}) \Rightarrow 2x_1 + x_2 - s = 40 \Rightarrow 2(0) + 0 - s = 40$$

$$\Rightarrow \underline{\underline{s=-20}} (\text{This is mathematically unacceptable})$$

Thus, in order to avoid the mathematical contradiction, we have to add artificial variable (A)

Artificial variable (A):

Artificial variable is a variable that has no meaning in a physical sense but acts as a tool to create an initial feasible LP solution.

Note:

Type of constraint	To put into standard form
$\leq$	Add a slack variable
$=$	Add an artificial variable
$\geq$	Subtract a surplus variable and add a slack variable

2.8.4.1. Big M-method

/Charnes Penalty Method/

The Big-M Method is a method which is used in removing artificial variables from the basis .In this method; we assign coefficients to artificial variables, undesirable from the objective function point of view. If objective function Z is to be minimized, then a very large positive price (called **penalty**) is assigned to each artificial variable. Similarly, if Z is to be maximized, then a very large negative price (also called **penalty**) is assigned to each of these variables.

Following are the characteristics of Big-M Method:

- High penalty cost (or profit) is assumed as M
- M is assigned to artificial variable A in the objective function Z.
- Big-M method can be applied to minimization as well as maximization problems with the following distinctions:

Minimization problems

-Assign +M as coefficient of artificial variable A in the objective function Z

Maximization problems:

-Here –M is assigned as coefficient of artificial variable A in the objective function Z

- Coefficient of S (slack/surplus) takes zero values in the objective function Z
- For minimization problem, the incoming variable corresponds to the **highest negative** value of **Cj – Zj**.
- Solution is optimal when there is no negative value of **Cj – Zj**.(For minimization case)

Example:

Minimize $Z=25x_1 +30x_2$

Subject to:

$20x_1+15x_2 \geq 100$

$2x_1+ 3x_2 \geq 15$

$x_1, \quad x_2 \quad \geq 0$

Solution

Step 1 Standardize the problem

Minimize $Z=25x_1 +30x_2 +0s_1+0s_2 +MA_1+MA_2$

Subject to:

$20x_1+15x_2- s_1+A_1 = 100$

$2x_1+ \quad 3x_2 -s_2+A_2 = 15$

$x_1, \quad x_2 \quad , s_1, s_2 ,A_1 ,A_2 \geq 0$

Step 2 Initial simplex tableaus

The initial basic feasible solution is obtained by setting $x_1= x_2= s_1= s_2=0$

No production, $x_1= x_2= s_1=0==>20(0) +15(0) - 0+A_1 = 100 ==> A_1 = 100$

$x_1= x_2= s_2=0==>0(0)+3(0) - 0+A_2 =15==> A_2 = 15$

Initial simplex tableau

C_j		25	30	0	0	M	M	
SV		X_1	X_2	S_1	S_2	A_1	A_2	Q
M	A_1	20	15	-1	0	1	0	100
M	A_2	2	3	0	-1	0	1	15
	Z_j	22M	18M	-M	-M	M	M	115 M
	$C_j - Z_j$	25 -22M	30- 18M	M	M	0	0	

RR (min ratio)

$100/20=5$

$15/2=7.5$

X_1 is entering variable because in the **$C_j - Z_j$** row X_1 have the highest negative value. And A_1 is leaving or outgoing variable because the RR is lowest positive in the A_1 row.

Note:

Once an **artificial variable** has left the basis, it has served its purpose and can therefore be removed from the simplex tableau. An artificial variable is never considered for re-entry into the basis.

2nd Simplex Tableau

C_j	25	30	0	0	M	
SV	X_1	X_2	S_1	S_2	A_2	Q
25 X_1	1	3/4	-1/20	0	0	5
M A_2	0	3/2	1/10	-1	1	5
Z_j	25	75/4+3/2M	-5/4+1/10M	-M	M	125+5 M
$C_j - Z_j$	0	45/4-3/2M	5/4-1/10 M	M	0	

$R'_1=R_1/20$

$R'_2=R_2-2 R'$

3rd Simplex Tableau

C_j	25	30	0	0	
SV	X_1	X_2	S_1	S_2	Q
25 X_1	1	0	-1/10	1/2	5/2
30 X_2	0	1	1/15	-2/3	10/3
Z_j	25	30	-1/2	-15/2	162.5
$C_j - Z_j$	0	0	1/2	15/2	

$R''_1=R'_1-3/4 R''_2$

$R''_2=R'_2/3/2$

$C_j - Z_j \geq 0 \implies$ Optimal solution is reached

$X_1=5/2$

$X_2=10/3$ and Minimum $Z=162.5$

Note:

As long as an “A” variable is available in the solution variable column, the solution is **infeasible**.

Example 2. Use the penalty (Big-M) method to solve the following LPP

Min $Z=5x_1 +3x_2$

Subject o:

$$x_1, \quad x_2 \quad \geq 0$$

Minimize $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$

Subject to: If no production

$$2x_1 + 2x_2 + A_1 = 10 \implies x_1 = x_2 = 0 \implies A_1 = 15 \text{ (Solution Value in the initial simplex tableau)}$$

$5x_1 + 2x_2 - s_2 + A_1 = 10 \implies x_1 = x_2 = s_2 = 0 \implies A_2 = 10$ (Solution Value in the initial simplex tableau)

$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

Initial Simplex tableau

C_j		5	3	0	0	M	M		
	SV	X_1	X_2	S_1	S_2	A_1	A_2	Q	RR
0	S_1	2	4	1	0	0	0	12	6
M	A_1	2	2	0	0	1	0	10	5
M	A_2	5	2	0	-1	0	1	10	2
	Z_j	7M	4M	0	M	M	M	20 M	
	$C_j - Z_j$	5 - 7M	3 - 4M	0	- M	0	0		

C_j		5	3	0	0	M	
	SV	X_1	X_2	S_1	S_2	A_1	Q

Department of Management
Page 90

2nd simplex

0	S ₁	0	16/5	1	2/5	0	8
M	A ₁	0	6/5	0	2/5	1	6
5	X ₁	1	2/5	0	-1/5	0	2
	Z _j	5M	6/5M +2	0	2/5M -1	M	10+6 M
	C _j - Z _j	0	-6/5M +1	0	-2/5M+1	0	

tableau

3rd simplex tableau

Type

C_j	5	3	0	0	M		
SV	X_1	X_2	S_1	S_2	A_1	Q	
3	X_2	0	1	5/16	1/8	0	2.5
M	A_1	0	0	-3/8	1/4	1	3
5	X_1	0	0	-1/8	-1/4	0	1
Z_j	5	3	$-3/8M + 5/6$	$M/4 - 7/8$	M	$12.5 + 3M$	
$C_j - Z_j$	0	0	$3/8M - 5/6$	$-M/4 + 7/8$	0		

equation

RR

20

12

-

here. 4th Simplex tableau

C_j		5	3	0	0	Q
	SV	X_1	X_2	S_1	S_2	
3	X_2	0	1	1/2	0	1
	S_2	0	0	-3/2	1	12
5	X_1	0	0	-1/2	0	4
	Z_j	5	3	-1	0	23
	$C_j - Z_j$	0	0	1	0	

$X_1=4, X_2=1, S_1=0, S_2=12$ and *Minimize* $Z=23$

Exercise

Find the optimal solution using Simplex method

1. *Min* $Z=10x_1 + 5x_2$

Subject to:

$2x_1 + 5x_2 \geq 150$

$3x_1 + x_2 \geq 120$

$x_1, x_2 \geq 0$

Ans: $x_1 = 450/13, x_2 = 210/13$ and *Min* $Z = \$540$

2. *Min* $Z=4x_1 + 5x_2$

Subject to:

$x_1 + 2x_2 \geq 80$

$3x_1 + x_2 \geq 75$

$x_1, x_2 \geq 0$

Ans: $x_1 = 14, x_2 = 33$ and *Min* $Z = \$221$

2.8.5. Special Cases in Simplex Method

1. Mixed constraints

Example

$\text{Max } Z=6x_1 +8x_2$

Subject to:

$x_2 \leq 4$

$x_1+ x_2 = 9$

$6x_1+ 2x_2 \geq 24$

$x_1, x_2 \geq 0$

Standard form

Maximize $.Z=6x_1 +8x_2 + 0 s_1 +0 s_2+ 0 s_3-M A_2- M A_3$

St:

$x_2 + s_1 = 4$
 $x_1+ x_2 + A_2 = 9$

AMU, CBE, Department of Management

Standard form

$6x_1 + 2x_2 - s_3 + A_3 = 24$

All Variables ≥ 0

Initial simplex tableau

C_j		6	8	0	0	-M	-M	
	SV	X_1	X_2	S_1	S_3	A_2	A_3	Q
0	S_1	0	1	1	0	0	0	4
-M	A_2	1	1	0	0	1	0	9
-M	A_3	6	2	0	-1	0	1	4
	Z_j	-7M	-3M	0	+M	-M	-M	24
	$C_j - Z_j$	7M+6	3M+8	0	-M	0	0	

Answer: At the 4th tableau: $X_1 = 5, X_2 = 4, S_3 = 14$ and Maximize $Z = 62$

Note:

For the initial basis, use **artificial** variables for constraints that have them. Otherwise, use a constraint **slack** variable. Hence, **surplus** variables will not appear in an initial solution.

Two incoming variables

Or Tie for entering variables/

In order to break this tie, the selection for the key column (entering variable) can be made arbitrary. However; the number of solution can be minimized by adopting the following rules:

If there is a tie between two decision variables, then the selection can be made arbitrary.

If there is a tie between a decision variable and a slack (or surplus) variable, then select the decision variable to enter into basis first.

If there is a tie between slack or surplus variable, then selection can be made arbitrary.

3. Alternate Optimal Solutions

If a non- basic variable corresponding to which $C_j - Z_j = 0$ the problem have multiple optimal solution.

Let us solve a small example:

Example1

As before, we add slacks, and we solve by the Simplex method, using tableau representation.

Max Z =6x₁+4x₂

Subject to: 2x₁ + 3x₂ ≤ 30

3x₁ + 2x₂ ≤ 24

x₁ + x₂ ≥ 3

x₁ & x₂ ≥ 0

Write it in standard form

As before, we add slacks s₁, s₂ and surplus s₃ and artificial A₁, and we solve by the simplex method, using tableau representation.

Max Z =6x₁+4x₂+0s₁+0s₂-MA₁

Subject to: 2x₁ + 3x₂ + s₁ = 30

3x₁ + 2x₂ + s₂ = 24

x₁ + x₂ + A₁ = 3

x₁, x₂, s₁, s₂, s₃ + A₁ ≥ 0

The optimal solution for this LP problem is presented in the following table

C _j		6	4	0	0	0	RHS
	BV	x ₁	x ₂	s ₁	s ₂	s ₃	
0	s ₁	0	5/3	1	-2/3	0	14
0	s ₃	0	-1/3	0	1/3	1	5
6	x ₁	1	2/3	0	1/3	0	8
	Z _j	6	4	0	2	0	48
	C _j - Z _j	0	0	0	-2	0	

↑

The optimal solution is x₁=8, x₂=0, s₁=14, and s₃= 5 and max Z=48

But the

row

Operations Research

2

$C_j - Z_j$

shows a value zero corresponding to a non basic variable X_2 . Thus

an alternative optimal solution can also be obtained by entering variable X_2 in to the basis and removing S_1 from the basis.

Alternative solution

C_j		6	4	0	0	0	RHS
	BV	x_1	x_2	s_1	s_2	s_3	
0	x_2	0	1	3/5	-2/3	0	42/5
0	s_3	0	0	1/5	1/5	1	39/5
6	x_1	1	0	-2/5	3/5	0	12/5
	Z_j	6	4	0	2	0	48
	$C_j - Z_j$	0	0	0	-2	0	

↑

The optimal solution is $x_1=12/5$, $x_2=42/5$, $s_1=0$, and $s_3= 39/5$ and max $Z=48$

3. Unbounded solution

In maximization LP problem, if $C_j - Z_j > 0$ ($C_j - Z_j < 0$ for a minimization case) for a column not in the basis and all entries in this column are negative, then for determining key row, we have to calculate minimum ration corresponding to each basic variable have negative or zero value in the denominator.

Negative value in denominator cannot be considered, as it would indicate the entry of a non basic variable in the basis with a negative value (an infeasible solution will occur). A zero value in the denominator would result a value of $+\infty$. This implies that the entering variable could be increased indefinitely with any of the current basic variables being removed from the basis.

Example solve the following LP problem

Maximize $Z= 3x_1+5x_2$

Subject to

$$x_1 - 2x_2 \leq 6$$

$$x_1 \leq 10$$

$$x_2 \geq 1$$

$$x_1 \& x_2 \geq 0$$

add slack s_1 and s_2 surplus s_3 and artificial A_1 , and we solve by the simplex method,

AMU, CBE, Department of Management

using tableau representation.

Page 95

Maximize $Z = 3x_1 + 5x_2 + 0s_1 + 0s_2 + -MA_1$

Subject to $x_1 - 2x_2 + s_1 = 6$

$x_1 \leq s_2 = 10$

$x_2 - s_3 + A_1 = 1$

$x_1, x_2, s_1, s_2, s_3 \text{ and } A_1 \geq 0$

The initial solution to this problem is presented below

C_j		3	5	0	0	0	-M	RHS	Min ratio
	BV	x_1	x_2	s_1	s_2	s_3	A_1		
0	s_1	1	-2	1	0	0	0	6	—
0	s_2	1	0	0	1	0	0	10	—
-M	A_1	0	1	0	0	-1	1	1	1
	Z_j	0	-M	0	0	M	-M	48	
	$C_j - Z_j$	3	5+M	0	0	-M	0		

↑

Variable x_2 enter in to the basis and variable A_1 leaves the basis

C_j		3	5	0	0	0	-M	RHS	Min ratio
	BV	x_1	x_2	s_1	s_2	s_3	A_1		
0	s_1	1	-2	1	0	0	0	6	—
0	s_2	1	0	0	1	0	0	10	—
-M	A_1	0	1	0	0	-1	1	1	1
	Z_j	0	-M	0	0	M	-M	-M	
	$C_j - Z_j$	3	5+M	0	0	-M	0		

↑

Second table

C_j		3	5	0	0	0	-M	RHS	Min ratio
	BV	x_1	x_2	s_1	s_2	s_3	A_1		
0	s_1	1	0	1	0	-2	2	6	—
0	s_2	1	0	0	1	0	0	10	—

Infeasible solution

Example solving the following LPP

$$x_1 \& x_2 \geq 0$$

$$x_1, x_2, s_1, s_2 \text{ and } A_1 \geq 0$$

C_j		6	4	0	0	-M	RHS
	BV	x_1	x_2	s_1	s_2	A_1	
4	x_2	1	1	1	0	0	5
-M	A_1	-1	0	-1	-1	1	3
	Z_j	4+M	4	4+M	M	-M	20-3M
	$C_j - Z_j$	2-M	0	-4-M	-M	0	

Since the $C_j - Z_j < 0$ the solution is optimal. But the solution not feasible for the given problem since it has $X_1=0$, $X_2=5$ (recall that in the second constraint $X_2 \geq 8$). The fact that artificial variable $A_1 = 3$ is in the solution also indicates that the final solution violates the second constraint ($X_2 \geq 8$) by 3 units.

2.8. Duality in linear programming

Introduction

For every LP formulation there exists another unique linear programming formulation called the '*Dual*' (the original formulation is called the '*Primal*'). The - Dual formulation can be derived from the same data from which the primal was formulated. The Dual formulated can be solved in the same manner in which the Primal is solved since the Dual is also a LP formulation.

The Dual can be considered as the '*inverse*' of the Primal in every respect.

- ➡ The columns coefficients in the Primal constrain become the row coefficients in the Dual constraints.
- ➡ The coefficients in the Primal objective function become the right hand side constraints in the Dual constraints.
- ➡ The column of constants on the right hand side of the Primal constraints becomes the row of coefficients of the dual objective function. The 'directions of the
- ➡ inequalities are reversed.
- ➡ If the primal objective function is a 'Maximization' function then the dual objective function is a 'Minimization' function and vice-versa.

The concept of duality is very much useful to obtain additional information about the variation in the optimal solution when certain changes are effected in the constraint coefficient, resource availabilities and objective function coefficients. This is termed as *post optimality or sensitivity analysis*.

Note:

The dual contains economic information useful to management, and it may also be easier to solve, in terms of less computation, than the primal problem.

Corresponding to every LPP, there is another LPP.

The given problem is called the primal.

The related problem to the given problem is known as the dual.

The dual of a dual is the primal

If the primal has optimal solution ,the dual will have optimal solution

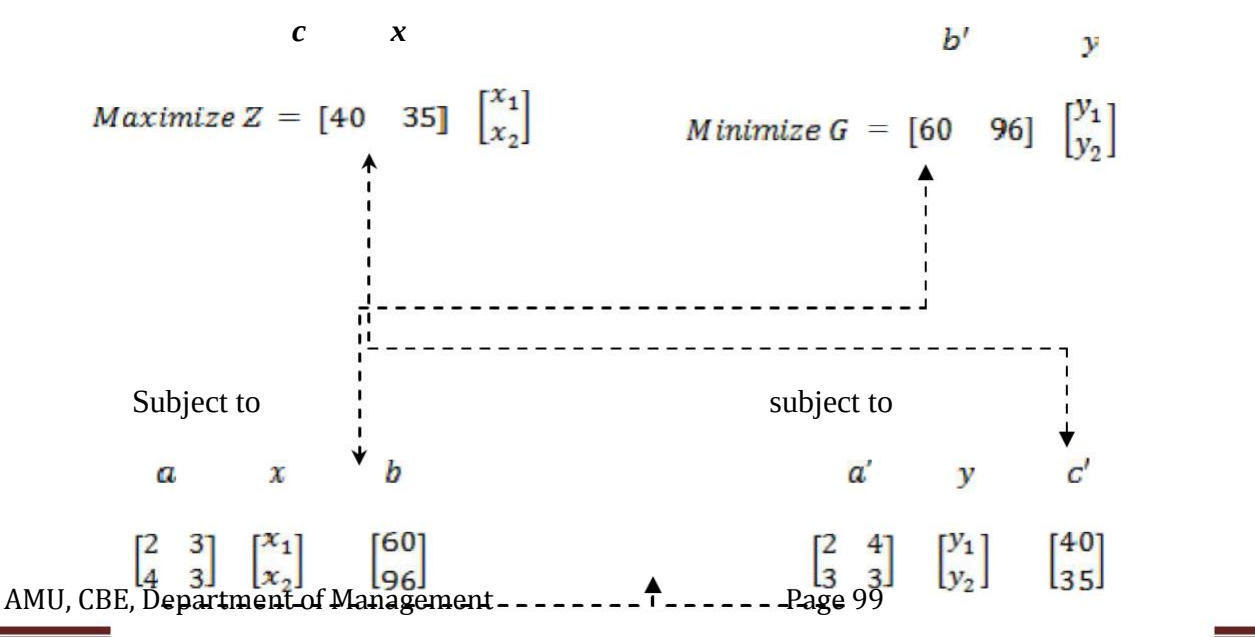
If the primal has no optimal solution, the dual will not have optimal solution.

Whether we follow the dual or primal system, the optimal solution will remain equal.

Table

Primal-Dual Relationship	
Primal	Dual
Objective is minimization	Objective is maximization and vice versal
$\geq$ type constraints	$\leq$ type constraints
N^0 of columns	N^0 of rows
N^0 of rows	N^0 of columns
N^0 of decision variables	N^0 of constraints
N^0 of constraints	N^0 of decision variables
Coefficient of Object function	Right hand side (RHS) value
RHS value	Coefficient of Object function

And the relationship also presented as follow by using chart



Duality Advantage

The dual form provides an alternative form

The dual reduces the computational difficulties associated with some formulation

3. The dual provides an important economic interpretation concerning the value of scarce resources used.

Dual Formation

Following are the steps adopted to convert primal problem into its dual.

Step1. For each constraint in primal problem there is an associated variable in dual problem.

Step2. The elements of right hand side of the constraints will be taken as the co-efficient of the objective function in the dual problem.

Step3. If the primal problem is maximization, then its dual problem will be minimization and vice versa.

Step4. The inequalities of constraints should be interchanged from $\geq$ to $\leq$ and vice versa and the variables in both the problems and non-negative.

Step5. The rows of primal problem are changed to columns in the dual problem. In other words the matrix A of the primal problem will be changed to its transpose (A) for the dual problem.

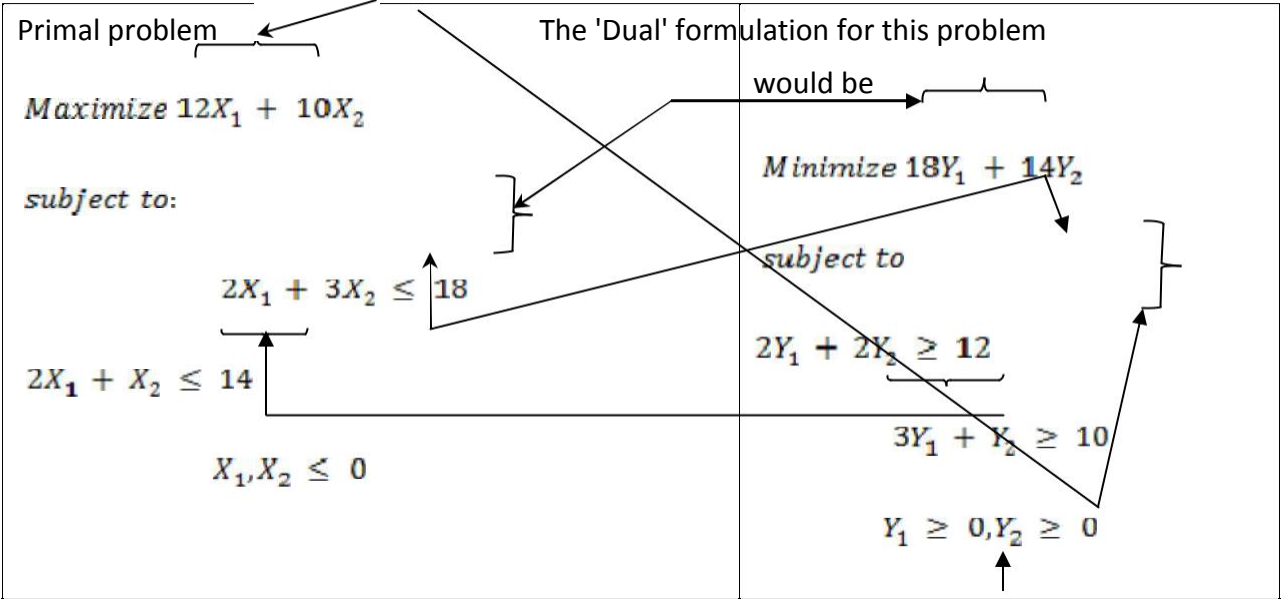
Step6. The co-efficient of the objective function will be taken the right hand side of the constraints of the dual problem.

Problems and Solutions

An example will clarify the concept basis. Consider the following 'Primal' LP formulation

Example 1

Mathematical formulation the primal and the dual linear programming problems of the given problem are



Note the following:

The column coefficient in the Primal constraint namely (2,2) and (3,1) have become the row co-efficient in the Dual constraints.

The co-efficient of the Primal objective function namely, 12 and 10 have become the constants in the right hand side of the Dual constraints.

The constants of the Primal constraints, namely 18 and 14, have become the coefficient in the Dual objective function.

The directions of the inequalities have been reserved. The Primal constraints have the inequalities of $\leq$; while the Dual constraints have the inequalities of $\geq$.

While the Primal is a 'Maximization' problem the Dual is a 'Minimization' problem

Example: 2

Write the duals to the following problems

Maximize $Z=5x_1+6x_2$

Subject to:

$2x_1+3x_2 \leq 3000$ (**Labor constraint**)

$5x_1 + 7x_2 \leq 1000$ (**Machine constraint**)

$x_1 + x_2 \leq 5000$ (**Market constraint**)

$x_1, x_2 \geq 0$

Solution

Represent primal in the conventional table as follows

Dual variables	x_1	x_2	Constraints
u_1	2	3	≤ 3000

u_2	5	7	< 1000
u_3	1	1	≤ 500
Maximize Z	5	6	

By referring the above table, dual for this can be stated as:

Minimize $Z^* = 3000 u_1 + 1000 u_2 + 500 u_3$

Subject to:

$2u_1 + 5u_2 + u_3 \geq 5$

$3u_1 + 7u_2 + u_3 \geq 6$

$u_1, u_2, u_3 \geq 0$

Note:

For maximizing, all constraints must be brought to “ $\leq$ ” form

For minimizing, all constraints must be brought to “ $\geq$ ” form

If they are not, use multiplication factor -1

“=” is an intersection of “ $\geq$ ” and “ $\leq$ ”

Example3.

A firm manufactures two products A and B on machine I and II as shown below:

Machine	<u>Product</u>		Available Hours
	A	B	
I	30	20	300
II	<u>5</u>	<u>10</u>	110
Profit/unit (\$)	<u>6</u>	<u>8</u>	

The total time available is 300hours and 110hours on **machine I and II** respectively

.Product A and B contribute \$6 and \$8 per unit respectively.

Formulate the LPP model for the primal and determine the optimal solution using simplex method.

Formulate the LPP model for the dual and determine the optimal solution from the final primal simplex tableau

Solution

Let X_1 =N^o of units of product A produced

X_2 =N^o of units of product B produced

Maximize $Z = 6X_1 + 8X_2$

Subject to:

$30X_1 + 20X_2 \leq 300$

$5X_1 + 10X_2 \leq 110$

$X_1, X_2 \geq 0$

The final simplex tableau is:

C_j		6	8	0	0	
	S_V	X_1	X_2	S_1	S_2	Q
6	X_1	$1_{1/10}$	0	1/20	-	4
8	X_2	$0_{3/20}$	1	-1/40		9
	Z_j	6	8	-1/10		96
	$C_j - Z_j$	0	0	-1/10	-	

The optimal solution is:

X_1 =4 units of product A produced

X_2 =9 units of product B produced and Maximize $Z = \underline{\$96}$

Let u_1 =Cost of one hour on machine I

u_2 =Cost of one hour on machine II

Minimize $Z^* = 300u_1 + 110u_2$

Subject to:

$30u_1 + 5u_2 \geq 6$

$20u_1 + 5u_2 \geq 8$

$u_1, u_2 \geq 0$

NOTE:

The value in the **Cj - Zj** row under columns of the slack /surplus variables with change in sign give directly the optimal values of the dual/primal basic variables.

Therefore,

Note: The column **s₁** corresponds to **u₁** an the column **s₂** corresponds to **u₂**

u₁=\$(-1/10) per hr on machine I

u₂=\$(-6/10) per hr on machine II

Maximize Z=Minimize Z*=\$96(Total minimum cost)

Note: The dual variables **u_j** are also called as *the shadow prices*

Exercise1.

Maximize Z= **300X₁+ 250X₂**

Subject to:

2 X₁+ X₂ ≤40

X₁+ 3X₂ ≤45

X₁ ≤12

X₁, X₂ ≥0

The final primal solution is:

C_j	300	250	0	0	θ		
SV	X_1	X_2	S_1	S_2	S_3	Q	
0	S_1	0	0	1	-1/3	-5/3	5
250	X_2	0	1	0	1/3	-1/3	11
300	X_1	1	0	0	0	1	12
Z_j	300	250	0	250/3	650/3	6350	
$C_j - Z_j$	0	0	0	-250/3	-650/3		

Required:

What are the solution of the dual variables (i.e. Shadow prices) **u₁, u₂** and **u₃**?

What is the optimal dual cost?

Solution:

I. Note: The column **s₁** corresponds to **u₁**, **s₂** to **u₂** and **s₃** to **u₃**.

$$u_j = -(C_j - Z_j)$$

Therefore, $u_1=0$, $u_2=250/3$ and $u_3=650/3$

Minimize $Z^*=\underline{\$6350}$

3.9. Sensitivity Analysis

Sensitivity Analysis is concerned with the study of ‘Sensitivity’ of the optimal solution of an LPP with discretion variables (changes) in parameters .The degree of sensitivity of the solution due to those variations can range from no change at all to a substantial change in the optimal solution of the given LPP. Thus, insensitivity analysis, we determine the range over which the LP model parameters can change without affecting the current optimal solution. The process of studying the sensitivity of the optimal solution of an LPP is called ***post-optimal analysis***.

The two sensitivity analysis approaches are:

I. Trial and error approach and

II. Analytical approach

4.1.1. Analytical approach

Five types of discrete changes in the original LP model may be investigated during the sensitivity analysis:

Changes of the coefficients of the objective function (c_j)

Changes of the RHS Quantity(b_j)

Changes of the input-output coefficient

Add/delete constraints

The addition of a new variable to the problem

A. Changes of the coefficients of the objective function (c_j)

Decision variables can be:

i. Basic (in the solution)

Non-basic (out-of the solution)

Note:

Instead of resolving the entire problem as a new problem with new parameters, we may take the original optimal solution table as an initial solution table for the purpose of knowing ranges both lower and upper within which a parameter may assume value.

The range of optimality is the range over which a basic decision variable coefficient in the objective function can change without changing the optimal solution mix. However, this change will change only the optimal value of the objective function.

Example:

Max. $Z=5x_1 +4.5x_2 +x_3$

Subject to:

$15\, x_1+15.8x_2 \leq 150$

$5x_1+6.4x_2+15x_3 \leq 77$

$2.8x_2+11.8x_3 \leq 36,$

The optimal tableau for this solution is:

C_j		5	4.5	1	0	0	0	
	SV	X_1	X_2	X_3	S_1	S_2	S_3	Q
5	X_1	1	1.053	0	0.067	0	0	10
1	X_3	0	0.67	1	-0.022	0.067	0	1.8
0	S_3	0	1.924	0	0.258	-0.773	1	15.12
	Z_j	5	5.342	1	0.311	0.067	0	51.8
	$C_j - Z_j$	0	-0.842	0	-0.311	-0.067	0	

Determine the **range of optimality** for the coefficients of the basic-decision variables.

Solution:

	X_2	S_1	S_2	← <i>Non-basic variables</i>
Analysis of basic decision variables				

The analysis will be conducted on products on X_1 and X_3 which are in the basic solution. Divide each $C_j - Z_j$ row entry for variables not in the solution (for instance, by X_2 , S_1 and S_2 values) by the associated variable a_{ij} from X_1 or X_3 row.

I. Analysis of X_1

Steps:

- Take the $C_j - Z_j$ row of the optimal solution of the non-basic variables
- Take the X_1 row of the non-basic variables

$$\frac{C_j - Z_j \text{ row}}{X_1 \text{ row}}$$

$C_j - Z_j \text{ row}$	-0.842	-0.311	-0.067
$X_1 \text{ row}$	1.053	0.067	0
$\underline{C_j - Z_j \text{ row}}$	-0.8	-4.64	$-\infty$
$X_1 \text{ row}$			

❖ Upper Limit

The **smallest positive number** in the $\underline{C_j - Z_j}$ row tells how much the profit X_1 of X_1 can be increased before the solution is changed.

Upper Limit= C_j (for X_1) +the smallest positive value of $\underline{C_j - Z_j}$ row

$=5+\infty=\infty$ $X_1 \text{ row}$

of

	X_2	S_1	S_2	Non-basic variables
$C_j - Z_j \text{ row}$	-0.842	-0.311	-0.067	
$X_3 \text{ row}$	0.67	-0.022	0.067	
$\underline{C_j - Z_j \text{ row}}$				
$X_1 \text{ row}$	-1.26	14.13	-1	

Note: C_j (for X_1) =5(Look in the OF the LP problem)

Lower Limit

The largest negative number closest (negative amount closest to 0)

Lower Limit= C_j (for X_1) +The largest negative value of $\underline{C_j - Z_j}$ row

$=5+ (-0.8)= \underline{4.2}$ $X_1 \text{ row}$

Therefore, the range of optimality for the coefficient of X_1 is $4.2 \leq C_j \text{ (for } X_1) < \infty$ (The coefficient of X_1 in the objective function can change between 4.2 and ∞ without changing the optimal solution mix $X_1=10$, $X_3=1.8$ and $S_3=15.12$)

Analysis of X_3

Upper Limit= C_j (for X_3)+The smallest positive value of $\underline{C_j - Z_j}$ row

$=1+ (-1) = \underline{\underline{0}}$ X_1 row

Note: C_j (for X_3) =5(Look in the OF of the LP problem)

Lower Limit= C_j (for X_3)+The largest negative value of $\underline{C_j - Z_j}$ row

$=1+ (-14.13) = \underline{\underline{15.13}}$ X_1 row

Therefore, the range of optimality for the coefficient of X_3 is $0 \leq C_j \text{ (for } X_3) < 15.13$
(The coefficient of X_3 in the objective function can change between 0 and 15.13 without changing the optimal solution mix $X_1=10$, $X_3=1.8$ and $S_3=15.12$).However, this change will change only the optimal value of the objective function(i.e.MaxZ will change)

Exercise:

Max.Z=50 x_1 +120 x_2

Subject to:

$x_1+4x_2 \leq 80$

$3x_1+x_2\leq 60$

$x_1, x_2 \geq 0$

Determine the range of optimality for the coefficient of the basic variables.

Optimal Solution

C_j						
	SV	X_1	X_2	S_1	S_2	Q
5	X_1	1/2	1	1/4	0	20
0	S_2	5/2	0	-1/4	1	40
Z_j		60	120	30	0	\$2,400
$C_j - Z_j$		\$-10	\$0	\$-30		

Ans: The range of optimality for X_2 's profit coefficient is: $\$100 \leq C_j \text{ (for } X_2) < \infty$

b. The range for the non-basic variables

If there is a variable C_j , not participating in the **optimal basis**, then, in order for the variable to be included in **the optimal solution**, its coefficient in the objective function will have to change from the existing C_j to a new level $C_j(\text{new})$.

$C_j(new) > Z_j$

The range of insignificance is the range over which C_j rates for non-basic variables can vary without causing a change in the optimal solution mix (variable) is called the range of insignificance.

Example:

Max.Z=5x₁ +4.5x₂ +x₃

Subject to:

15 x₁+15.8x₂ ≤150

5x₁+6.4x₂+15x₃ ≤ 77

2.8x₂+11.8x₃ ≤ 36

x₁, x₂ , x₃ ≥ 0

The optimal tableau for this solution is:

C _j		5	4.5	1	0	0	0	
SV		X ₁	X ₂	X ₃	S ₁	S ₂	S ₃	Q
5	X ₁	1	1.053	0	0.067	0	0	10
1	X ₃	0	0.67	1	-0.022	0.067	0	1.8
0	S ₃	0	1.924	0	0.258	-0.773	1	15.12
Z _j		5	5.342	1	0.311	0.067	0	51.8
C _j - Z _j		0	-0.842	0	-0.311	-0.067	0	

Calculate the range of insignificance for the coefficient of non-basic variable(X₂)

Solution

C_j(for X₂)=4.5 and Z_j(for X₂) =5.342

C_j(new for X₂)>5.342==> C_j(new for X₂).If the profit contribution of X₂ is greater than 5.342,then X₂ will be included in the solution.

Thus, ∞< C_j(new for X₂)≤ 5.342 is the range of insignificance for X₂.

C_j (new for X₂) can vary with in the given range without causing a change in the optimal solution mix(X₁=10, X₁=0, X₃=1.8, S₁= S₂=0 and S₃=15.12).

B. Change in the Right Hand—Side Quantity (RHS)

Or

Change in the availability of resource (Capacity) (b_j)

Shadow prices:

==>How much should a firm be willing to pay to make additional resources available?

Shadow prices signify the change in the optimal value of the objective function for **1 unit increases** in the value of the RHS of the constraint that represent the availability of scarce resources.

The negative of the number of $C_j - Z_j$ row in **its slack variable columns** provide as with shadow prices. **Or:** shadow prices are found in the Z_j row of the final simplex tableau in the slack variable columns.

RHS ranging is the range over which shadow prices remain valid.

Example:

Max. $Z=3x_1+4x_2$

Subject to:

$3x_1+5x_2 \leq 15$

$2x_1 + x_2 \leq 8$

$x_2 \leq 2$

$x_1, x_2 \geq 0$

The optimal tableau is given as:

C_j		3	4	0	0	0	
SV		X_1	X_2	S_1	S_2	S_3	Q
3	X_1	1	0	-0.143	0.714	0	3.57
0	S_3	0	0	-0.286	0.428	1	1.143
4	X_2	0	1	0.286	-0.428	0	0.857
$C_j - Z_j$		0	0	-0.714	-0.428	0	

Required:

- Determine the shadow price for each constraint
- Determine the RHS ranges over which the shadow prices are valid

Analysis of the 1st constraint (S₁)

Q	S_1	$\underline{Q/S_1}$
3.57	-0.143	24.96
1.143	-0.286	3.99
0.857	0.286	3.00

Lower Limit=b_j-the smallest positive number in the Q/ S_j column

Upper Limit=b_j-the largest negative number in the Q/ S_j column

Lower Limit=b₁-the smallest positive number in the Q/ S₁column

=15-3=12

Upper Limit=b₁-the largest negative number in the Q/ S₁ column

=15-(-3.99) =18.99

Therefore, $12 \leq b_1 \leq 18.99$ (The range of resource 1 over which the shadow price \$0.714 per unit is valid).

Analysis of the 2nd constraint (S₂)

Q	S_2	$\underline{Q/S_2}$
3.57	0.714	5
1.14		
3	0.428	2.67
0.85		
7	-0.428	-2

Lower Limit=b₂-the smallest positive number in the Q/ S₂ column

=8-(2.67)=5.33

Upper Limit=b₂-the largest negative number in the Q/ S₂ column

=8-(-2) =10

Therefore, $5.33 \leq b_2 \leq 10$ (The range of resource 2 over which the shadow price \$0.428 per unit is valid).

❖ *Analysis of the 3rd constraint (S₃)*



<u>Q</u>	<u>S₃</u>	<u>Q/ S₃</u>
3.57	0	-
-		
1.143	1	1.143
0.857	0	-

Lower Limit=b₃-the smallest positive number in the Q/ S₃ column

=2-(1.143)= 0.857

Upper Limit=b₃-the largest negative number in the Q/ S₃ column

=2-(-∞) =∞

Therefore, **0.857 ≤ b₃ ≤ ∞** (The range of resource 3 over which the shadow price \$0 per unit is valid).

Exercise

Solve the following LPP using simplex method. A firm that manufactures both lawn mowers and snow blowers:

X₁ =the number of lawn mowers

X₂ =the number of snow blowers

Max.Z=30x₁+80x₂

Subject to:

x₁+4x₂ ≤ 1000 Labor hours available

6x₁ + 2x₂ ≤ 1,200lb of steel available

x₂ ≤ 20 snow blower engine available

x₁, x₂ ≥ 0

a. What is the best product mix? What is the optimal profit?

Answer:

x₁=100, x₂=200 and profit =\$19,000

b. What are the shadow prices? When the optimal solution has been reached, which resource has the highest marginal value?

Answer:

The shadow price for 1 additional labor=\$15

The shadow price for 1 additional pound of steel=0

The shadow price for 1 additional snow blowers engine made available =\$20

Thus, snow blower engine have the highest marginal value at the optimal solution.

c. Over what range in each of the RHS values are these shadows valid?

Answer:

The shadow price for labor hours is valid from 800 hours to 1,066.66 hours

The shadow price for pounds of steel is valid from 1,000pounds up to an infinite number of pounds. The shadow price for snow blower engines ranges from 180 engines up to 250 engines

d. What are the ranges over which the objective function coefficients can vary for each of the two decision variables?

Answer:

Without changing the current solution mix, the profit coefficient for the mowers can range from \$0 to 40, while the coefficient for the blowers can range from \$60 to infinity.

Chapter Summary

Linear Programming is a branch of mathematical programming which is designed to solve *optimization problems* where all the constraints as well as the objectives are expressed as *linear function*. It was developed by George B. Denting in 1940. Its earlier application was solely related to the activities of the second' World War. However soon its importance was recognized and it came to occupy a prominent place in the industry and trade.

The basic steps in formulating a linear programming model are presented as follows:

Step I Identification of the decision variables. The decision variables (parameters) having a bearing on the decision at hand shall first be identified, and then expressed or determined in the form of linear algebraic functions or in equations.

Step II Identification of the constraints. All the constraints in the given problem which restrict the operation of a firm at a given point of time must be identified in this stage.

Step III formulate the objective function. In the last stage, the objective which is required to be optimized (i.e., maximized or memorized) must be dearly identified and expressed in terms of the pre-defined decision variables.

There are two types of finding a solution for Linear programming problems this are Graphic solution and Simplex solution method.

The graphic solution procedure is one of the methods of solving two variable linear programming problems. Steps in graphic solution method:-

Step I *Defining the problem.* Formulate the problem mathematically. Express it in terms of several mathematical constraints & an objective function.

Step II *Plot the constraints graphically.* Each inequality in the constraint equation has to be treated as an equation.

Step III *Locate the solution space.* Solution space or the feasible region is the graphical area which satisfies all the constraints at the same time. Such a solution point (x, y) always occurs at the corner Points of the feasible Region the feasible region is determined as follows:

For "greater than" & "greater than or equal to" constraints, the feasible region or the solution space is the area that lays above the constraint lines.

For" Less Then" &" Less than or equal to" constraint, the feasible region or the solution space is the area that lays below the constraint lines.

Step IV *Selecting the graphic solution technique.* Select the appropriate graphic technique to be used for generating the solution.

Some of the important terms commonly used in linear programming are:

Solution: Values of the decision variable $x_i (i=1,2,3,...)$ satisfying the constraints of a general linear programming model is known as the solution to that linear programming model.

Feasible solution: Out of the total available solution a solution that also satisfies the non-negativity restrictions of the linear programming problem is called a feasible solution.

Basic solution: For a set of simultaneous equations in Q unknowns (P, Q) a solution obtained by setting (P - Q) of the variables equal to zero & solving the remaining P equation in P unknowns is known as a basic solution. The variables which take zero values at any solution are detained as non-basic variables & remaining are known as-basic variables, often called basic solution.

Basic feasible solution: A feasible solution to a general linear programming problem which is also basic solution is called a basic feasible solution.

Optimal feasible solution: Any basic feasible solution which optimizes (i.e. maximize or minimizes) the objective function of a LP models is known as the optimal feasible solution to that linear programming model.

Degenerate Solution: A basic solution to the system of equations is termed as degenerate if one or more of the basic variables become equal to zero.

The second method to find a solution for LPP is using the Simplex method. It is an *iterative or “step by step”* method or repetitive algebraic approach that moves automatically from one basic feasible solution to another basic feasible solution improving the situation each time until the optimal solution is reached at.

The summery of the extra variable to be added in the given LP problem to convert it in to standard form is given in the following table.

Chapter concept quiz

Part I: Multiple choice questions

Mathematical model of LPP is important because:

- It helps in converting the verbal description and numerical idea in to mathematical expression
- Decision makers prefer to work with formal models
- It captures the relevant relationship among decision factors
- It enables the use of algebraic technique

Minimization of objective function in LP model means

- Value occurs at allowable set of decisions
- Lowest value is chosen among allowable decisions
- Highest value is chosen among allowable decisions
- Both A&B

Which of the following is not a characteristics of Linear programming model

- Alternative course of action
- Limited objective function
- Limited amount of constraints
- Work when all the information is known
- Non negativity condition on the value of decision variables
- All

Constraints in an LP model represents

- Requirements
- Balancing limitations and requirements

One of the following is not the distinguishing feature of an LP models

- Relationship among all variables is linear
- It has single objective and constraints
- Value of decision variables is nonnegative
- All except A

Each constraint in an LP model is expressed as an

A. Inequality with $\geq$ sign	C. Equation with = sign
B. Inequality with $\leq$ sign	D. All of the above

LP is a?

- Constrained optimization technique
- Technique for economic allocation of limited resources
- Mathematical technique
- All

Which of the following is limitation of LP

- Linear Relationship
- Constant Value of objective & Constraint Equations.
- No Scope for Fractional Value Solutions.

Degree of Complexity.

Multiplicity of Goals.

All

The role of artificial variable in the simplex table is?

To aid in finding an initial solution

To obtain optimal dual prices in the final simplex table

To start phases of simplex method

All

For a minimization problem, the objective function coefficient for an artificial variable is?

A. $-M$

C. Zero

B. $+M$

D. None

A variable which does not appear in the basic variable column of simplex table is ?

A. Never equal to zero

Called a basic variable

B. Always equal to zero

None

We must add artificial variable to write the standard form of LPP is?

A. Equality constraint

C. Both A and B

B. For greater than constraint

D. None

If a negative value appears in the RHS value then

A. The solution is optimal

C. The solution is unbounded

B. The solution is infeasible

D. All

If dual has an unbounded solution, primal has

A. No feasible solution

C. feasible solution

B. unbounded solution

D. None

the RHS constant of a constraint in a primal problem appears in the corresponding dual as

a coefficient in the objective function

a RHS constant of constraint

an input- output coefficient

none

shadow price indicates how much one unit change in the resource value will change the

optimality range of an objective function

optimal value of the objective function

value of basic variables in the optimal solution

none

the dual of a dual is

A. A primal

C. Both A and B

B. A dual

D. None

for any primal and its dual

optimal value of the objective function is same

primal will have an optimal solution if and only if dual does not too

both dual and primal cannot be infeasible

all of the above

to ensure best marginal increase in the objective function value, a resource value may be increased whose shadow p[rise is comparatively

A. Large

C. Neither A nor B

B. Small

D. Both A and B

the entering variable in the sensitivity analysis of objective function coefficient is always

decision variable

non basic variable

basic variable

slack variable

none

while performing sensitivity analysis, the upper bound infinity on the value of the RHS of the constraint means that

there is slack in constraint

the constraint is redundant

the shadow price for that constraint is zero

None

Part II: Solve the following problems

1. **Maximize** $Z=10X_1+15X_2$

2. **Min.** $Z=3X_1+2X_2$

Subject to:

$$2X_1+X_2 \leq 26$$

$$2X_1+4X_2 \leq 56$$

$$-X_1+X_2 \leq 5$$

$$X_1, X_2 \geq 0$$

Subject to:

$$5X_1+X_2 \geq 10$$

$$X_1 + X_2 \geq 6$$

$$X_1+ 4X_2 \geq 12$$

$$X_1, X_2 \geq 0$$

A manufacturer produces two different models; **X** and **Y**, of the same product .The raw materials r_1 and r_2 are required for production. At least 18 Kg of r_1 and 12 Kg of r_2 must be used daily. Almost at most 34 hours of labor are to be utilized .2Kg of r_1 are needed for each model **X** and 1Kg of r_1 for each model **Y**. For each model of **X** and **Y**, 1Kg of r_2 is required. It takes 3 hours to manufacture a model **X** and 2 hours to manufacture a model **Y**. The profit realized is \$50 per unit from model **X** and \$30 per unit from model **Y**. How many units of each model should be produced to maximize the profit?

A manufacturing firm produces two machine parts **P₁** and **P₂** using milling and grinding machines .The different machining times required for each part, the machining times available on different machines and the profit on each machine part are as given below:

	Manufacturing time		Maximum time
	Required (min)	available per week (minimum)	
Machine	P_1	P_2	
<i>Lathe</i>	<i>10</i>	<i>5</i>	<i>25,000</i>
<i>Milling Machine</i>	<i>4</i>	<i>10</i>	<i>2000</i>
<i>Grinding Machine</i>	<i><u>1</u></i>	<i><u>1.5</u></i>	<i>450</i>
<i>Profit per unit (\$)</i>	<i><u>\$50</u></i>	<i><u>\$100</u></i>	

Determine the number of pieces of P_1 and P_2 to be manufactured per week to maximize profit.

Consider the following 'Primal' LP formulation

Maximize $12x_1 + 10x_2$

Subject to $2x_1 + 3x_2 \leq$

$18 \quad 2x_1 + x_2 \leq 14$

$x_1, x_2 \geq 0$

Formulate the 'Dual' for this problem

CHAPTER THREE

TRANSPORTATION AND ASSIGNMENT MODEL

Introduction

All linear programming can be solved by the Simplex method. However due to their specialized structure, certain classes of linear programming problems lend themselves to solutions by other techniques that are computationally more efficient than the simplex methods. These types of linear problems can be represented by the so called *transportation and assignment model*. Let us first describe the transportation model and the assignment model will be discussed later in this chapter.

3.1. Transportation model

Basic transportation problem was stated by F.L. Hitchcock in 1941, and later expanded by T.C. Koopmans and W.W. Cooper. Several extensions of transportation model and methods have been subsequently developed. The transportation problem is a particular case of linear programming problem.

The structure of transportation problem involves a large number of shipping routes from several supply origins to several demand destinations. The objective is to determine the number of units which should be shipped from an origin a destination in order to satisfy the required quantity of goods or services at each demand destination, within the limited quantity of goods or services available at each supply origin, at the minimum transportation cost and/or time.

The transportation algorithm discussed in this chapter is applied to minimize the total cost of transporting a homogeneous commodity (product) from supply origin to demand destination. However, it can also be applied to the maximization of some total value or

utility, for example, financial resources are distributed in such away that the profitable return is maximized. Note that the assumption of homogeneous commodity implies no difference in commodity characteristics among origins.

3.1.1. Formulation of Transportation model (table)

In order to develop a model of transportation problem, it is necessary to know the following information:

The *sources of supply (origin)* which may include production facilities, warehouses, or supply points, characterized by available capacities.

The *destinations* which may include consumption facilities, warehouses or demand points, characterized by required levels of demand.

Supply quantity (capacity) of each origin which is stated in rows.

Demand quantity of each destination, which is stated in column

Unit transportation cost (in birr) for each route.

The total quantity available for shipment is equal to the total quantity demanded

Example1.

A company has three production facilities S1, S2, and S3 with production capacity of 7, 9, and 18 units (in 100s) per week of product respectively. These units are to be shipped to four ware houses D1, D2, D3, and D4 with requirement of 5, 8, 7 and 14 units (in100s) per week respectively. The transportation costs in birr per unit between factories to warehouses are given in the table below:

<u>Source</u>	<u>Ware houses (demand or requirements)</u>			
<u>(Production facilities)</u>	<u>D1</u>	<u>D2</u>	<u>D3</u>	<u>D4</u>
Ware house1 (S1)	19	30	50	10
Ware house 2 (S2)	70	30	40	60
Ware house 3 (S3)	40	8	70	20

Required: formulate the transportation table (model)

Solution

The First step is identifying the information needed to formulate the transportation table. Generally, three information are needed to formulate the transportation table: demand (also called destination or requirement), supply (also called source or capacity), and unit transportation cost. These in formations are arranged as follows:

** Data about supplies*

<u>Source (production facilities)</u>	<u>supply (capacity)</u>
S1	7
S 2	9
S3	18

**data about demand*

<u>warehouses</u>	<u>demand(requirement</u>
D1	5
D2	8
D3	7
D4	14

**data about unit transportation cost*

<u>Source</u>	<u>Ware houses(demand or requirements)</u>			
<u>(Production facilities)</u>	<u>D1</u>	<u>D2</u>	<u>D3</u>	<u>D4</u>

Ware house1 (S1)	19	30	50	10
Ware house 2 (S2)	70	30	40	60
Ware house 3 (S3)	40	8	70	20

The next step is formulating the transportation table.

To formulate the table, list the demand vertically (in column) and supply horizontally (in rows). Thus, the production facilities (sources) are listed down the left side of the table, and their respective supply's quantities are listed down the right side of the table. The destinations (warehouses) are listed across the top of the table and their respective demands are listed across the bottom of the table. The unit shipping costs are shown in the upper right hand corner of each cell, which represent the shipping route.

The following table arranges the information in to a transportation table.

Transportation table

To From	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	34 34

*Remark: when the total supply equals total demand, the problem is called balanced transportation problem, otherwise an unbalanced transportation problem. The unbalanced

transportation problem can be made balanced by adding a dummy supply (row) or a dummy demand (column) as the need arise. We will see this in detail later in this chapter.

3.1.2. Finding initial feasible solution

A solution to a transportation problem consists of quantities that are assigned to the various routes (i.e. cells in the table). These values can range from zero, which implies that no units will be shipped over that route, to a maximum that equals the smaller of two quantities: the row (supply) and column (demand) totals.

Remark: the quantity shipped cannot exceed the available supply in rows, and it should not exceed the amount of demand (column total). Such solution is a feasible solution. The starting point of the transportation method is a feasible solution. A feasible solution is one in which assignments are made in such a way that all supply and demand requirements are satisfied.

In this section, we shall discuss the following three different methods to obtain an initial solution:

North- west corner method
Least cost method, and
Vogel's approximation (or penalty) method.

The initial solution obtained by any of the three methods must satisfy the following conditions:

- ✓ The solution must be feasible i.e. it must satisfy all the supply and demand constraints
- ✓ The number of occupied cell should equal one less than the sum of the number of rows(m) and the number of columns(n) in a transportation table i.e. $(m + n) - 1$. For example, in the case of a table with 3 rows and 3 columns, the number

of occupied cell should be: $(3 + 3) - 1 = 5$ in order to be able to use the transportation algorithm.

Any solution that satisfies the above conditions is called *non degenerate basic feasible solution*, otherwise, *degenerate solution*.

Dear student, now let us see the above three methods of finding initial feasible solution one by one beginning with the NWCM.

A) North-West Corner Method (NWCM) of finding initial feasible solution

The north –west corner method is a systematic approach for developing an initial basic feasible solution. Its *chief advantages* are that it is simple to use and easy to understand. Its *chief drawback* is that it does not take transportation cost into account. Consequently, such solution may require much additional effort to obtain the optimal solution.

The north-west corner method gets its name because the starting point for the allocation process is the upper left hand (North- West) corner of transportation table.

This method can be summarized as follows:

Step1. Start with the cell at the upper left (north-west) corner of the transportation table and allocate as many units as possible to that cell. This will be the smaller of the row supply and the column demand. Adjust the row and column quantities to reflect the allocation.

Step2. - If allocation made in step 1 is equal to the supply available in the first row, then move vertically down to the cell in the second row and first column.

If allocation made in step1 is equal to the demand of the first destination (in first column), then move horizontally to the cell in the first row and second column and apply step 1 again for next allocation.

Step3. Continue the procedure step by step till an allocation is made in the south east corner cell of the transportation table.

Example2. Let us return to Example1 transportation table and see how to find the initial feasible solution using NWCM.

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	34 34

Required: consider the above table and then find the initial basic feasible solution (total transportation cost) using NWCM.

Solution

To find the initial basic feasible solution using NWCM, track the following steps.

Beginning in cell (S1, D1) which is the north-west corner cell in the given transportation table. The value for row S1 (i.e.7) and column D1 (i.e.5) are compared. The smaller of the two, i.e. 5 is assigned as the first allocation; otherwise it will violate the feasibility condition. This means that 5 unit of commodities are to be transported from production facility 1 (S1) to warehouse 1 (D1). However, this allocation leaves a supply of 2 ($7-5=2$) units of commodity at S1.

Staying in row S1, move horizontally to cell (S1, D2) where supply is now 2 and demand is 8. Allocate 2 units to this cell, exhausting the supply of row S1 and leave 6 units of demand in D2 column.

Staying in column D2, move down to cell S2 D2, where supply is now 9 and demand is 6. Allocate 6 units to this cell, exhausting the demand in column D2 and leaving 3 units of supply in row S2.

Staying in row S2, move to cell (S2, D3) and allocate 3 units to this cell, exhausting that row's supply and leaving 4 units in column D3.

Staying in column D3, move down to cell (S3, D3) and allocate 4 units to this cell, exhausting D3 columns and leaving 14 units of supply.

Staying in column S3 move to cell (S3, D4) and allocate 14 units to this cell, exhausting both the rows and columns quantities.

	D1	D2	D3	D4	Supply
S1	19 5	30 2	50	10	7
S2	70	30 6	40 3	60	9
S3	40	8	70 4	20 14	18
Demand	5	8	7	14	34 34

(NB: cells with bold quantity indicates the occupied cell)

No of occupied cell = $m + n - 1 = 3 + 4 - 1 = 6$, so the table is not degenerate solution.

The total transportation cost of the initial solution derived by the NWCM is obtained by multiplying the quantity in the occupied cells with the corresponding unit costs and adding. Thus,

$$\text{Total transportation cost} = 19 \times 5 = 95$$

$$30 \times 2 = 60$$

$$30 \times 6 = 180$$

$$40 \times 3 = 120$$

$$70 \times 4 = 280$$

$$20 \times 14 = 280$$

Birr 1015

In terms of minimizing total transportation cost, this solution may or may not be optimal. We shall make that determination very shortly under ‘test for optimality’ section. At this point, our target is simply computing basic initial feasible solution (i.e. total cost =1015).

B) Least Cost Method (LCM) of finding initial feasible solution

As noted earlier, the main drawback of the north –west corner method is that it does not consider cell (route) costs in making the allocation. Consequently, if this allocation is optimal, that can be attributed to chance rather than the method used. To overcome this drawback, we can use the LCM (which consider the unit transportation cost) to find the initial feasible solution.

This approach, also called the minimum –cost method, uses lowest cell as the basis for selecting route. Generally, this method takes in to account the minimum unit cost of transportation for obtaining initial solution and can be summarized as follows:

Step1. Select the cell with the lowest unit cost in the entire transportation table and allocate as much unit as possible to this cell and eliminate (line out) that row or column in which either supply or demand is exhausted.

If both a row and column are satisfied simultaneously, only one may be crossed out arbitrarily

In case the smallest unit cost cell is not unique, then select the cell where maximum allocation can be made.

Step2. after adjusting the supply and demand for all uncrossed –out rows and columns repeat the procedure with the next lowest unit cost among the remaining rows and columns of the transportation table and allocate a quantity to this cell that is equal to the lower of the available supply of the row and the demand for the column and eliminate (cross out) that row or column in which either demand or supply is exhausted.

Step3. Repeat the procedure until the entire available supply at various sources and demand at various destinations is satisfied. The solution obtained need to be non-degenerate.

Example3

Taking data of the above example, the initial solution using **LCM** and total transportation cost is calculated as follows:

The cell with lowest unit cost is 8 at cell (S3, D2). The maximum units which we can allocate to this cell are selected from (18 and 8) whichever is smallest. Therefore, at this cell we must allocate 8. This meets the complete demand of D2 and leaves 10 units with S3. *Thus, crossed out column D2.*

In the reduced table without column D2 (note: D2 cannot be used because it has been crossed out), the next smallest unit transportation cost is 10 in cell (S1, D4). The maximum unit which can be allocated to this cell can be 7(i.e. compare 14 and 7 and then assign the smallest unit). This exhausts the capacity of S1 and leaves 7 units with D4 as unsatisfied demand. *Thus, eliminate S1 row.*

The next smallest cost, without considering S1, is 20 units at cell (S3, D4). The maximum allocation in this cell could be 7 units. This satisfies the entire demand of D4 and leaves 3 units with S3 as remaining supply. *Thus line out D4.*

The next smallest unit cost cell is not unique. That is, there are two cells (S2, D3) and (S3, D1), having the same unit transportation cost of 40 birr. Allocate 7 units in cell (S2, D3) first because it can accommodate more units as compared to cell (S3, D1) and allocate 3 units (only supply left with S3) in cell (S3, D1). The remaining demand of 2 units of D1 is fulfilled from S2. Since supply and demand

at each origin in destination is exhausted, the initial solution is arrived at. The following table summarizes the discussion.

	D1	D2	D3	D4	Supply
S1	19	30	50	10 7	7
S2	70 2	30	40 7	60	9
S3	40 3	8 8	70	20 7	18
Demand	5	8	7	14	34 34

Number of occupied cell = $m + n - 1 = (3 + 4 - 1) = 6$ and the solution is not degenerated.

Total transportation cost = $7 \times 10 = 70$

$70 \times 2 = 140$

$40 \times 7 = 280$

$40 \times 3 = 120$

$8 \times 8 = 64$

$20 \times 7 = \underline{140}$

Birr 814

C) Vogel's approximation Method (VAM) of finding initial feasible solution

Vogel's approximation (penalty or regret) method is a heuristic method and is preferred to the other two methods described above. In this method each allocation is made on the bases of the opportunity (or penalty or extra) costs that would have been incurred if allocation in certain cells with minimum unit transportation costs were missed. In this method allocations are made so that the penalty cost is minimized. The advantage of this method is that it gives an initial solution which is nearer to an optimal solution or is the optimal solution itself. In VAM method, we look for cells having lowest cost and next lowest cost in each row and column.

The steps in VAM are as follows:

Step1. Calculate penalties for each row /column by taking the difference between the smallest and next smallest unit transportation cost in the same row/column and note down their difference. This difference indicates the penalty or extra cost which has to be paid if one fails to allocate to the cell with the minimum unit transportation cost.

Step2. Select the row or column with the largest penalty (with maximum differences) and allocate as much unit as possible in the cell having the least cost in the selected row or column satisfying the rim condition. If there is a tie in the value of penalties, it can be broken by selecting the cell where maximum allocation can be made.

Step3. Adjust the supply and demand and cross out the satisfied row or column and redraw the reduced matrix. If a row and column are satisfied simultaneously, only one of them is crossed out and the remaining row or column is assigned a zero supply or demand. Any row /column with zero supply or demand should not be used in computing future penalties and hence should be eliminated.

Step4. Repeat step 1 to 3 until the entire available supply at various sources and demand at various destinations are satisfied.

Example4

Taking data of the above example, the initial solution using **VAM** and total transportation cost is calculated as follows:

As shown in the following table, differences (penalty costs) for each row and column have been calculated. In the first round, the maximum penalty, 22 occur in column D2. Thus, the cell (S3, D2) having the least transportation cost 8 is chosen for allocation. The maximum possible allocation in this cell is 8 (compare demand i.e. 8 and supply i.e.18 and assign the smallest unit) and it satisfies demand in column D2. Adjust the supply of S3 from 18 to 10 ($18-8=10$) and cross out column D2.

Calculate the new row and column penalties except for column D2 because its demand has been satisfied. The penalties are:

D1=21 (i.e. the smallest 19 subtracted from the next smallest 40 units)
 D3=10 (i.e. the smallest 40 subtracted from the next smallest 50 units)
 D4=10 (i.e. the smallest 10 subtracted from the next smallest 20 units)
 S1 = 9 (i.e. the smallest 10 subtracted from the next smallest 19 units)
 S2 =20 (i.e. the smallest 40 subtracted from the next smallest 60 units)
 S3 = 20(the smallest 20 subtracted from the next smallest 40 units)

NB: Column D2 should not be considered as it is eliminated (crossed out).

The second round allocation is made in column D1 with target penalty 21 in the same way as in the first round as shown in cell (S1, D1) of the table.

In the third round, the maximum penalty 50 occurs at S3. The maximum possible allocation of 10 units is made in cell (S3, D4) having least transportation cost 20 as shown in the table.

The process is continued with new allocations till a complete solution is obtained.

The initial solution using VAM is in the table.

	D1	D2	D3	D4	Supply	Row difference

S1	19	30	50	10	7	9	9	40	40
	5			2					
S2	70	30	40	60	9	10	20	20	20
			7	2					
S3	40	8	70	20	18	12	20	50	--
		8		10					
Demand	5	8	7	14	34				
					34				
column difference	21	22	10	10					
	21	--	10	10					
	--	--	10	10					
	--	--	10	50					

Number of occupied cell = $m + n - 1 = 3 + 4 - 1 = 6$ and solution is not degenerated

$$\text{Total transportation cost} = 19 \times 5 = 95$$

$$10 \times 2 = 20$$

$$40 \times 7 = 280$$

$$60 \times 2 = 120$$

$$8 \times 8 = 64$$

$$20 \times 10 = \underline{200}$$

Birr 779

1.1.3. Evaluating Initial Feasible Solutions for Optimality

Once the initial solution is obtained, the next step is to check its optimality. An optimal solution is one where there is no other set of transportation routes (allocations) that will further reduce the total transportation cost.

The test of optimality for initial feasible solution involves a cost evaluations of unoccupied or empty cell (i.e. routes to which no units have been allocated) to see if an improved solution is possible. Thus, we have to evaluate each unoccupied cell in the transportation table in terms of an opportunity of reducing total transportation cost.

In this chapter, we shall consider two methods for cell evaluation: *the stepping stone method and the MODI method*.

Mr. A. Charnes developed stepping stone methods for solving transportation problems. Subsequent improvements led to the developments of MODI methods in 1955. Stepping stone methods, even though now superseded by MODI method, is taken up for discussion in this section, for better understanding the logic behind its solutions. Now let us see first the stepping stone method and later we will see the MODI in detail.

A) Evaluation using stepping stone method

The stepping stone method involves tracing a series of closed paths in the transportation table, using one such path for each empty cell. The name stepping stone relates to an analogy of crossing a pond or stream by moving from stone to stone; in the case of transportation problem the *stones* are the *completed cell*.

The steps for evaluating the optimality of initial feasible solution using stepping stone method include the following:

Step1. Determine improvement index for each unoccupied cell.

Trace a 'closed path' that begins at the unoccupied cell, and moves alternately in horizontal and vertical directions pivoting only on occupied cells and terminates on the unoccupied cells.

The plus and minus sign indicates the necessary adjustments for satisfying the row and column requirements. See the following example.

Now let us take the initial feasible solutions found using LCM in Example 5 and test for optimality using the stepping stone method.

In this example an unoccupied cells includes:

The index for each unoccupied cell can be determined by tracing a closed path.

<u>Unoccupied cell</u>	<u>Closed path with alternate '+' and '-' sign</u>
S1, A	+5-9+5-3 = -2
S1, B	+7-9 +4-2=0
S2, A	+6-4+5-3 = 4
S3, B	+9-5+4-2= 6

Step2. If a better solution exists, determine which cell should be occupied (entering cell). For minimization problems, a better solution exists if there are any negative improvement indices. An optimal solution has been found when all improvement indices are *nonnegative (zero or positive)*. Once the optimal solution has been found, the existence of one or more improvement indices at zero level indicates alternative optimal solutions.

NB: If more than one negative index exists simultaneously, occupied (assign) to the cell with largest negative number (i.e. largest number with minus sign). Based on the above example, cell (S1, A) has negative indices (i.e.-2) indicating that total transportation cost can be further reduced by **2Q** (where Q is the number of units assigned to this cell).

Step3. Determine the departing cell and the number of units to assign the entering cell

The departing cell is identified by the *smallest cell in a minus position* on the closed path for the entering cell. The entering cell is assigned the number of units previously given the departing variable.

Based on the above example, the cells with minus signs are :(S1,C) & (S3 , A) with a quantity of 500 unit and 600 units respectively.

The *departing cell* is the cell with the smallest quantity in a negative position. Therefore, cell (S1, C) is the departing cell (since 500 is less than 600). Thus, the number of unit assigned to the new entering cell (i.e.S1, A) is **500** units.

Step4. Develop the new solution, and return to step 1

The new solution can be determined by adding 500 units to the cell with + sign (i.e. cell S1, A and S3, C) in the closed path and by deducting 500 from the cell with minus sign (i.e. S1, C, and S3, A).

Assigning the entering cell(S1,A) with 500units

Adjust the supply side by subtract the 500 unit from cell (S1, C) i.e. $500 - 500 = 0$

Adjust the demand side by adding 500 unit to cell (S3,C) i.e. $200 + 500 = 700$

Adjust the supply side by deducting 500 unit from cell (S3, A) i.e. $600 - 500 = 100$

(S1, A), (S2, B), (S2, C), (S3,A), (S3, C). See the following table.

	A	B	C	Supply
S1	(+)5 500	7	(-)9	500
S2	6 	2 500	4 100	600
S3	(-)3 100	9	(+)5 700	800
Demand	600	500	800	1900

The optimum total transportation cost = $500 \times 5 = 2500$

$$500 \times 2 = 1000$$

$$100 \times 4 = 400$$

$$100 \times 3 = 300$$

$$700 \times 5 = \underline{3500}$$

Birr 7700

To prove whether this figure is an optimal solution or not see the following. We stated that total transportation cost can be reduced by $2Q$ where Q is the number of units assigned to this cell. Now we identified that $Q=500$ units and the total transportation cost reduction = $500 \times 2 = \text{birr } 1000$

Therefore, the optimal total transportation cost = $8700 - 1000 = \text{birr } 7700$

B) Modified Distribution Method(MODI method)

Unlike stepping stone method, the MODI algorithm eliminates the need to trace closed paths for each unoccupied cell. The MODI method of evaluating a transportation solution for optimality involves the use of Index numbers that are established for the row and columns. These are based on the unit cost of the occupied cells. The index number can be used to obtain the cell evaluation for empty cells without the use of stepping stone-paths. There is one index number for each column and one for each row.

The steps to evaluate unoccupied cells using MODI method are as follows:

Step1. For an initial basic feasible solution with $m + n - 1$ occupied cells, calculate U_i for rows($i=1,2,3,\dots,m$) and V_j for columns($j=1,2,3,\dots,n$).

-Start the process by assigning a value of *zero* for the index number of row1.

-then complete the calculation of U_i 's and V_j 's for other rows and columns using the relations: $c_{ij} = U_i + V_j$ where c_{ij} is the transportation cost of the cell.

Step2. For unoccupied cell calculate an opportunity cost by using the relation:

$$d_{ij} = c_{ij} - (u_i + v_j) \text{ for all } i \text{ and } j$$

- if $d_{ij} > 0$, then current basic feasible solution is optimal

-if $d_{ij} = 0$, then current basic feasible solution will remain unaffected but an alternative solution exists

If one or more $d_{ij} < 0$, then an improved solution can be obtained by entering unoccupied cell (i,j) in the basis. An unoccupied cell having the largest negative value of d_{ij} is chosen for entering in to the solution mix (new transportation schedule).

Step4. Construct a closed path for the unoccupied cell with largest negative opportunity cost. Start the closed path with the selected unoccupied cell and mark a plus sign (+) in this cell, trace a path along the rows (or columns) to unoccupied cell, mark the corner with minus sign (-) and continue down the column or row to unoccupied cell and mark the corner with plus sign and minus sign alternatively. Close the path back to the selected unoccupied cell.

Step5. Select the *smallest quantity* amongst the cells marked with *minus sign* on the corners of closed path. Allocate this value to the selected unoccupied cell and add it to other occupied cells marked with plus signs and subtract it from the occupied cells marked with minus sign

Step6. Obtain a new improved solution by allocating units to the unoccupied cell according to step 5 and calculate the new total transportation cost.

Step7. Test the revised solution further for optimality. The procedure terminates when all $d_{ij} \geq 0$ for unoccupied cell.

Remark:

The path starts and ends at the selected unoccupied cell. It consists of successive horizontal and vertical lines whose end points must be occupied cells, except for the end point associated with entering unoccupied cell.

It is immaterial whether the loop (path) is traced in a clockwise or anti-clockwise direction. However, for a given solution only one loop can be constructed for each unoccupied cell.

Any two adjacent cells of the ordered set lie either in the same row or in the same column.

Closed loops (path) may or may not be square in shape.

All cells that have a plus or minus sign, except the starting unoccupied cell, must be occupied cell.

Every loop has an even number of cells and at least four.

Example 6

For better understanding turn back to Example 5 with the initial feasible solution found using LCM method. The following table is the initial feasible solution found using LCM (see Example 5)

	A	B	C	Supply	U_i
S1	5	7	9	500	$U_1=0$
			500		
S2	6	2	4	600	$U_2=-5$
		500	100		
S3	3	9	5	800	$U_3=-4$
	600		200		
Demand	600	500	800	1900 1900	
V_j	$V_1=7$	$V_2=7$	$V_3=9$		

Required: Obtain an optimal solution using MODI method

Solution

To test the initial feasible solution of LCM by using MODI methods followed the following steps.

Step1. Calculate the index for U_i (row), V_j (column) for occupied cell.

NB: the index of $U_1=0$

<u>Occupied cell</u>	<u>$c_{ij} = U_i + V_j$</u>	<u>index</u>
S1, C	$9 = U_1 + V_3$	$U_1 = 0$
	$9 = 0 + V_3$	$V_3 = 9$
S2, B	$2 = U_2 + V_2$	
	$2 = -5 + V_2$	$V_2 = 7$
S2, C	$4 = U_2 + V_3$	
	$4 = U_2 + 9$	$U_2 = -5$
S3, A	$3 = U_3 + V_1$	
	$3 = -4 + V_1$	$V_1 = 7$
S3, C	$5 = U_3 + V_3$	
	$5 = U_3 + 9$	$U_3 = -4$

Step2 calculate the opportunity cost for unoccupied cell using the relation:

$$d_{ij} = C_{ij} - U_i - V_j$$

<u>Unoccupied cell</u>	<u>$d_{ij} = C_{ij} - U_i - V_j$</u>	<u>index (d_{ij})</u>
S1, A	$5 - 0 - 7$	-2

S1,B	7 - 0 - 7	0
S2,A	6 - (-5) - 7	4
S3,B	9 - (-4) - 7	6

Step3 examine the sign of each.

if $d_{ij} > 0$, then current basic feasible solution is optimal

if $d_{ij} = 0$, then current basic feasible solution will remain unaffected

if $d_{ij} < 0$, then an improved solution can be obtained by reassignment

if one or more d_{ij} values < 0 , then improved solution can be obtained by entering unoccupied cell (i,j) having the largest negative value.

Based on the above data, cell (S1, A) has a negative index (d_{ij}).

This indicates that, if we occupied cell (S1, A), there will be birr two reduction of transportation cost per unit.

Thus, total reduction of cost = $2Q$ (where Q is number of unit assigned to this cell)

Step4. Construct a closed path for the unoccupied cell with largest negative opportunity cost starting the closed path with the selected unoccupied cell (i.e cell S1,A).

At (S1,A), the closed path(loop) is: $+(S1,A), -(S1,C), +(S3,C), -(S3,A)$.

See the following table.

	A	B	C	Supply	U_i
S1	5 (+)500	7	9 (-)	500	$U_1=0$

Operations Research

S2	6	2	4	600	U2=-5
		500	100		
S3	3	9	5 (+)	800	U3=-4
	100	(-)	700		
Demand	600	500	800	1900 1900	
Vj	V1=7	V2=7	V3=9		

Step5. Select the *smallest quantity* amongst the cells marked with *minus sign* on the corners of closed path and allocate this value to cell (S1, A). Then, add it to cell (S3, C) and subtract it from cell (S3, A and S1, C).

Based on the above table, 500 and 600 are quantities found in cell marked with minus sign. So, select 500 units and adjust the demand and supply as follows

assigning the entering cell (S1, A) with 500units

adjust the supply side by subtract the 500 unit from cell (S1, C) i.e. $500 - 500 = 0$

adjust the demand side by adding 500 unit to cell (S3,C) i.e. $200 + 500 = 700$

adjust the supply side by deducting 500 unit from cell (S3, A) i.e. $600 - 500 = 100$

The optimum total transportation cost can be calculated as follows:

<u>Route</u>	<u>unit cost</u>	<u>quantity shipped</u>	<u>cost</u>
S1, A	5	500	2500

S2, B	2	500	1000
S2, 3	4	100	400
S3, A	3	100	300
S3, C	5	700	<u>2100</u>

Total cost 7700

Exercise

1. Based on the data given under Example 5:

find the initial feasible solution using NWCM and VAM method

test the optimality of the initial feasible solution found in a above using stepping stone methods and MODI method

1.1.4. SPECIAL CASE IN TRANSPORTATION PROBLEM

In this section we will consider some special cases in solving transportation problems. The most common special cases includes the following

- unbalanced supply and demand
- degeneracy
- alternative optimal solution
- unacceptable routes

Now let as see each in detail.

a) Unbalanced supply and demand

For a feasible solution to exist, it is necessary that the total supply must equal total demand. Unbalanced case may be of two types: supply > demand or demand > supply.

Case1. Demand more than Supply

If total demand exceeds total supply, a dummy row (called a dummy supply centre) can be added to the transportation table to account for the excess demand.

The unit transportation cost for the cells in the dummy row is set equal to zero.

Example 1

Three cement factories are located at A, B, C are to be supplied with carbonates stones from mines X, Y, and Z. any mine can supply to any factory. The only condition is to minimize the overall transportation cost. The total requirements (demand) and capacities (supply of mines) are indicated in the table below, along with cost of transportation in birr per ton.

<i>i.</i>		<i>ii.</i>		<i>iii</i>			
<i>Mines capacity per day</i>		<i>factory requirements per</i>		<i>cost of transpo</i>			
<i>In Lorries loads in tones</i>		<i>day in lorry loads in tons</i>		<i>ration in birr</i>			
					A	B	C
X	20	A	40	X	2	4	4
Y	50	B	25	Y	8	12	8
Z	<u>30</u>	C	<u>85</u>	Z	2	8	12
	100		150				

Required: formulate the transportation table and find the initial feasible solution using NWCM.

Since demand and supply are not equal (i.e. demand exceeds supply) feasible solution will not exist. This problem can be solved by a balanced situation and proceed as per method explained earlier.

In order to balance supply and demand, create a dummy source of mine (row) having a capacity by which supply is less than demand in the initial solution. In this case it is 50 (i.e. $150 - 100 = 50$). Dummy source is included in the table by adding one extra row. Dummy source will have distribution cost = 0. Then find the initial feasible solution using NWCM. See table below.

The solution using cost cost 20	To From	A	B	C	Supply	
	X	2 20	4	4	20	
	Y	8 20	12 25	8 5	50	
	Z	2	8	12 30	30	
	Dummy	0	0	0 50	50	
	Demand	40	25	85	150	150
						150
						X, A 2 40
	Y, A	8	20		160	
	Y, B	12	25		300	
	Y, C	8	5		40	
	Z, C	12	30		360	
	Dummy, C	0	50		<u>0</u>	

initial feasible
NWCM is:

Route unit
quantity shipped

Total cost 900

Now proceed as above to till optimum solution is obtained using either *stepping stone* method or *MODI* method.

Case 2 *Supply more than demand*

If total supply exceeds total demand, an additional column (called dummy demand centre) can be added to the transportation table to absorb the excess supply. The unit transportation cost for these cells in this column is set equal to zero because these represent product items that are not being made and not being sent.

Example 2

In the above example given in Example1 of this section, let as have the following supply and demand. Unit cost is assumed to be the same.

i.		ii.		iii		
<i>Mines capacity per day</i>		<i>factory requirements per</i>		<i>cost of transpo</i>		
<i>In Lorries loads in tones</i>		<i>day in lorry loads in tons</i>		<i>ration in birr</i>		
				A	B	C
X	20	A	40	X	2	4 4
Y	50	B	25	Y	8	12 8
Z	<u>80</u>	C	<u>35</u>	Z	2	8 12
150		100				

Required: formulate the transportation table and find the initial feasible solution using NWCM.

Sine total supply is greater than total demand by 50 units, add a dummy column with 0 unit cost and 50 units of demand. See the following table.

To From	A	B	C	Dummy	Supply
X	2 20	4	4	0	20
Y	8 20	12 25	8 5	0	50
Z	2	8	12 30	0 50	80
Demand	40	25	35	50	150 150

Route	<u>unit cost</u>	<u>quantity shipped</u>	<u>cost</u>
X, A	2	20	40
Y, A	8	20	160
Y, B	12	25	300

Y, C	8	5	40
Z, C	12	30	360
Z, dummy	0	50	<u>0</u>
Total cost			900

Now proceed as above to till optimum solution is obtained using either *stepping stone* method or *MODI* method.

b) Degeneracy

Degeneracy occurs when the number of occupied cell contains is *less than* $(m+n-1)$. In such cases the current solution cannot be improved because it is not possible to draw a closed path for every occupied cell. Also the values dual variables U_i and V_j which are used to test the optimality ca not be computed. Thus, we need to remove the degeneracy to improve the given solution.

The degeneracy can arise either in developing *the initial solution* or *at some intermediate solution*.

Case1. Degeneracy at the initial solution

To resolve degeneracy at the initial solution consider the following points.

Add an artificial solution ' ε ' (Greek letter epsilon) or Δ (delta) to one or more of the unoccupied cell to get ' $m + n - 1$ ' number of occupied cells. The cell with Δ is considered as an occupied cell and is modified in the same manner as other occupied cells.

' ε or Δ is assumed to be a very small quantity close to zero

Generally, in a minimization transportation problem, it is better to allocate Δ to unoccupied cells that have lowest transportation costs. In maximization problems it should be allocated to a cell that has a high pay off value.

The placement of Delta (Δ) cannot be in just any unoccupied cells. It has to be placed in either the column or the row associated with the cell which simultaneously full filled the supply and demand constraints.

The quantity of Δ is considered to be so small that if it is transferred to an occupied cell it does not change the quantity of allocation. Hence it does not affect the total transportation cost. Finally we can eliminate Δ from the table.

Example. A manufacturer wants to ship 200 loads of his products from two factories (factory1 and factory2) to three warehouses (warehouse A, B, and C) as shown below.

To From	A	B	C	Supply
X	5 75	10 Δ	8	75
Y	4	15 60	12 65	125
Demand	75	60	65	200 200

Required: consider the above table and find:

The initial feasible solution using NWCM.

Test for optimality using stepping stone method.

Solution

- i. Number of occupied cell is less than $(m+n-1)$. Therefore, it is degenerate. To overcome this problem adds Δ to unoccupied cell. *Note that* Δ has to be placed in either the column or the row associated with the cell which simultaneously full filled the supply and demand constraints. In this case cell (X, B) is the appropriate cell to assign Δ . The cell with Δ is considered as an occupied cell and is modified in the same manner as other occupied cells.

Therefore, the initial feasible solution using NWCM is:

$$(5 \times 75) + (15 \times 60) + (12 \times 65) = \text{birr } 2055$$

- ii. To test the optimality of the initial solution found above, let us use stepping stone method.

To \ From	A	B	C	Supply
X	(-) 5 75	(+) 10 Δ	8	75
Y	(+) 4	(-) 15 60	12 65	125
Demand	75	60	65	200 200

Unoccupied cell

Closed path with alternate '+' and '-' sign

X, C

$$+8 - 12 + 15 - 10 = 1$$

Y, A

$$+4 - 15 + 10 - 5 = -6$$

Since cell (Y,A) has a negative index, we must occupy this cell. This enables us to save birr6 per unit of assignment at this cell. Therefore, total cost reduction=6Q.

See the reallocations in the following table

To From	A	B	C	Supply
X	5 75	10 Δ	8	75
Y	4 60	15	12 65	125
Demand	75	60	65	200 200

The quantities in the cell with minus sign are 75 and 60. Since the smallest is 60, assign this units at cell (Y, A) and adjust the respective demand and supply amount.

Now, the newly occupied cells are:

(X, A) =75 unit, (Y, A) = 60 units, and (Y, C) = 65 units

Therefore, the optimum solution is:

$$(75 \times 5) + (4 \times 60) + (12 \times 65) = \text{birr } 1395$$

Case2. Degeneracy at some intermediate solution

To resolve degeneracy which occurs during optimality test, the quantity may be allocated to one or more cells which have become unoccupied recently to have $m+n-1$ number of occupied cells in the new solution.

Example

Goods have to be transported from sources S1, S2, and S3 to destination D1, D2, and D3. The transportation cost per unit, capacity of the sources and requirements of the destinations are given in the following table.

To From	D1	D2	D3	Supply
S1	8	5	6	120
S2	15	10	12	80
S3	3	9	10	80
Demand	150	80	50	280 280

Required: determine the transportation Schedule so that cost is minimized.

Solution

Using the NWCM, the non degenerate initial feasible solution is given in the following table:

To From	D1	D2	D3	Supply
S1	8 120	5	6	120
S2	15 30	10 50	12	80
S3	3	9 30	10 50	80
Demand	150	80	50	280 280

Total cost= $(120 \times 8) + (15 \times 30) + (10 \times 50) + (9 \times 30) + (10 \times 50) = \text{birr } 2680$

To test the optimality of the above solution found in NWCM using MODI method, calculate U_i , V_j and d_{ij} . As usual as shown below.

To From	D1	D2	D3	Supply	U_i
S1	8 120	5	6	120	$U_1=0$
S2	15 30 (-)	(+)10 50	12	80	$U_2=7$
S3	(+)3 30	(-)9 30	10 50	80	$U_3=6$

Demand	150	80	50	280 280	
	V1=8	V2=3	V3=4		

Calculate the index for U_i (row), V_j (column) for occupied cell.

NB: the index of $U_1=0$

<u>Occupied cell</u>	<u>$c_{ij} = U_i + V_j$</u>	<u>index</u>
S1, D1	8 = $U_1 + V_1$	$U_1=0$
	8 = $0 + V_1$	$V_1=8$
S2, D1	15 = $U_2 + V_1$	
	15 = $U_2 + 8$	$U_2=7$
S2, D2	10 = $U_2 + V_2$	
	10 = $7 + V_2$	$V_2=3$
S3, D2	9 = $U_3 + V_2$	
	9 = $U_3 + 3$	$U_3=6$

$$\begin{aligned} S3, D3 \quad 10 &= U3 + V3 \\ 10 &= 6 + V3 \quad V3 = 4 \end{aligned}$$

Then calculate the opportunity cost for unoccupied cell using the relation:

$$d_{ij} = C_{ij} - U_i - V_j$$

<u>Unoccupied cell</u>	<u>$d_{ij} = C_{ij} - U_i - V_j$</u>	<u>index (dij)</u>
S1,D2	5 - 0 - 3	2
S1,D3	6 - 0 - 4	2
S2, D3	12 - 7 - 4	1
S3,D1	3 - 6 - 8	-11

Since the unoccupied cell (S3,D1) has the largest negative opportunity cost of -11, therefore, cell (S3, D1) is entered in to the new solution mix. The closed path for (S3.D1) is shown in the above table. The maximum allocation to this cell is 30. However, when this amount is allocated to (S3,D1) both cells with minus sign i.e. (S2,D1) and (S3,D2) become unoccupied because these two have same allocation(30 units each).thus, the number of positive allocations became less than the required number(i.e. $m+n-1$)

Hence this is a degenerate solution as shown below.

To From	D1	D2	D3	Supply
S1	8 120	5	6	120

S2	15	10 80	12	80
S3	3 30	9 Δ	10 50	80
Demand	150	80	50	280 280

To remove the degeneracy a quantity Δ is assigned to one of the cells that has become occupied so that there are $m+n-1$ occupied cells. Assign Δ to either (S2, D1) OR (S3, D2) and proceed with the usual solution procedure.

C) Alternative optimal solution

Sometimes transportation problems have multiple optimal solutions. In such instance, it can be useful for a manager to be aware of alternative solutions because this gives the manager an option of bringing non-quantitative considerations in to the decision. The existence of alternative optimal solutions can be determined by an inspection of the opportunity cost (d_{ij}) for the unoccupied cells. If an unoccupied cell in an optimal solution has opportunity cost of zero, an alternative optimal solution can be formed with another set of allocations without increasing the total transportation cost.

Example

A company has received a contract to supply gravels from its three gravel pits (located in towns W, X and Y) to three new construction projects (located in town A, B and C). The transportation cost per unit, capacity of the sources (gravel pits) and requirements of the destinations (construction projects) are given in the following table.

To From	A	B	C	Dummy	Supply
W	4	8	8	0	76
X	16	24	16	0	82
Y	8	16	24	0	77
Demand	72	102	41	20	235 235

(NB: Since demand is less than supply, add dummy column with a demand of 20 units.)

Required: determine the transportation Schedule so that cost is minimized.

Solution

First let us find the initial feasible solution using VAM

(Note that if the column difference and row differences are equal; select the cell in which you can assign the maximum amount with the minimum transportation cost)

	A	B	C	Dummy	Supply	Row difference
W	4	8 76	8	0	76	4 4 0 -
X	16	24 21	16 41	0 20	82	16 8 8 8

Y	8 72	16 5	24	0	77	8 8 8 8
Demand	72	102	41	20	235	
Column difference	4 4	8 8	8 8	0 10 10		

The initial feasible solution= $(8 \times 72) + (8 \times 76) + (24 \times 21) + (16 \times 5) + ((16 \times 41) + (0 \times 20)) = \text{birr } 2424$

Then calculate the opportunity cost (d_{ij}) as shown below.

First

	A	B	C	dummy	Supply	U_i
W	4	8 76	8	0	76	$U_1=0$
X	16(+) ----- 21	24 (-) ----- 21	16 41	0 20	82 21	$U_2=16$
Y	8 (-) ---- 72	16 (+) ----- 5	24	0	77 5	$U_3=8$
Demand	72	102 26	41	20	235	
V_j	$V_1=0$	$V_2=8$	$V_3=0$	$V_4=-16$		

calculate the index for U_i (row), V_j (column) for occupied cell.

NB: the index of $U_1=0$

The initial feasible solution = $(8 \times 72) + (8 \times 76) + (24 \times 21) + (16 \times 5) + ((16 \times 41) + (0 \times 20)) = \text{birr } 2424$

<u>Occupied cell</u>	<u>$c_{ij} = U_i + V_j$</u>	<u>index</u>
W, B	$8 = U_1 + V_2$	$U_1 = 0$
	$8 = 0 + V_2$	$V_2 = 8$
X, B	$24 = U_2 + V_2$	
	$24 = U_2 + 8$	$U_2 = 16$
X, C	$16 = U_2 + V_3$	
	$16 = 16 + V_3$	$V_3 = 0$
X, dummy	$0 = U_2 + V_4$	
	$0 = 16 + V_4$	$V_4 = -16$
Y, A	$8 = U_3 + V_1$	
	$8 = 8 + V_1$	$V_1 = 0$
Y, B	$16 = U_3 + V_2$	
	$16 = U_3 + 8$	$U_3 = 8$

Then, calculate the opportunity cost for unoccupied cell using the relation:

Y, dummy	$0 - 8 - (-16)$	8	$d_{ij} = C_{ij} - U_i - V_j$
<u>Unoccupied cell</u>	<u>$d_{ij} = C_{ij} - U_i - V_j$</u>	<u>index (d_{ij})</u>	
W, A	$4 - 0 - 0$	4	
W, C	$8 - 0 - 0$	8	
W, dummy	$0 - 0 - (-16)$	16	
X, A	$16 - 16 - 0$	0	
Y, C	$24 - 8 - 0$	16	

The opportunity costs in all unoccupied cells are positive except for cell (X,A) which has a zero opportunity cost. This means, if (X, A) is occupied, no change in the transportation

cost would occur. To determine the alternative solution, form a closed path for cell(X,A) as shown above. Quantity of 21 and 72 are in the cell with minus sign. So, the maximum quantity which can be allocated to cell (X, A) is 21. after this change, the new solution is shown in the following table.

	A	B	C	Dummy	Supply	U _i
W	4	8 76	8	0	76	U ₁ =0
X	16 21	24	16 41	0 20	82 21	U ₂ =16
Y	8 51	16 26	24	0	77 5	U ₃ =8
Demand	72	102 26	41	20	235	
V _j	V ₁ =0	V ₂ =8	V ₃ =0	V ₄ =-16		

Total cost=(16x21)+(8x51)+(8x76)+(16x26)+(16x41)+(0x20)= birr 2424

Since all d_{ij} values are positive or zero, this solution is optimal with a minimum total transportation cost of birr 2424 which is the same as in the previous solution.

d) Unacceptable route

In some cases, certain source-destination combinations may be unacceptable. This may be due to road hazards (like snows, floods etc.), traffic regulations, equipment break downs, labor problems or skill requirements and so on. In order to prevent that route from appearing in the final solution, the manager could assign a unit cost to that cell that is large enough to make that route uneconomical and hence prohibit its occurrence. One

rule of thumb would be to assign a cost that is 10 times the largest cost in the table. Then the revised problem could be solved using NWCM, LCM or VAM method for an initial solution and either stepping stone or MODI method can be used to evaluate the initial solution and possible reallocation. This section is left as an exercise for the reader.

3.1.4. Maximization transportation problem

In general, transportation model is used for cost minimization problems. However, some transportation-type problems concern profits or revenues rather than costs. In this case the objective is to maximize rather than to minimize.

The algorithm for solving maximization problem is the same as that for the minimization problems except that it adds one additional step at the start:

First identify the cell with the largest profit and subtract all the other cell profits from that value. These values reflect the opportunity costs that would be incurred by using routes with unit profits that are less than the largest unit profit.

Then replace the original unit profits with these opportunity costs and solve in the usual way for the minimum opportunity cost solution. This will be identical to maximizing the total profit.

When the optimal distribution plan has been identified, use the original cell values (i.e. profits) to compute the total profit for that plan.

Example

A company has three manufacturing plants and three warehouses. These data along with the unit profit are given below.

Manufacturing plant	warehouse	<u>unit profit</u>			
<u>(Sources)</u>	<u>(Destination)</u>	<u>D1</u>	<u>D2</u>	<u>D3</u>	
S1 =100	D1 = 50	S1	4	2	8
S2 = 200	D2 = 150	S2	5	1	9
S3 = 200	D3 = 300	S3	7	6	3

Required:

- formulate this problem as the transportation problem to maximize profit
- find the initial feasible solution
- iii. test for optimality and find the optimum solution

NB: the steps for developing an initial feasible solution, evaluation of optimality and reallocation are identical to those used for cost minimization.

Solution

- i. To formulate the transportation table of maximization, follow similar procedures as discussed for minimization cases.

To \ From	D1	D2	D3	Supply
S1	4	2	8	100
S2	5	1	9	200
S3	7	6	3	200
Demand	50	150	300	500

To find the initial feasible solution, first identify the largest unit profit and then subtract all cells from that value. In this case the largest unit profit is 15.

Then subtract each cell from 15 to get the opportunity cost as shown below.

	<u>D1</u>	<u>D2</u>	<u>D3</u>		<u>D1</u>	<u>D2</u>	<u>D3</u>
S1	9-4=5	9-2=7	9-8=1	S1	5	7	1
S2	9-5=4	9-1=8	9-9=0	S2	4	8	0

$$9 - 6 = 3$$

$$9 - 3 = 6$$

To \ From	D1	D2	D3	Supply
S1	5	7	1	100
S2	4	8	0	200
S3	2	3	6	200
Demand	50	150	300	500

Find the initial feasible solution using VAM, LCM or NWCM method. Let us use NWCM.

To \ From	D1	D2	D3	Supply
S1	5 50	7 50	1	100
S2	4	8 100(-)	0 100	200
S3	2	3 (+)3	6 (-)6 200	200
Demand	50	150	300	500 500

Since occupied cell= $m+n-1=5$, it is not degenerate.

Initial solution= $(5 \times 50) + (7 \times 50) + (8 \times 100) + (0 \times 100) + (6 \times 200) = \text{birr } 2600$

iii. Test for optimality and find the optimum solution.

As usual let us use the stepping stone method to test for optimality.

The index for each unoccupied cell can be determined by tracing a closed path.

<u>Unoccupied cell</u>	<u>Closed path with alternate '+' and '-' sign</u>
S1, D3	$+1-0+8-7=2$
S1, D1	$+4-8+7-5=-2$
S3, D1	$+2-6+0-8+7-5=0$
S3, D2	$+3-6+0-8=-11$

The index for cell (S1, D1) and (S3, D2) has a negative index which indicates that further cost reduction (in this case profit improvement) is possible. Since cell (S3, D2) has the largest number with minus sign (-11), occupied this cell. Thus the total reduction of cost = **11Q** (where Q is the amount allocated to cell (S3, D2))

For the optimum assignment see the clothed path in the above table. The quantity 200 and 100 are assigned in the cell with minus sign. Therefore, select the smallest (i.e.100) to be assigned at cell (S3, D2). The revised allocation is shown below.

To From	D1	D2	D3	Supply
S1	5 50	7 50	1	100

Operations Research

S2	4	8	0	200
			200	
S3	2	3	6	200
		100	100	
Demand	50	150	300	500

The optimum total transportation cost can be calculated as follows:

<u>Route</u>	<u>unit cost</u>	<u>quantity shipped</u>	<u>cost</u>
S1, D1	5	50	250
S1, D2	7	50	350
S2, D3	9	200	0
S3, D2	3	100	300
S3, D3	6	100	<u>600</u>

We stated that the total transportation cost reduction is 11Q. Since Q=100, the total cost reduction is birr 1100.

Therefore, the optimum solution is 2600-1100= birr 1500.

Using this optimal allocation the *total profit* can be computed by using the original cell values (unit profits).

<u>Route</u>	<u>unit profit</u>	<u>quantity shipped</u>	<u>cost</u>
S1, D1	4	50	200
S1, D2	2	50	100
S2, D3	9	200	1800
S3, D2	3	100	300

S3, D3

6

100

600

Total cost = birr3000

At the optimum allocation total profit = birr3000

3.2. ASSIGNMENT PROBLEM

The assignment problem is a special case of Transportation problem in which the number of sources and destinations are the same, and the objective is to assign the given job(tasks) to most appropriate machine(person)so as to optimize the objective function like minimizing cost or maximizing profit.

The problem of assignment arise because available resources such as men, machine etc have varying degree of efficiency for performing different activities. Therefore cost, profit, or time of performing the different activities is different. Thus, the problem is: ‘How the assignment is made so as to optimize the given objective’.

Assumptions

number of jobs is equal to number of machines or persons

each man or machine is loaded with one and only one job

each man or machine is independently capable of handling any of the job being presented

Loading criteria must be clearly stated such as “minimizing operating time”, or “maximizing profit”, or “minimizing production costs” or “minimizing production cycle time” etc.

the data matrix of the assignment model is similar to transportation model except that the supply of each resource and demand at each of the destinations is taken to be one i.e. assignment is made on the bases of 1:1

3.2.1. Solution method for assignment problem

Usually the solution of assignment problem is calculated based on the principles of reducing the given cost matrix to a matrix of opportunity costs. Opportunity cost shows

the relative penalties associated with assigning resources to an activity as opposed to making the best (or least assignment).

The Hungarian method can be summarized in the following steps.

Step1. Develop the table from the given problem

If the numbers of rows are not equal to the number of columns and vice-versa, a dummy row or dummy column must be added. The assignment costs for dummy cells are always zero.

Step2. Find the opportunity cost table using row reduction and column reduction.

Procedures for row reduction:

- Identify the minimum value in each row

- Subtract the minimum value in each row from all the values in that row

- Use the resulting values to develop the new table

Procedure for column reduction

- identify the lowest cost in each column

- subtract the lowest cost value in each column from each of the values in that column

- use the resulting values to form a new table

Step3. Determine whether an optimal assignment can be made.

The technique for determining whether an optimal assignment is possible at this stage consists of drawing straight lines (vertically and horizontally) through the opportunity cost table in such a way as to minimize the number of lines necessary to cover all zero entries (i.e. draw a line through a column or row which has the largest number of zeros)

If the number of lines equals the number of either the number of rows or columns, an optimal assignment is made. If the number of lines less than the number of rows or

column, an optimal assignment cannot be determined, and the opportunity cost table must be revised.

Step4. Develop the new revised the opportunity cost table

If it is not possible to determine an optimal assignment in step 2, the opportunity cost table must be modified. This can be accomplished as follows:

From among the cells not covered by any line, chose the smallest elements. Call this value ' k '.

Subtract k from every element in the cell not covered by a line

Add k to every element in the cell covered by the two lines i.e. intersection of two lines.

Element in cells covered by one line remain unchanged.

Step5. You should repeat the process of finding the minimum number of covering lines.

Repeat steps 3 and 4 until number of lines equal number of rows or column.

Step 6. Make an assignment

If number of line is equal with either number of columns or rows, the optimum assignment is possible. Here, the question is “*how to assign?*”

To avoid confusion in making the assignments, begin by identifying a *row* or *column* that has only *one zero*, and make an assignment to that cell (box that cell). Since no other assignments can be made in this row or column (row or column which contain the assigned cell), we cross them off. With crossed off column and row, look for a row or column in which there is only one zero and assigned it. Continue until all zeros are either crossed or assigned.

Example1

The district court assigns four judges (A, B, C, and D) to four court dockets (i, ii, iii, and

~~iv). Estimated time (in days) each court will take to clear each docket are given below. AMU, CBE,~~

The objective of the assignment is to minimize the total time required to complete all of the cases on the four dockets. Based on the given data, determine the number of days each judge would take to clear each docket.

Estimated days to clear docket

	<u>Docket</u>			
<u>judge</u>	<u>i</u>	<u>ii</u>	<u>iii</u>	<u>iv</u>
A	14	13	17	14
B	16	15	16	15
C	18	14	20	17
D	20	13	15	18

Solution

Step1. Develop the assignment table

The number of rows = number of column=4. so no need of adding dummy row or column Step2. Find the opportunity time.

Row reduction: the minimum time (day) in row A, B, C, and D are: 13, 15 14, and 13 respectively. Subtract these elements in their respective row. This shows the additional time that would be incurred if the lowest time assignment is not made either in terms of rows or columns. For example, judge A can handle docket *ii* with the lowest processing time of 13 days. Therefore, if judge A were assigned docket *i*, the additional time would be, $14-13=1$ day. Similarly, if judge A were assigned docket *iii* and *iv*, the additional time would be $4(17-13=4)$ and $1(14-13=1)$. We can perform similar calculation for the other rows.

Original estimated days to clear dockets

estimated days after row reduction

<u>Docket</u>				row	<u>docket</u>				
judge	i	ii	iii	<i>minimum</i>	judge	i	ii	iii	iv

A	14	13	17	14	13
B	16	15	16	15	15
C	18	14	20	17	14
D	20	13	15	18	13

A	1	0	4	1
B	1	0	1	0
C	4	0	6	3
D	7	0	2	5

Column reduction: the procedure of column reduction is similar with row reduction.

Estimated days after column reduction

	<u>Docket</u>			
<u>judge</u>	<u>i</u>	<u>ii</u>	<u>iii</u>	<u>iv</u>
A	1	0	4	1
B	1	0	1	0
C	4	0	6	3
D	7	0	2	5

	<u>Docket</u>			
<u>judge</u>	<u>i</u>	<u>ii</u>	<u>iii</u>	<u>iv</u>
A	0	0	3	1
B	0	0	0	0
C	3	0	5	3
D	6	0	1	5

Column

Minimum 1 0 1 0

Step3. *Determine whether an optimal assignment can be made*

	<u>docket</u>			
	i	ii	iii	iv
A	0	0	3	1
B	0	0	0	0
C	3	0	5	3
D	6	0	1	5

Now compare the number of line with the number of columns or rows. Since the minimum number of covering lines (i.e. 3) is not equals with the number of rows or columns (i.e.4), at this stage optimum assignment is not possible.

(NB: note that there is another way of covering the first column by vertical line instead of the line drawn through the first row. This will not change the final result. Here the point is that, only three lines drawn would be needed.) Step4. Develop the new revised the opportunity cost table

Since number of line is not equal with number of rows/columns, we must revise the opportunity time. See the following:

From among the cells not covered by any lines (i.e.3, 6, 5, 13, & 5), chose the smallest element (i.e. 1). So $k=1$

Subtract **1** from every element in the cell not covered by a line

Add k to every element in the cell covered by the intersection of two lines i.e. cell(A, ii)=0+1 and cell (B, ii)=0+1

Element in cells covered by one line remain unchanged: i.e. cell (A, i) ,(A, iii), (A,vi), (B,i) ,(B, iii) (B,iv), (C, ii), and (D, ii)

The revised opportunity cost table is shown below

	<u>docket</u>			
	i	ii	iii	vi
A	0	1	3	1
B	0	1	0	0
C	2	0	4	2
D	5	0	0	4

Step5. You should repeat the process of finding the minimum number of covering lines.

Draw the line and compare it with the number of rows or columns.

We can also draw a line through row D instead of column iii.

Now the number of line is 4 which are equal with the number of row or columns. Therefore the next step is making the assignment.

Step6. Make an assignment

The procedure for identifying the assignment cell is to select the row or column in which there is only one zero. In this example, the candidates are: row **A**, row **C** & column **iv**.

Thus, first assign row **A** of column **i** with a box (). Since no other assignments can be made in row **A** and column **i** cross them with 'X' mark. With row **A** and column **i** crossed off, look for a row or column in which there is only one zero.

Then cell at row **B** column **iv** can be assigned, and cross row **B** and column **iv**. Next assign cell row **D** column **iii**. Finally assign row **C** column **ii**.

docket

	i	ii	iii	vi
A	0	1	3	10
B	0	1	0	0
C	2		4	2
D	5	0		4

Step7. Find the total days needed.

The total days associated with this solution is obtained by adding original day in the occupied cells as shown below.

<i>Final assignments</i>	<i>days</i>
Judge A to docket i	14
Judge B to docket iv	15
Judge C to docket ii	14
Judge D to docket iii	<u>15</u>

Total estimated days 58

Example 2

A computer centre has three expert programmers. The centre wants three application programs to be developed. The head of the computer centre, after studying carefully the programs to be developed, estimate the computer time in minutes required by the expert for the application programs as follows:

		<u>Programmers</u>		
		A	B	C
Programs	1	120	100	80
	2	80	90	110
	3	110	140	120

Required: assign the programmers to the programs in such a way that the total computer time is minimum.

Solution

Step1. Since numbers of rows are equal with number of column, dummy is not needed.

Step2. First conduct row reduction and column reduction

Estimated time row reduction

		<u>Programmers</u>			row	programmers		
		A	B	C	<u>minimum</u>			
Programs	1	120	100	80	80	Programs1	A	B C
	2	80	90	110	80		40	20 0
						2	0	10 30

AMU, CBE, Department of Management
Programs

3 110 140 120 110

3 0 30 10

Second conduct column reduction

Estimated time after column reduction

		Programmers'					programmers		
		A	B	C			A	B	C
Programs	1	40	20	0	Programs	1	40	10	0
	2	0	10	30		2	0	0	30
	3	0	30	10		3	0	20	10

Column

Minimum 0 10 0

Step3. *Determine whether an optimal assignment can be made*

		Programmers		
		A	B	C
Programs	1	40	10	0
	2	0	0	30
	3	0	20	10

At this stage optimal assignment is possible because the numbers of lines are equal with the number of row (columns)

Step5. Make an assignment

Since there is only one zero at row 1 assign this cell.

	A	B	C
Programs 1	40	10	0
2	0	0	30
3	3	20	10

Final assignments *minutes*

Program 1 to programmer C 80

Program 2 to programmer B 90

Program 3 to programmer A 110

Total 280

3.3. Special cases in assignment problem

1. Multiple optimal solutions

While making an assignment in the reduced assignment matrix, it is possible to have two or more ways to strike off a certain number of zeros. Such a situation indicates multiple optimal solutions with the same optimal value of objective function. In such cases the more suitable situation may be considered by the decision maker.

2. Maximization cases in assignment problem

There may arise situations when the assignment problem calls for maximization of profit, revenue, sales etc. as the objective function. In this case, one extra step must be added to start of the process: identify the largest value (pay off element) in the assignment table and then subtract all the elements of the table from the largest element.

Example

A marketing manager has three sales men and three sales districts. Considering the capabilities of the sales men and the nature of districts, the marketing manager estimates that cell per month (in '000'birrs) for each sales man in each district would be as follows:

		<u>Districts</u>		
		1	2	3
Sales men	A	14	22	30
	B	20	18	40
	C	11	12	50

Required: find the assignment of salesmen to districts that will result maximum sales.

Solution

First convert the given maximization problem in to minimization problem by subtracting all the elements of the table from the largest element (i.e. 50).

The new cost data (which is equivalent with maximization of the original data) is given below:

		<u>Districts</u>		
		1	2	3
Sales men	A	36	28	20
	B	30	32	10
	C	39	38	0

** First reduce the row*

**Table after row r deduction*

		Operations Research								
		<u>Districts</u>			row	<u>district</u>				
		1	2	3	minimum			1	2	3
Sales men	A	36	28	20	20	sale men	A	16	8	0
	B	30	32	10	10		B	20	22	0
	C	39	38	0	0		C	39	38	0

* Second reduce the column

Table before column deduction

		District		
		1	2	3
sale men	A	16	8	0
	B	20	22	0
	C	39	38	0

table after column reduction

		District		
		1	2	3
sale men	A	0	0	0
	B	4	14	0
	C	23	30	0

Column

Minimum 16 8 0

Then as usual, cover all zeros (from the table found after reducing the table) by lines as shown below.

District

		1	2	3
sale men	A	0	0	0
	B	4	14	0
	C	23	30	0

Since the number of table is less than the number of rows or columns, develop a revised opportunity table. To develop the revised table: add 4 to cell (A,3) and deduct 4 from uncovered elements i.e. cell (B,1), (B,2) (C,1) and (C, 2) and cover zeros by line.

Operations Research

		<u>District</u>		
		1	2	3
Sale	A	0	0	4
Men	B	0	10	0
	C	19	26	0

since the number of lines (3) is equal with the number of column or rows (3) optimum assignment can be possible at this stage.

		<u>District</u>		
		1	2	3
Sale	A	0	0	4
Men	B	0		
	C	19	26	0

* there is only one zero in row C. Thus assign this cell and cross row C and column 3.

* now row B has only one zero. Assign this cell and cross off row B and column 1.

* Finally assign row cell (A, 2)

Optimum assignments

sales (in '000')

Salesman A to district 2	22
Salesman B to district 1	20
Salesman C to district 3	<u>50</u>

Total 280 Thus, total sale = 280,000 birr.

unbalanced assignment problem

Any assignment problem can be solved if and only if the opportunity cost matrix is a square matrix i.e. the number of column must be equal with the number of rows. That is why solving the assignment problem begins with checking whether the matrix is square or ton.

If the given matrix is not a square matrix, the assignment problem is called unbalanced problem. In such cases, a dummy row(s) or column(s) are added in the matrix (with zero as a

~~cost elements) to make it a square matrix. After making the given assignment matrix AMU, CBE,~~

a square matrix, similar procedures can be used to solve the problem. See the Example given under next section (section 4)

Restriction on assignment

In certain instances, a particular matching or pairing may be either undesirable or otherwise unacceptable. For example, an employee may not have the skills necessary to perform a particular job or a machine may not be equipped to handle a particular operation. In this cases, the cost of performing that particular activity by a particular resources (say man, machine etc) is considered to be very large (written as M) so as to prohibit the entry of this pair of resource- activity in to the final solution.

Example

In the modification of a plant lay out of a factory 4 new machines M1; M2, M3 and M4 are to be installed in a machine shop. There are 5 vacant places A, B, C, D and E available. Because of limited space, machine M2 cannot be placed at C and M3 cannot be placed at A. The cost of locating a machine at a place (in '00's of birr) is as follows:

		<i>Location</i>				
		A	B	C	D	E
<i>Machine</i>	M1	9	11	15	10	11
	M2	12	9	--	10	9
	M3	--	11	14	11	7
	M4	14	8	12	7	8

Required: find the optimal assignment schedule and the minimum cost.

Solution

* *First balance the problem*

Since the number of rows is not equal with number of column, add one dummy row to get a square cost matrix.

Second assign a high cost, denoted by M to the pair (M2, C) and (M3,A). The cost matrix so obtained is shown below.

	<u>Location</u>				
	A	B	C	D	E
M1	9	11	15	10	11
M2	12	9	M	10	9
M3	M	11	14	11	7
M4	14	8	12	7	8
M5	0	0	0	0	0

Now apply the assignment procedures to find the optimum assignment i.e. first reduce the row and then the column. Next draw the line to cover zeros. If all zeros are covered by a line and the numbers of lines are equal with the number of rows or columns, assign the cell. See the following.

	<u>Location</u>				
	A	B	C	D	E
M1	9	11	15	10	11
M2	12	9	M	10	9
M3	M	11	14	11	7
M4	14	8	12	7	8
M5	0	0	0	0	0

	<u>location</u>				
	A	B	C	DE	
M1	0	2	6	1	2
M2	3	0	M	1	0
M3	M	4	7	4	0
M4	7	1	5	0	1
M5	0	0	0	0	0

Since the number of lines (5) is equal with the number of rows or columns (5), the optimal solution can be obtained at this stage.

	<u>Location</u>				
	A	B	C	D	E
M1	0	2	6	1	2
M2	3	0	M	1	0
M3					0
M4	M	4	7	4	0
M5	7	1	5		1

The optimum assignment and the total minimum costs are:

<u>optimum assignment</u>	<u>cost (in '00')</u>
MachineM1 to location A	9
MachineM2 to location B	9
MachineM3 to location E	7
MachineM4 to location D	7
MachineM5 to location C	0

Total cost 32 birr

Thus the total cost is 3200 birr.

We examine all the rows starting from row 'A' one-by-one until a row containing only a single zero element is located (assigned). Here rows 'C' and 'D' have only one zero element in the cell (C ,ii) and (D, ii). Assignment is made first at cell (C, ii) and these cells. All zeros in the assigned columns are now crossed off as shown below

second examine the column

We now examine each column starting from column1 (start from the first column).there is one zero in column iii and column iv in the cell (A, iii) and (A, iv). Assignment is made in these cells. Thus, cell

If the numbers of rows are not equal to the number of columns and vice-versa, a dummy row or dummy column must be added. The assignment costs for dummy cells are always zero.

Step2. Find the opportunity cost table using row reduction and column reduction.

Procedures for row reduction:

Identify the minimum value in each row

Subtract the minimum value in each row from all the values in that row

Use the resulting values to develop the new table

Procedure for column reduction

identify the lowest cost in each column

subtract the lowest cost value in each column from each of the values in that column

use the resulting values to form a new table

Step3. Make assignment in the opportunity cost matrix.

The procedure for making assignment is as follows:

Examine the rows successively until a row with exactly one unmarked zero is obtained. Make an assignment to this single zero by making square () around it.

For each zero value that becomes assigned, eliminate (strike off) all other zeros in the same row and /or column

Repeat these two steps for each column and also with exactly single zero value cell that has not been assigned

If a row and/or column has two or more unmarked zeros and one cannot be chosen by inspection, then choose the assigned zero cell arbitrarily

Continues these process until all zeros in rows/columns are either enclosed (assigned) or struck off with 'X' mark.

Step4. Optimality criteria

If the number of assigned cells is equal to the number of rows /columns, then it is an optimal solution. The total cost associated with this solution is obtained by adding original cost figures in the occupied cells. If a zero cell was chosen arbitrarily in step 3, there exists an alternative optimal solution. But if no optimal solution is found, then go to step 5.

Step5. Revising the opportunity table

Draw a set of horizontal and vertical lines to cover all the zeros in the revised cost table obtained from step 3 by using the following procurers:

For each row in which no assignment was made, mark a tick (✓)

Examine the marked rows. If any zero cell occurs in those rows, mark a ✓ to the respective columns that contain those zeros.

Examine the marked columns. If any assigned zero occurs in those columns, tick the respective rows that contain those assigned zeros.

Repeat this process until no more rows or columns can be marked.

Draw a straight line through each marked column and each unmarked row.

If the number of lines drawn (or total assignment) is equal to the number of rows (or rows) the current solution is the optimal solution, otherwise, go to step 6.

Step6. Develop the new revised opportunity cost table

From among the cells not covered by any line, chose the smallest elements. Call this value ' k '.

Subtract k from every element in the cell not covered by a line

Add k to every element in the cell covered by the two lines i.e. intersection of two lines.

Element in cells covered by one line remain unchanged.

Step7. Repeat steps

Repeat steps 3 to 6 until an optimal solution is obtained.

Example1

The district court assigns four judges (A, B, C, and D) to four court dockets (i, ii, iii, and iv). Estimated time (in days) each court will take to clear each docket are given below. The objective of the assignment is to minimize the total time required to complete all of the cases on the four dockets. Based on the given data, determine the number of days each judge would take to clear each docket.

Estimated days to clear docket

	<u>Docket</u>			
<u>judge</u>	<u>i</u>	<u>ii</u>	<u>iii</u>	<u>iv</u>
A	14	13	17	14
B	16	15	16	15
C	18	14	20	17
D	20	13	15	18

Solution

Step1. Develop the assignment table

The number of rows = number of column=4. so no need of adding dummy row or column

Step2. Find the opportunity time.

Row reduction: the minimum time (day) in row A, B, C, and D are: 13, 15, 14, and 13 respectively. Subtract these elements in their respective row. This shows the additional time that would be incurred if the lowest time assignment is not made either in terms of rows or columns. For example, judge A can handle docket *ii* with the lowest processing time of 13 days. Therefore, if judge A were assigned docket *i*, the additional time would be, $14-13=1$ day. Similarly, if judge A were assigned docket *iii* and *iv*, the additional time would be $4(17-13=4)$ and $1(14-13=1)$. We can perform similar calculation for the other rows.

Original estimated days to clear dockets *estimated days after row reduction*

	<u>Docket</u>						<u>docket</u>				
	<u>judge</u>	<u>i</u>	<u>ii</u>	<u>iii</u>	<u>iv</u>	<u>minimum</u>	<u>judge</u>	<u>i</u>	<u>ii</u>	<u>iii</u>	<u>iv</u>
A		14	13	17	14	13	A	1	0	4	1
B		16	15	16	15	15	B	1	0	1	0
C		18	14	20	17	14	C	4	0	6	3
D		20	13	15	18	13	D	7	0	2	5

Column reduction: the procedure of column reduction is similar with row reduction.

Estimated days after column reduction

	<u>Docket</u>						<u>Docket</u>				
	<u>judge</u>	<u>i</u>	<u>ii</u>	<u>iii</u>	<u>iv</u>		<u>judge</u>	<u>i</u>	<u>ii</u>	<u>iii</u>	<u>iv</u>
A		1	0	4	1		A	0	0	3	1
B		1	0	1	0		B	0	0	0	0

C 4 0 6 3

D 7 0 2 5

C 3 0 5 3

D 6 0 1 5

Column

Minimum 1 0 1 0

Step3. Make an assignment in the opportunity time matrix

first examine the row

We examine all the rows starting from row 'A' one-by-one until a row containing only a single zero element is located (assigned). Here rows 'C' and 'D' have only one zero element in the cell (C, ii) and (D, ii). Assignment is made first at cell (C, ii) and, all zeros in the assigned column and now must be crossed off.

second examine the column

We now examine each column starting from column1 (start from the first column and then to the next). In this example, there is one zero in column *iii* and column *iv* in the cell (A, iii) and (A, iv). Assignment is made in these columns (first column *iii* as per their order and then the next column). Then, all zeros in the assigned cells column and row must be crossed off. See the following.

Estimated days after column reduction

<u>Docket</u>				
	<u>i</u>	<u>ii</u>	<u>iii</u>	<u>vi</u>
A	0	0	3	1
B	0	0	0	0
C	3	0	5	3
D	6	0	1	5

Chapter Summery

The transportation problem is a particular case of linear programming problem. The structure of transportation problem involves a large number of shipping routes from several supply origins to several demand destinations.

In order to find initial solution of a transportation methods the following methods used:

- ✓ North- west corner method
- ✓ Least cost method, and
Vogel's approximation (or penalty) method.

The north –west corner method is a systematic approach for developing an initial basic feasible solution. Its *chief advantages* are that it is simple to use and easy to understand.

List cost approach, also called the minimum –cost method, uses lowest cell as the basis for selecting route.

Vogel's approximation (penalty or regret) method is a heuristic method and is preferred to the other two methods described above.

The assignment problem is a special case of Transportation problem in which the number of sources and destinations are the same, and the objective is to assign the given job (tasks) to most appropriate machine (person) so as to optimize the objective function like minimizing cost or maximizing profit.

Solution of assignment problem is calculated based on the principles of reducing the given cost matrix to a matrix of opportunity costs. Opportunity cost shows the relative penalties associated with assigning resources to an activity as opposed to making the best (or least assignment).

Review exercise

Suppose that a firm has three factories / *sources of supply* / & four warehouses/point of demand/. The firm's production capacity at the three factories, the demand for the four destination centers located at various regions & the cost of shipping each unit from the factories to the warehouses through each route is given as follows:

Destinations

	W_1	W_2	W_3	W_4	<i>Factory Capacity</i>
F_1	3	2	7	6	5000
F_2	7	5	2	3	6000
F_3	2	5	4	5	2500
Demand	6000	4000	2000	1500	13500

Required:

- a. Develop an initial feasible solution using NWCM & Compute the total cost
- b. Develop an initial feasible solution using least-cost method & compute the total cost.

Three garment plants are available for monthly production of four styles of men's shirts. The capacities of the three plants are 45,000, 93,000 and 60,000 shirts. The number of shirts required in style "a" through "d" is 28,000, 65,000, 35,000 & 70,000, respectively. The profits, in \$ per shirt, at each plant for each style are shown below.

Table: The garment plants' profit.

STYLE PLANT	a	b	c	D
1	8	12	-2	6
2	13	4	3	10
3	0	7	11	8

How many shirts of each type to produce in each plant so that profit is maximized?

A car rental company has one car at each of five depots **a**, **b**, **c**, **d** and **e**. A customer in each of the five towns **A**, **B**, **C**, **D** and **E** requires a car. The distance in (in kilometers) between the depots and towns where the customers are, is given in the following distance matrix:

Depots

	<i>a</i>	<i>B</i>	<i>c</i>	<i>d</i>	<i>e</i>
<i>A</i>	160	130	175	190	200
<i>B</i>	135	120	130	160	175
<i>C</i>	140	110	155	170	185
<i>D</i>	50	50	90	80	110
<i>E</i>	55	35	70	80	105

How should the cars be assigned to the customers so as to minimize the distance traveled?

Chapter 4

Decision theory

Learning objectives:

After completing this chapter, you should be able to:

Outline the characteristics of a decision theory to decision making

Describe and give example of decision making under the three conditions (certainty, risk and complete uncertainty)

Construct and understand the pay off table

Make conclusion by using maximin, maximax, minimax regret, insensit reason and expected value criteria.

Find out the expected value of perfect information

4.1. Introduction

Making appropriate decision is the most vital aspects in management. Infact certain authors have defined management as decision making. Decision making is an action. Every action has a reaction. Some decision initiates a set of activities; some put an end a certain activities. Some decision can be withdrawn without any consequential actions and losses, but majority of the decisions are not. Therefore, the success or failure of an individual or organization experiences, depends to large extent on the ability of making appropriate decisions

Making an appropriate decision requires an enumeration of feasible and viable alternatives (courses of actions or strategies), the projection of consequence associated with different alternatives, and a measure of effectiveness (or an objectives) by which the most preferred alternative is identified.

Decision theory provides an analytical and systematic approach to the study of decision making. Decision models useful in helping decision makers make the best possible decisions are classified according to the degree of certainty. The scale of certainty can range from complete certainty to complete uncertainty. The region which falls between these two extreme points corresponds to the decision making under risk (probabilistic problems).

Irrespective of the types of decision model, there are certain essential terms which are common in decision making. Let us define these terms:

Decision alternatives: these are *act, action, option, or strategies* available for decision maker before any decision is made. Decision alternatives are *under the control and known to* decision maker.

State of nature: a possible future condition (*consequence or event*) resulting from the choices of a decision alternatives depends up on certain factors *beyond the control of* decision maker. These factors are depends on the environment. Decision maker can at the most attach subjective probability of occurrence of such events.

Payoff: a numerical value resulting from each possible combination of *alternatives and state of nature* is payoff. The payoff might be profits, revenues costs, or other measures of values. The payoff values are always conditional values because of unknown state of nature.

Degree of certainty. There can be different degree of certainty. One extreme is complete certainty and the other is complete uncertainty. The latter exists when the likelihood of the various state of are unknown. Between these two extremes is risk, a term that implies that probabilities are known for the state of nature. We will see these in detail in the next section.

A tabular arrangement of these conditional outcome (payoff) values is known as payoff matrix (or decision table or payoff table). Rows represent the state of nature and columns represent decision alternatives to be considered.

4.2. Steps in decision making theory

The decision making process involves the following steps:

Step1. Identify and define the problem

Step2. List all possible future events (state of nature) which are beyond the control of the decision maker

Step3. Identify all courses of actions (alternatives) which are under the control of the decision maker

Step4. Express the payoff resulting from each pair of course of action and state of nature. Normally payoffs are expressed in a monetary value.

Step5. Apply appropriate decision theory model to select the best courses of action.

4.3. The payoff table

A tabular arrangement conditional outcome (payoff) values is known as payoff matrix (or decision table or payoff table). It includes a list of the alternatives, the possible state of nature, and the payoff associated with each of the alternatives-state of nature combinations. Conventionally, rows represent the state of nature and columns represent decision alternatives to be considered. Please note that pay off can be *profits, revenues costs, or other measures of values*.

Example 1

A firm manufactures three types of products (A, B, C). The fixed cost and the variable cost of producing each product are given below.

	<u>Fixed cost</u>	<u>variable cost</u>
Product A	25000	12
Product B	35,000	9
Product C	53,000	7

The likely demand (units) of the products is given below.

Poor demand	3000
Moderate demand	7000
High demand	11,000

The sales price of each type of product is birr 25.

Required: prepare the payoff table.

Solution

In this case the payoff is expressed in profit. So first calculate the profit for each alternative. There are three products and three demands. Therefore, there are 9(3x3=9) pairings or pay off cells.

Let D1, D2, and D3 be poor, moderate, and high demand respectively, then payoff is given by:

$$\text{Payoff} = \text{sales revenue} - \text{cost}.$$

The calculation for payoff (in '000's) for each pair of alternative demand (course of action) and the types of product (state of nature) are shown below:

$$\text{Payoff} = (p \ q) - [(v \ q) + FC]$$

$$D1,A = (3000 \times 25) - [(12 \times 3000) + 25000] = 14000$$

$$D1,B = (3000 \times 25) - [(9 \times 3000) + 35000] = 13000$$

$$D1,C = (3000 \times 25) - [(7 \times 3000) + 53000] = 1000$$

$$D2,A = (7000 \times 25) - [(12 \times 7000) + 25000] = 66000$$

$$D2,B = (7000 \times 25) - [(9 \times 7000) + 35000] = 77000$$

$$D2,C = (7000 \times 25) - [(7 \times 7000) + 53000] = 73000$$

$$D3,A = (11000 \times 25) - [(12 \times 11000) + 25000] = 118000$$

$$D3,B = (11000 \times 25) - [(9 \times 11000) + 35000] = 141000$$

$$D3,C = (11000 \times 25) - [(7 \times 11000) + 53000] = 145000$$

Thus, the pay off values in the payoff table (in '000's) is shown below.

Product type (State of nature)	demand (alternative courses of action)		
	D1	D2	D3
Product A	14	66	118
Product B	13	77	141
Product C	1	73	145

4.4. Decision making environment

Knowledge of the likely hood of each of the state of nature can play an important role in selecting a course of action. Generally there are three decision making environments: certainty, uncertainty and risk.

4.4.1. Decision under certainty

Decision under certainty is the case where the decision maker has a perfect knowledge (information) about the state of nature. This situation does not need any techniques or managerial tool for the manager to take his decision. In this case, the decision maker will simply select an alternative that yields the largest return (payoff) for the known future (state of nature). However, in practice, decision under certainty is very rare. So this situation is not of our interest and we left it for the reader.

4.4.2. Decision making under uncertainty

This is a situation in which decision maker has neither previous knowledge of the outcomes of his decisions, nor is in a position to attribute a probability of occurrence of such outcomes. Still various options are opened to you to make decision in the absence of perfect knowledge which depends on a variety of criteria. In this section we will introduce you only four approaches for decision making under uncertainty.

a) maximaxi or minimini(criteria of optimism)

In this section, the decision maker ensures that he should not miss the opportunity to achieve the largest possible profit (maxmaxi) or lowest possible cost (minimini). Thus, he selects the alternatives that represent the maximum of the maxima (or minimum of the minima) payoffs (outcomes).

The working method is summarized as follows:

Locate the maximum (or minimum) payoff values corresponding to each alternative.

Then,

Select an alternative with best anticipated payoff value(maximum for profit and minimum for cost)

Example 2

To illustrate the maximaxi criteria, let us refer the example given under *Example 1*.

Required: based on Example1, which strategy (decision alternative) should the manufacturing company adopt?

Solution

In order to determine the strategy (alternative action) first determine the maximum value for each column. Then, the column with the largest value is considered as the best strategy.

<i>Product type</i>	<i>demand (alternative courses of action)</i>		
<i>(State of nature)</i>	<i>D1</i>	<i>D2</i>	<i>D3</i>
Product A	14	66	118
Product B	13	77	141
Product C	1	73	145
<i>Column maximum</i>	14	77	145
			↑ Maximaxi

The maximum of column maxima is 145. Hence the manufacturing company should adopt alternative D3 (sale in the high demand).

b) Maximin or minimaxi(criteria of pessimism).

In this criterion, the decision makers ensures that he would earn no less or (pay no more) than some specified amount. Thus he selects the alternative that represents the maximum of the minima (in case of profit) and the minimum of the maximum (in case of loss). The application is simple. First select the minimum from each column and select the largest values from the *minimum column* rows.

Example3

<i>Product type</i>	<i>demand (alternative courses of action)</i>		
<i>(State of nature)</i>	<i>D1</i>	<i>D2</i>	<i>D3</i>
Product A	14	66	118
Product B	13	77	141
Product C	1	73	145
<i>Column minimum</i>	<i>1</i>	<i>66</i>	<i>118</i>

maxmin

c) Laplace criteria (equally likely or insufficient reason)

Since the probabilities of state of nature are not known, it is assumed that all state of nature will occur with equal probability, i.e. each state of nature is assigned an equal probability.

The working method is summarized as follows:

- compute the expected(average) value payoff for each alternative by adding all the pay offs and dividing by the number of possible states of nature
- Select the best expected payoff value (maximum for profit or minimum for cost).

minimax regret

This criteria also called opportunity loss decision criteria or savage criterion because decision makers feels regret after adopting a wrong course of action (alternative) resulting an opportunity loss of payoff. Thus, the decision maker always intends to minimize this regret. The work method is as follows:

- from the given pay off matrix, develop an opportunity loss(or regret)matrix as follows:

Subtract all other entries (payoff values) in that row from this value.

For each course of action(alternative) identify the worst (maximum regret value)

Select the course of action with the smallest anticipated opportunity loss value. For better understanding of the concept let us see the following Example

Example

Given the following payoff tables, determine which alternatives would be chosen using each of these decision criteria.

- i. maximaxi, ii) maximin iii) minimax regret iv) laplace criterion

State of nature	alternative			
	a	b	c	d
1	12	17	22	14
2	18	10	16	14
3	15	14	10	14

Solution

- i. The maximaxi approach to select the best alternative using maximaxi approach; first select the maximum values from each column. Then, the best strategy is the minimum column row with the highest values. See the following computation.

Strategy				
State of nature	(alternative)			
	a	b	c	d
1	12	17	22	14
2	18	10	16	14

State of nature	(alternative) Strategy			
	a	b	c	d
1	(22-12=10)	(22-17=5)	(22-22=0)	(22-14=8)
2	(18-18=0)	(18-10=8)	(18-16=2)	(18-14=4)
3	(15-15=0)	(15-14=1)	(15-10=5)	(15-14)=1)

Thus, the regret (opportunity loss) table is:

State of nature	(alternative) Strategy			
	a	b	c	d
1	10	5	0	8
2	0	8	2	4
3	0	1	5	1

Worst regret (Column 10 8 5 8
↑
Maximum)

Minimum regret

The minimum value from the worst regret row is 4. Thus select strategy C.

iv. Laplace criterion

To select the best alternative using Laplace approach, first we must determine the average payoff for each alternative. Then select the minimum values from each column. Then, the

best strategy is the column with the highest values. To illustrate the laplace criterion application, let us refer the above Example once again.

		Strategy			
State of		(alternative)			
nature		a	b	c	d
1	12	17	22	14	
2	18	10	16	14	
3	15	14	10	14	
Average payoff		$(12+18+15)/3$	$(17+10+14)/3$	$(22+16+10)/3$	$(14+14+14)/3$
		= 15	=13.67	=16	=14
				↑	
				Best payoff	

Since the largest payoff is obtained from alternative C, select this strategy.

Hurwicz (criterion of realism)

4.4.3. Decision making under risk

In this case also the decision maker has to make decision when outcomes are not certain. However, unlike the previous case, he has sufficient information to assign probability values to the likely occurrence of each of the states. Knowing the probability distribution of the state of nature, the best decision is to select that course of action (decision alternatives) which has the largest expected payoff value.

The most widely used criterion for evaluating various courses of action (alternatives) under risk includes: expected monetary value (EMV), expected opportunity loss (EOL), expected value of perfect information (EVPI).

Now let us see each with Examples

expected monetary value

Expected monetary value is the sum of the payoffs for each course of action multiplied by the probabilities associated with each state of nature.

Steps for calculating EMV

Construct a payoff matrix listing all possible courses of action and state of nature. Enter the conditional payoff values associated with each possible combination of courses of action and state of nature along with the probabilities of the occurrences of each state of nature.

Calculate the EMV for each course of action by multiplying the conditional payoffs by the associated probabilities and add these weighted values for each course of action.

Select the course of action that yields the optimal values.

Example

Mr. Abebe quite often flies from Addis Ababa to Debrezeit. He can use the airport bus but it costs him birr 25 and there is a 0.8 chance that he will miss the bus. He can also stay in hotel but it costs him birr 270 with a 0.96 chances of being on time for the flight. Still he can use Taxi which costs him birr 350 with a 0.99 chance of being on time for flight. If Mr. Abebe catches the plane, he will conclude a business transaction which will produce a profit of birr 10,000 otherwise, he will lose it.

Required: using EMV method, which mode of transportation should Mr. Abebe use?

Solution

Computations of EMV for the three courses of action are shown below.

State of nature	<u>courses of action</u>		
	<u>Bus</u>		
	Cost	probability	expected value
Catch the flight	$10,000 - 25 = 9975$	$1 - 0.8 = 0.92$	$9975 \times 0.92 = 9177$
Miss the flight	-25	0.8	$-25 \times 0.8 = -2$
EMV	$9177 - 2 = \text{birr}$	9175	

State of nature	<u>courses of action</u>		
	<u>Stay in hotel</u>		
	Cost	probability	expected value
Catch the flight	$10,000 - 270 = 9730$	0.96	$9730 \times 0.96 = 9341$
Miss the flight	-270	$1 - 0.96 = 0.04$	$-270 \times 0.04 = -11$
EMV	$9341 - 11 = \text{birr}$	9330	

State of nature	<u>courses of action</u>		
	<u>Taxi</u>		
	Cost	probability	expected value
Catch the flight	$10,000 - 350 = 9650$	0.99	$9650 \times 0.99 = 9553.50$
Miss the flight	-350	$1 - 0.99 = 0.01$	$-350 \times 0.01 = -3.50$
EMV	$9553 - 3.50 = \text{birr}$	9550	

Thus, comparing the EMV of the three courses of action, the alternative with the highest EMV is the third alternative i.e. using Taxi. So Mr. Abebe must use taxi to maximize his expected monetary value.

i. Expected opportunity loss (EOL)

An alternative approach to maximize EMV is to minimize the EOL. This approach is nearly identical to the EMV approach except that a table of opportunity losses used rather than a table of payoffs. Hence the opportunity losses for each alternative are weighted by the probabilities of their respective state of nature to compute the EOL. The alternative with the smallest EOL is selected as the best choice.

Example

A company manufactures goods for a market in which the technology of the product is changed very rapidly. The R&D department has produced a new product which appears to have potential for commercial exploitation. A further birr 60,000 is required for development testing. The company has 100 customers and each customer might purchase at the most one unit of the product. Market research suggests that a selling price of birr 6000 for each unit with total variable costs and selling estimates are birr 2000 for each unit.

From the previous experience, it has been possible to derive a probability distribution relating to the proportion of customers who will buy the product as follows:

Proportion of customers	0.04	0.08	0.12	0.16	0.20
Probability	0.10	0.10	0.20	0.40	0.20

Required: given the above data,

show the expected opportunity loss value using the opportunity loss table
state whether or not the company should develop the product

Solution

Given: selling price= birr 6000, variable cost = birr 2000,

Number of unit sold=100p, cost of testing 60,000

Let 'p' be the proportion of customers who purchase the product,

S1 be *develop the new product* alternative

S2 be *do not develop the new product*

N state of nature, then

First determine the conditional profit payoff as follows

The conditional profit is= (contribution to profit X number of unit sold) -
development test

$$(6000-2000) \times (100p) -$$

$$60000 = 400000p - 60,000$$

The conditional profit values (payoffs) for each pair of state of nature (proportion of customers who will buy the product) and alternative (to develop or not to develop the new product) is shown in the following table.

Operations Research

(Proportion of customers, p)	<u>conditional profit = 400000p - 60,000</u>	
	<u>courses of action</u>	
	S1	S2
	(Develop)	(Do not develop)
0.04	-44,000	0
0.08	-28000	0
0.12	-12000	0
0.16	4000	0
0.20	20,000	0

<i>State of nature</i>					
(Proportion of customers, p)		<u>Conditional profit</u>		<u>opportunity loss</u>	
State of nature	prob.	S1	S2	S1	S2
0.04	0.1	-44,000	0	44,000	0
0.08	0.1	-28000	0	28000	0
0.12	0.2	-12000	0	12000	0
0.16	0.4	4000	0	0	4000
0.20	0.2	20000	0	0	20,000

Using the given estimate of probabilities of associated with each state of nature, the expected opportunity loss (EOL) for each alternative is given below:

**EOL of developing the product (S1) is:*

$$0.1(44000) + 0.1(28000) + 0.2(12000) + 0.4(0) + 0.2(0) = \text{birr } 9600$$

**EOL of do not develop the product (S2) is:*

$$0.1(0) + 0.1(0) + 0.2(0) + 0.4(4000) + 0.2(20,000) = \text{birr}5600.$$

Since the company wants to minimize the expected opportunity loss, it should select courses of action S2 (do not develop the product) with minimum EOL.

Expected value of perfect information(EVPI)

In decision making under risk each state of nature is associated with the probability of its occurrence. However, if the decision maker can acquire perfect (complete and accurate) information, about the occurrence of various state of nature, then he will be able to select a course of action that yields the desired payoff for whatever state of nature that actual occurs.

EVPI represents the maximum amount of money the decision maker has to pay to get information about the occurrence of various state of nature before a decision has to be made.

Example

A company needs to increase its production beyond its existing capacity. It has narrowed the alternatives to two approaches to increase the production capacity: expansion, at a cost of birr 8 million or modernization at a cost of birr5 million. Both approaches would require the same amount of time for implementation. Management believe that over the required payback period, demand will either be high or moderate. Since high demand is considered to be somewhat less likely than moderate demand, the probability of high demand has been set at 0.35. If the demand is high, expansion would gross an estimated additional birr 12 million but modernization only an additional birr 6 million, due to lower maximum production capability. On the other hand, if the demand is moderate, the comparable figure would be birr 7 million for expansion and birr 5 million for modernization.

Required: given the above data:

calculate conditional profit

If the company wishes to maximize EMV, should it modernize or expand?

Calculate EVPI

Construct the conditional opportunity loss table and also calculate EOL.

Solution

Given: - state of nature: high demand (N1) and moderate demand (N2)

-alternative action (courses of action): expand (S1) and modernize (S2)

The calculation for conditional profit values are shown below:

NB: since the probability that the demand is high estimated at 0.35, the probability of moderate demand must be $(1-0.35) = 0.6$

State of nature (Demand)	<u>conditional profit (in '000000's</u>	
	expand (S1)	modernize (S2)
High demand (N1)	$12-8=4$	$6-5=1$
Moderate demand (N2)	$7-8=-1$	$5-5=0$

The calculation of EMV for each courses of action S1 and S2 is given below:

$$\text{EMV (S1)} = 0.35(4) + 0.65(-1) = 0.75\text{million}$$

$$\text{EMV (S2)} = 0.35(1) + 0.65(0) = 0.35\text{milliun}$$

To maximize EMV, the company must choose courses of action S1 (expand). Thus, the EMV=0.75million

To calculate EVPI, we shall first calculate expected payoff with perfect information (EPPI). To calculate EPPI, we chose optimal courses of action for each state of nature , multiply its conditional profit by the given probability to get weighted profit, and then sum these weights as shown below:

State of nature (Demand)	probability	optimal course of action	conditional profit	weighted profit
High demand	0.35	S1	4	$4 \times 0.35 = 1.40$
Moderate DD	0.65	S2	0	$0 \times 0.65 = 0$
				EPPI = 1.40

The optimal EMV is 0.75million corresponding to the courses of action S1. Then,

$$EVPI = EPPI - EMV (S1)$$

$$1.40 - 0.75$$

$$= \text{birr } 0.65 \text{ million}$$

The opportunity loss values are shown below:

State of nature (Demand)	probability	<u>conditional</u>		<u>profit</u>	<u>conditional</u>	<u>loss</u>
		S1	S2	S1	S2	
High demand	0.35	4	1	0	3	
Moderate DD	0.65	-1	0	1	0	

$$EOL(S1) = 0.35(0) + 0.65(1) = 0.65 \text{ million}$$

$$EOL(S2) = 0.35(3) + 0.65(0) = 1.05 \text{ million}$$

Since decision makers seek to minimize the EOL, he must select alternative S1.

Decision threes

In the discussion of decision problems until now our concern has been with the single stage a problem where in the decision maker has to select the best course of section on the basis of whatever information is achievable at appoint of time. We shall now consider the decision situations that involve multiple stages. Also called the sequential decision problems, they are characterized by a sequence of decisions in which following each decision, a chance event occurs which in turn affects the next decision.

In analyzing multiple stage decision situations we have to evaluate the decision procedure in a backward manner by evaluating the best course of action at the later stages to decide the best action at the earlier stages. For this purpose the decision tree or the decision flow diagram as it is sometimes called is a very effective device.

A decision tree is a graphic representation of the sequence of action event combination available the decision maker. It depicts in a systematic manner all possible sequences of decisions and consequences. Each such sequence is shown by a distinct path through h the tree. A decision tree enables the decision maker to see the various element of this

problem in proper perspective and in a systematic manner. It may be mentioned that the criterion on the basis of which the decision are made in the decision tree approach is generally the expectation principle. Thus we may choose the alternative that maximizes the expected profit, or the alternative that minimizes the expected cost... and so on.

Example1.

An oil company has recently acquired rights in a certain area to conduct surveys and test drillings to lead to lifting oil if it is found in commercially exploitive quantities. The area is considered to have good potential for finding oil in commercial quantities. At the outset the company has the choice to conduct further geological tests or to Carry out a drilling program immediately. On the known conditions, the company estimates that there is a 70:30 chance of future tests showing a success.

Whether a tests show the possibility of ultimate success or not or even if no test are undertaken at all, the company could still pursue its drilling program or alternatively consider selling its rights to drill in the area. Therefore, However, if it carries out the drilling program, the likely hood of final success or failure is considered depends on the forgoing stages. Thus;

If successful tests have been carried out the expectation of success in drilling is given as 80:20.

If the test indicates failure, then the expectation of success in drilling is given as 20:80.

If no tests have been carried out at all the expectation of success in drilling is given as 55:45.

Costs and revenues have been estimated for all possible outcomes and the net present value of each is as follows:

Outcomes	Net present value (birr million)
Success	
With prior list	100
Without prior list	120
Failure	
With prior list	-50
Without prior list	-40

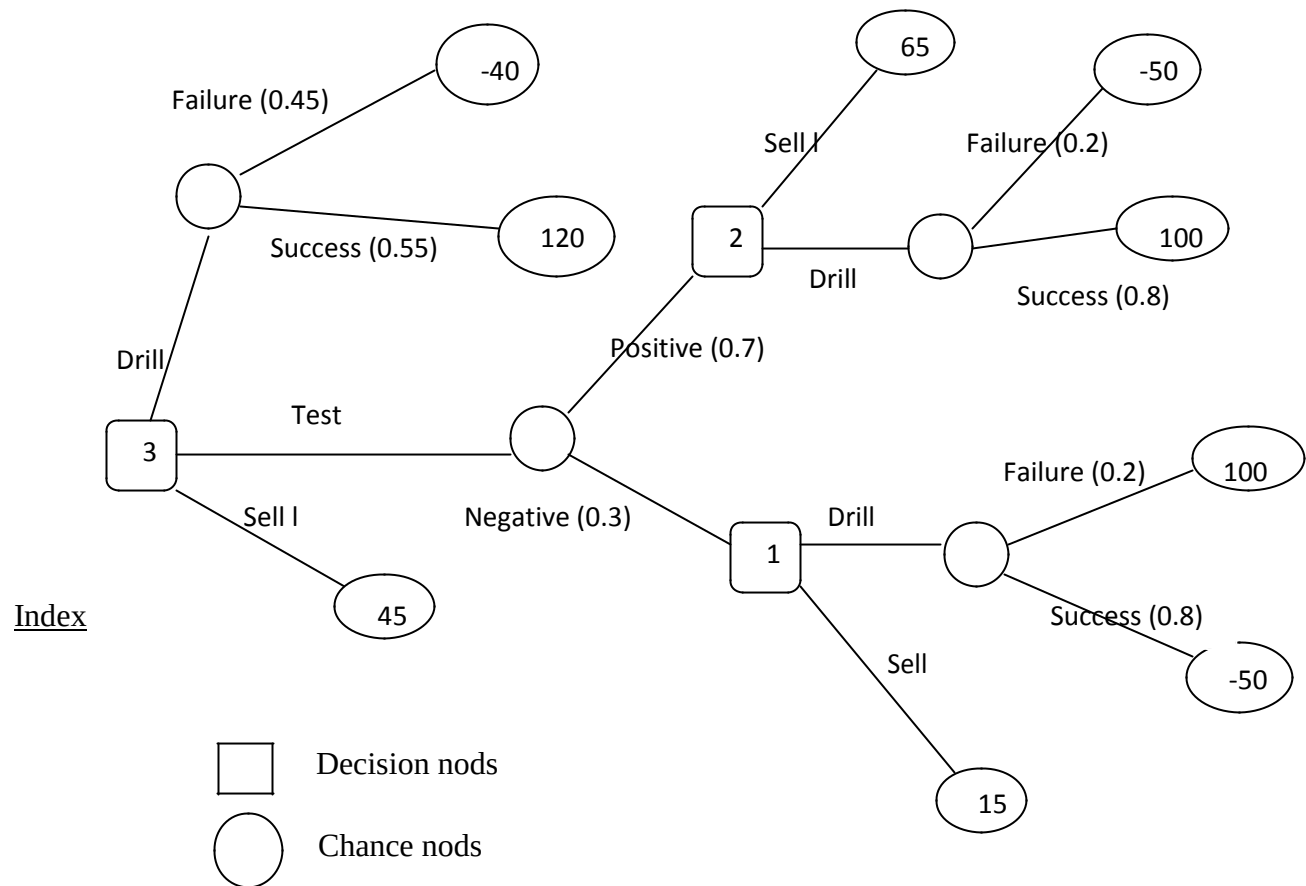
Sale of exploitation right:

Prior test show success

Prior test show failure

Without prior test

Draw up a decision (probability) tree diagram to represent the above information: and
Evaluate the tree in order to advice the management of the company on its best course of
action.



As you observe that the tree has several branches which originate from squares or from circles. A square represents a decision node or decision fork at which the decision maker has to take a decision, while a circle represents a chance node or chance fork at which events (i.e. the state of nature) are branched out.

The method of solution: as mentioned earlier the decisions have to be evaluated in a backward manner by evaluating the best course at the later stages so as to decide on the best course of actions on the early stage. Thus, the solution to the problem is obtained by working backwards, from right to left, through the tree.

Operations Research

Evaluation of decision node 1

Alternatives	Out come	Probability	Conditional value	Expected value
1. Drill	Success	0.2	100	20
	Failure	0.8	(-50)	(40)
	Total			(20)
2. Sell		1.0	15	15

The expected value of the drilling option is a loss of Birr 20 million while if the site is sold, we can get birr 15 million. On the basis of the criterion of maximization of the expected profit, our decision would be to sell the site.

Evaluation of decision node 2

Alternatives	Out come	Probability	Conditional value	Expected value
1. Drill	Success	0.8	100	80
	Failure	0.2	(-50)	(10)
	Total			70
2. Sell		1.0	65	15

The two alternative course of action, selling and drilling have expected value equal to 65 million and birr 70 million, respectively. Obviously, therefore, provided that a positive result is indicated by the test, the best course would be to go in for oil drilling.

Now at this stage we have two conditional decisions, sell the site if negative is obtained

on the test and drill in case the test indicates a positive. At each node the branch at which AMU,

we have not to move. Next, we move to the decision node where a decision has to be taken whether to drill at the outset.

Evaluation of decision node

Alternatives	Out come	Probability	Conditional value	Expected value
1. Drill	Success	0.55	120	66
	Failure	0.45	(40)	(18)
	Total			48
2. Test	Positive	0.7	70	49
	Negative	0.3	15	4.5
	Total			53.5
3. Sell	1.0		45	45

The expected value of the alternative of carrying out a test is birr 53.5 million, which is the highest of the three. Therefore, it is better to test before carried out. If it proves negative, the right should be sold to give a return of birr 15 million. To proceed with relining, if that happens, would expectedly lead to a loss. However, if the test proves positive the drilling should be under taken.

Chapter Summary

Knowledge of the likely hood of each of the state of nature can play an important role in selecting a course of action. There are three decision making environments: certainty, uncertainty and risk.

Decision under certainty is the case where the decision maker has a perfect knowledge (information) about the state of nature

Decision making under uncertainty is a situation in which decision maker has neither previous knowledge of the outcomes of his decisions nor is in a position to attribute a probability of occurrence of such outcomes.

The most widely used criterion for evaluating various courses of action (alternatives) under risk includes: expected monetary value (EMV), expected opportunity loss (EOL), expected value of perfect information (EVPI),

Expected monetary value is the sum of the payoffs for each course of action multiplied by the probabilities associated with each state of nature.

Expected value of perfect information (EVPI) represents the maximum amount of money the decision maker has to pay to get information about the occurrence of various state of nature before a decision has to be made.

A decision tree is a graphic representation of the sequence of action event combination available the decision maker. It depicts in a systematic manner all possible sequences of decisions and consequences.

Review question

Choose the best answer

A type of decision making environment is

- a. Certainty
- b. Uncertainty
- c. Risk
- d. All

The concept of utility is used to

- a. Measure the utility of money
- b. Take in to account a version of risk
- c. Both A and B
- d. None

Which of the following criteria is not applicable to decision making under risk

Maximize expected return

Maximize return

Minimize expected regret

Knowledge of likely hood occurrence of each state of nature

Which of the following criteria is not used for decision making under uncertainty

Maximin

Maximax

Minimax

Minimize expected loss

The expected value of perfect information (EVPI) is

- Equal to expected regret of the optimal decision under risk
- The utility of additional information
- Maximum expected opportunity loss
- None

Solve the following problem

An investor is given the following investment alternatives and percentage rate of return.

	State of nature (market condition)		
	Low (%)	Medium (%)	High (%)
Regular sharing	2	5	8
Risky sharing	-5	7	15
Property	-10	10	20

Over the past 300 days, 150 days have been medium market conditions and 60 days have had high market increases.

On the basis of these data, state the optimal investment strategy for the investor. According to the f given information the probabilities of low medium and high market conditions would be 90/300 Or 0.30, 150/300 or 0.50 and 60/300 or 0.20, respectively.

- Calculate the expected payoffs for each of the alternatives
- Calculate the expected return
- In which alternative the investor invest hiss money?

Answer b, regular share 0.047, b. risky shares 0.050 and property 0.060. c. since the expected of 6% highest for property, the investor should invest in this alternative.

CHAPTER 5

NETWORK MODEL

Learning objectives

After going through this chapter, you should be able to:

- Define network, project management and network diagram
- State and explain the characteristics and phases of project management
- Describe the major difference between PERT and CPM
- Explain the major conventions of network and steps to draw the network diagram
- Demonstrate project time cost trade-off

5.1. Introduction

Network model is a technique used for planning, and scheduling large projects in the field of construction, maintenance, fabrication, purchasing computer system, distribution, transportation, facility allocation, and many others. There are several reasons for the wide range of applications. First many problems that can be formulated as LP models can also be visualized with the context of **networks**. **Second** network models of problems are often easier to formulate than alternative LP models of the same problem. Third the pictorial representation of network models makes them relatively easy to understand, particularly by the nontechnical decision makers. Finally the advances in network optimization procedure have resulted in some amazing results in solving network model.

Network diagram is a practical representation of the various events and activities.

Network modeling: Techniques that enable complex projects to be scheduled, taking into account the precedence of each activity.

The different systems satisfying the definition of network in the physical world

Physical situation	Nodes	Branches	Flow
Highway systems	Intersections	Roads	Vehicles
Communication systems	Switching points	Wires	Messages
Fluid supply systems	Pumping stations	Pipes	Fluid
Production systems	Work centers	Handling routs	Jobs
Project Management	Decision points	Activities	Time
Airway systems	Air port	Air lines	Aircraft

PROJECTS

Definition: A **project** is a set of proposals for the investment of resources in to a clearly identified set of actions that are expected to produce future benefits that exceeds costs.

A project is a combination of interrelated activities all of which must be executed in a certain order for its completion.

Examples of a project:

New business establishment

Expansion of existing business

Launching new product

Cost reduction project

Total Quality of Management (TQM), etc

Characteristics of a project:

The duration of a project lasts long

A project is complex in nature involving many interrelated activities and participants

Delay in completion time may be very costly

Project activities are sequential

Phases of project management:

Project management consists of three phases;

1. project planning phase

In order to visualize the sequencing or preceding requirements of the activities in a project, it is helpful to draw a network diagram. For this the following steps are adopted:

Identify various activities (tasks or work elements) to be performed in the project, that is, develop a breakdown structure (WBS).

Determine requirement of resources such as man, material, machines, money, e.t.c. for carrying out activities listed above.

Assign responsibility for each work package. The work package corresponds to the smallest work effort defined in a project and from the set of elements which are the base for planning, scheduling and controlling the project

Allocate resources to work package.

Estimate cost and time at various level of project completion.

Develop work performance criteria

Establish control channels for project personnel.

2. Project scheduling phase

Once all packages have been identified and given unique names or identification, scheduling of the project, i.e. when each of the activities required to be performed, is taken up. The various steps are listed below:

Identify all people who will be responsible for each task.

Estimate the expected duration of each activity, taking in to consideration the resource required for their execution in most economic manner.

Specify the interrelationship (i.e. precedence relationship) among various activities.

Develop a network diagram showing the sequential interrelationship between various activities.

Based on the time estimate, calculate the total project duration, identify critical path; calculate floats; carry out resource smoothing (leveling) exercise for critical (or scarce) resources taking in to account resource constraints if any.

3. project control phase

It refers to evaluating actual progress (status) against the plan. If significant difference are observed then the scheduling and resource allocation decision are changed to update and revise the uncompleted part of the project.

There are two basic planning and control technique that utilize network to complete predetermined Project or schedule. These are Program Evaluation and Review Technique (PERT) and Critical Path Method.

PERT is an acronym for "Program Evaluation and Review Technique". This was created as a means to plan and accelerate the development of the Polaris Ballistic Missile. In USA the defense department developed a nuclear missile to be launched from beneath the ocean's surface by a mobile submarine, which would be an effective deterrent against aggression by an enemy. This paved way to plan how to design, develop and plan the different stages in the production of a missile and how quickly this task could be completed. A planning and scheduling technique named PERT gave the answer to these questions.

In any new venture, uncertainties are bound to creep in. PERT incorporated these uncertainties into a model, which provides a reasonable answer to these uncertainties. There are certain statistical aspects scheduling large projects consisting of numerous activities whose completion times are uncertain and are independent of one another. PERT is an event-oriented technique. By 'event' we mean reaching a certain stage of completion of the project.

Another technique, Critical Path Method, abbreviated as CPM, has emerged simultaneously. It is also a network technique but it is concerned with obtaining the trade-off, between cost and completion date for large projects. In any project consisting AMU, CBE, of several Department activities of Management each activity

If we employ more persons or skilled people or given overtime to the workers, the activity could be completed in a reduced duration known as crash duration. But this involves an increased cost in the form of additional resources. With CPM the amount needed to complete the various activities is assumed to be known with certainty. So, the direct costs for the activities increase and hence the cost of the project also increases. By reducing the activity duration of some or all possible completed ahead of the schedule. This will naturally reduce the overhead cost for the entire project. On one hand the direct expenses increase, if we shorten the activity duration, but, the indirect expenses for the project are reduced. We have to strike a balance or an optimum time schedule, or at least cost schedule is to be obtained. This is the purpose of the Critical Path Method. Thus CPM is not concerned with uncertain job times as in PERT. PERT is useful in research and developmental projects, whereas CPM is mostly used in construction projects, or in situations already handled, so that the details like the normal completion time, crash duration and cost of crashing are already known.

The following are the suggested applications when PERT or CPM is found useful.

- ➡ The construction of a building or of a highway.
- ➡ Planning and launching a new product.
- ➡ Scheduling maintenance for a project.
- ➡ The manufacture and assembly of a large machine tool.
- ➡ To conduct a music or drama festival.
- ➡ Preparation of budget for a company.

5.2. Basic Difference between PERT and CPM

PERT

It assumes a probability distribution for the duration of each activity. Thus completion time estimates for all of the activities are needed

To perform PERT analysis on a project, the emphasis is given on the completion of a task rather than the activities required to perform to reach a particular event or task. Thus, it is also called *event-oriented technique*.

It is used for one time project involving activities of none-repetitive nature (i.e. activities which may never have been performed before) in which time estimates are uncertain, such as redesigning an assembly line or installing a new information system

It helps in identifying critical areas in a project so that necessary adjustment can be made to meet the schedule completion date of the project.

CPM

This technique was developed in connection with a construction and maintenance project in which duration of each activity was known with certainty.

It is suitable for establishing a trade-off for optimum balancing between schedule time and cost of the project.

It is used for completion of project involving activities of repetitive nature.

5.3. Network Conventions

The project can be sub-divided into a set of predictable independent activities each of which has a clear beginning and ending.

Each activity can be sequenced as to its predecessors or successors.

The network is not cyclical

i.e: Each activity is executed once and only once during the life of the project.

*Activity times may be estimated either as a **single point estimate (CPM)** or as a **3- point estimate (PERT)***

***CPM**- Critical Path Method*

***PERT**- Program Evaluation and Review Technique*

The duration of the activities is independent of each other.

Basic terms

Network: it is the graphic representation of logically and sequentially connected arrows and nodes representing activities and events of a project. Networks are also called arrow diagram.

Events: Events of the network represent project milestone, such as *the start* or *the completion* of an activity (task) or activities, and occur at a particular instant of time at which some specific part of the project has been or is to be achieved. **Event** An event is a point in time that marks the beginning or ending of activity. Events are commonly represented by **circles (nodes)** in the network diagram.

The events can be further classified into the following two categories (types):

i. Merge event

An event which represents the **joint completion** of more than one activity is known as **merge event**.

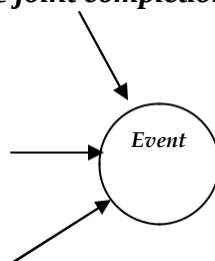


Fig: merge event

Burst event

An event which represents the **initiation (beginning)** of more than one activity is known as **burst event**.

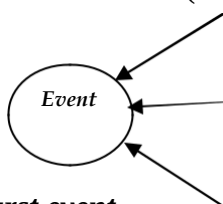


Fig: Burst event

Events in the network diagram are identified by numbers. Each event should be identified by a number higher than that allotted to its immediately preceding event to indicate progress of work. The numbering of events in the network diagram must **start** from **left** (start of the project) to **right** (completion of the project) and **top** to the **bottom**. Care should be taken that there is no duplication in the numbering.

Activities: Activity is a time-consuming job or task that is a key subpart of the total project. Activities of the net work represent project operations or tasks to be conducted. As such each activity except dummy consumes time and resources and incurs costs. An arrow is commonly used to represent an activity with its head indicating the direction of progress in the project.

Activities are identified by the numbers of their starting (tail or initial) event and ending (head or terminals) event. An arrow (→) extended between two events; the tail event i represents the start of the activity and the head event j , represents the completion of the activity.

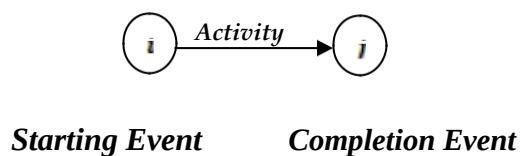


Fig: Activity -Node relationship in network diagram

Activities can be further classified into the following three categories:

Predecessor activity: Predecessor activity is an activity which must be completed before one or more other activities start.

Successor activity: Successor activity is an activity which started immediately after one or more of other activities are completed.

Dummy activity

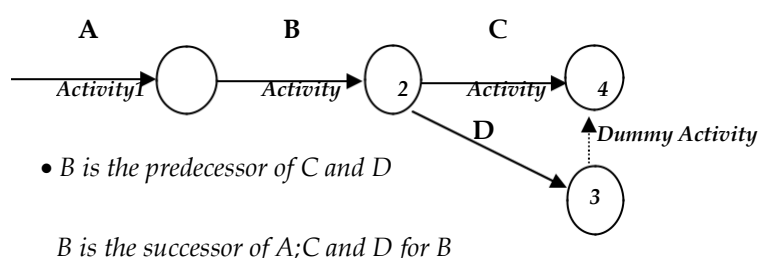
Dummying activity is an activity which does not consume either any resource or time.

A **dummy activity** in the network is added only to represent the given precedence relationships among activities of the project and is needed when

Two or more parallel activities in a project have same head and tail events, or

Two or more activities have some (but not all) of their immediate predecessor activities in common.

A dummy activity is depicted by **dotted line** in the network diagram.



Network model use the following two type of network to show precedence requirements of the activities in the project.

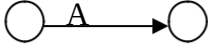
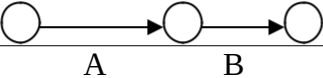
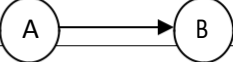
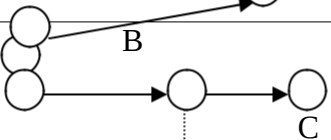
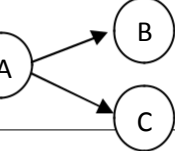
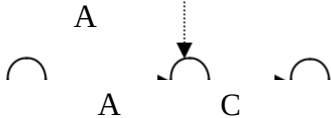
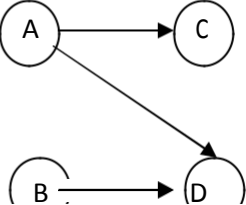
Activity-on-Node (AON) Network in this type of precedence network each node (or circle) represent a specific task while the arcs represent the ordering between tasks. AON network diagrams place the activities when the nodes and the arrows are used to indicate sequencing requirement. Generally, these diagrams have no particular starting and ending nodes for the whole project. The lack of dummy activities in this diagram always makes them easier to draw and to interpret.

Activity-on-Arrow (AOR) network: in this type of precedence network at the each end of the activity arrow is a node (or circle). This node represents points in time or instance, when an activity is starting or ending. The arrow itself represents the passage of time required for that activity to be performed.

These diagrams have a single beginning node from which all activities with no predecessor may start. The diagram then works its way from left to right, ends with a single ending node, where all activities with no followers come together. Three important advantage of using AOA are:

- a. Many computer programs are based on AOA network
- b. AOA diagram can be superimposed on a time scale with the arrows drawn, the correct length to indicate the time requirement.
- c. AOA diagram give a better sense of the flow of time throughout the project.

The following diagram illustrates AON and AOA.

	AOA network	AON network
1. Activity A		
2. B must follow A		
3 C must follow A&B		
4 C must follow A, and D must follow A&B		

5.4. Rules of Network Construction

Following are some of the rules which have to be followed while constructing a network:

In network diagram, arrows represent activities and circles the events. The length of an arrow is of no significance.

Each activity should be represented by one arrow and must start and end in a circle called event. The tail of an activity represents the start and head the completion of work. The event numbered 1 denotes start of the project and is called initial event. All activities emerging (or taking off) from event 1 should not be preceded by any other activity or activities. Event carrying the highest number denotes the completion events. A network should have only one initial event and only one terminal event.

*The general rule of numbering the event is that the number at an activity's head should always be larger than that at its tail. That is, events should be numbered such that for each **activity (i, j)**, $i < j$.*

An activity must be uniquely identified by its starting and completion event which implies that

- An event number should not get repeated or duplicated*
- Two activities should not be identified by the same completion event*
- Activities must be represented either by their symbols or by the corresponding ordered pair of starting-completion events.*

The logical sequence (or interrelationship) between activities must follow the following rules :

An event cannot occur until all the incoming activities into it have been completed.

An activity cannot start unless all the preceding activities, on which it depends, have been completed.

Though a dummy activity does not consume either any resource or time, even then it has to follow the rules 6(a) and 6(b).

Errors and Dummies In Network

Looping and Dangling

Looping (cycling) and dangling are considered as faults in a network. Therefore, these must be avoided.

Looping

A case of endless loop in a network which is also known as **looping** is shown in figure below, where activities **A**, **B**, and **C** form a cycle.

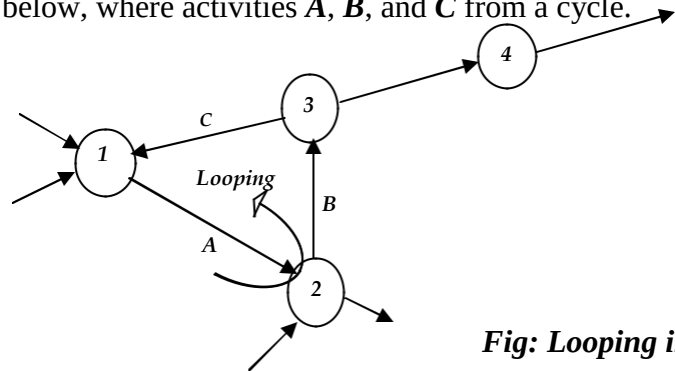


Fig: Looping in network diagram

In this case it is difficult to number three events associated with activity **A**, **B**, and **C** so as to satisfy **rule 6** of constructing the network.

2. Dangling

A case of disconnect activity before the completion of all activities which is also known as **dangling**.

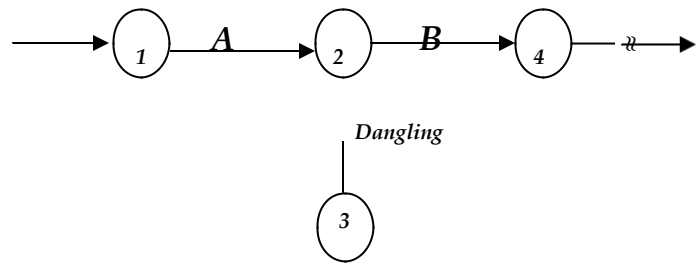


Fig: Dangling in network diagram

In this case, activity **C** does not give any result as per the rules of the network. The of dangling may be avoided by adopting **rule 5** constructing the network.

Dummy or Redundant Activity.

The following is the one of the cases in which the use of dummy activity may help in drawing the network correctly as per the various rules

When two or more parallel activities in a project have the same head and tail events:

i.e: two events are connected with more than one arrow.

In figure a, activities **B** and **C** have a common predecessor activity **A**. At the same time, they have activity **D** as a common successor. To get the correct network a dummy activity for the ending event of **B** to show that **D** may not start before both **B** and **C** are completed is shown in figure b). A dummy which is used in such a case to establish relationship proper logical relationships is also known as *logic dummy activity*.

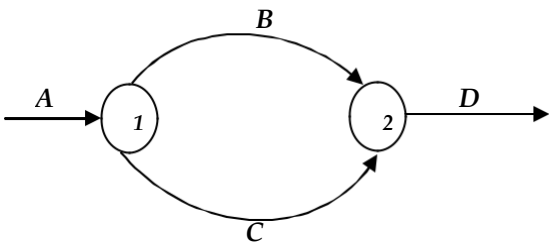


Fig: a) Parallel Activities

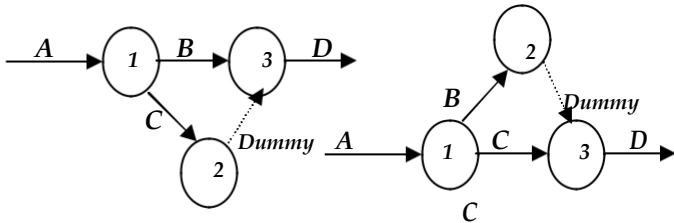


Fig: b) Dummy Activity

Example

An assembly is to be made from two parts **X** and **Y**. Both parts must be turned on a lathe and **Y** need not be polished. The sequence of activities together with their predecessors is given below:

Activity	Description	Predecessor Activity
A	Open work order	—
B	Get material for X	A
C	Get material for Y	A
D	Turn X on Lathe	B
E	Turn Y on Lathe	B, C
F	Polish Y	E
G	Assemble X and Y	D, F
H	Pack	G

Draw a network diagram for the project.

Solution

The network diagram for the project is show below:

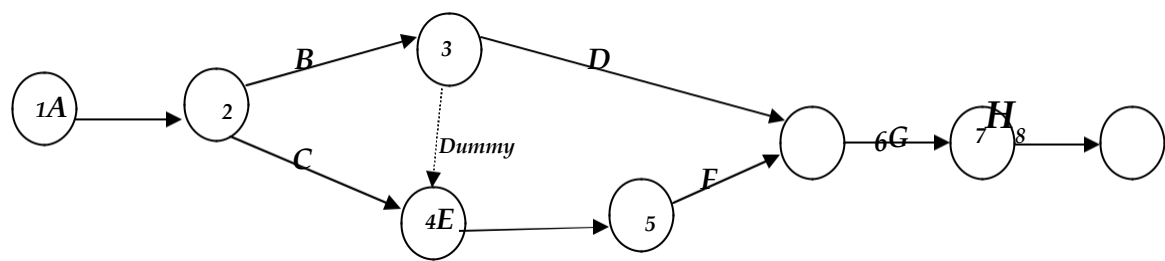


Fig: Network Diagram

Listed in the table are the activities and sequencing necessary for a maintenance job on the heat exchangers in a refinery.

Activity	Description	Predecessor Activity
A	Dismantle pipe connections	-
B	Dismantle header, closure, and floating front	A
C	Remove tube bundle	B
D	Clean bolts	B
E	Clean header and floating head front	B
F	Clean tube bundle	C
G	Clean shell	C
H	Replace tube bundle	F, G
I	Prepare shell pressure test	D,E,H
J	Prepare tube pressure test and reassemble	I

Required:

Draw a network diagram for the project

Solution

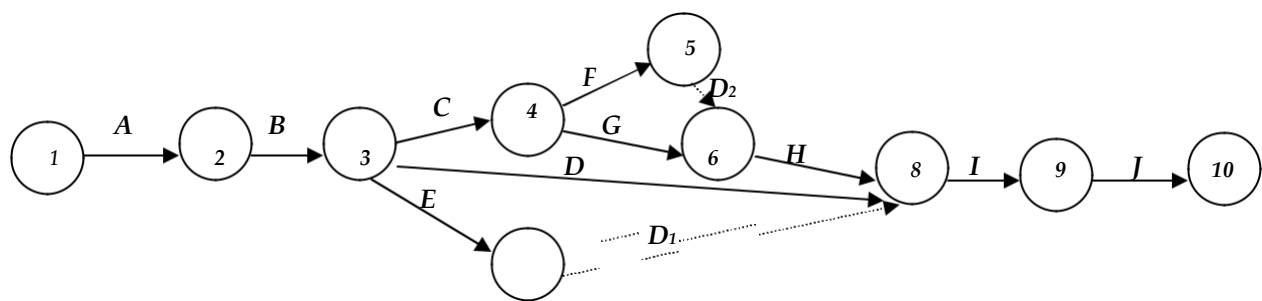


Fig: Network Diagram

3. The following table gives the activities in a construction project:

<u>Activity</u>	<u>Immediate Predecessor</u>
A	-
B	-
C	-
D	A
E	C
F	A
G	D,B,E

Required: Draw an arrow diagram for the project

Solution:

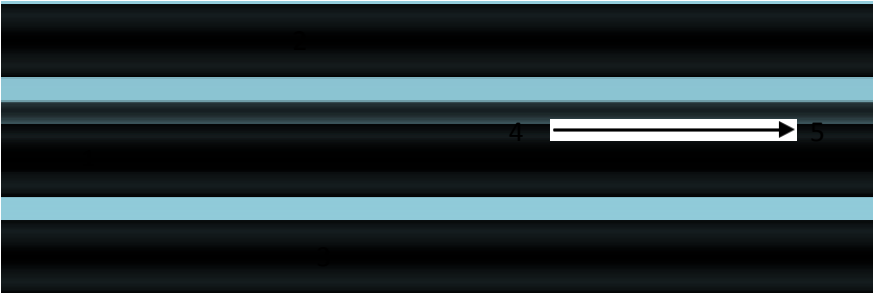


Fig. Net work diagram

Example A project has the following activities. The relationships among the activities are given below. Construct the network.

A is the first operation.

B and C can be performed parallel and are immediate successors to A.

D, E and F follow B.

G follows E.

H follows D, but it cannot start till E is complete.

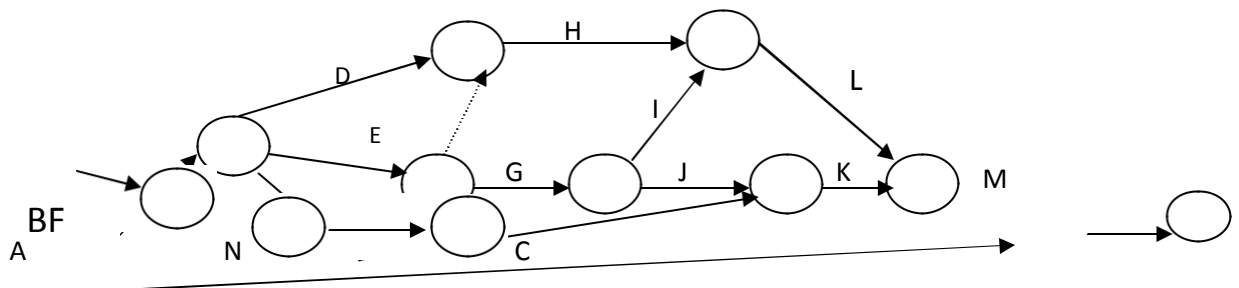
I and J succeed G.

F and J precede K.

H and I precede L.

M succeeds L and K.

The last operation N succeeds M and C.



6.5. Critical path Analysis

The objective of critical path analysis is to estimate the total project duration & to assign starting & finishing times to all activities involved in the project. This helps in checking actual progress against the scheduled duration of the project.

The duration of individual activities may be uniquely determined (in case of CPM) or may involve the three time estimates (in case of PERT) out of which the expected duration of an activity is computed. Having done this, the following factors should be known to prepare project scheduling.

Total completion time of the project

Earliest & latest start time of each activity

Float for each activity

i.e. the amount of time by which the completion of an activity can be delayed without delaying the total project completion time.

iv. Critical activities & critical path.

Consider the following notation for the purpose of calculating various times of event & activities.

E_i = Earliest occurrence time of an event, i. It is the earliest time at which an event can occur without affecting the total project time.

L_i = Latest occurrence time of event i. It is the latest time at which an event can occur without affecting the total project time

ES_{ij} = Earliest start time for activity (i, j). It is the time at which the activity can start without affecting the total project time

LS_{ij} = Latest start time for activity (i, j). It is the latest possible time by which an activity must start without affecting the total project time

EF_{ij} = Earliest finish time for activity (i, j). It is the earliest possible time at which an activity can finish without affecting the total project time.

LF_{ij} = Latest finish time for activity (i, j). It is the latest time by which an activity must get completed without delaying the project completion.

G. t_{ij} Duration of activity (i, j).

Expected time estimate

In a network diagram, there should only be one initial event & one final event, while other events are numbered consecutively with integers 1, 2... n such that $i < j$, for any two event i & j connected by an activity which starts at i & finish at j

For calculating the above mentioned times, there are two methods; namely;

Forward pass method &

Backward pass method

Forward Pass Method (For Earliest Event Time)

In this method calculation begins from the initial event 1, proceed through the network visiting event in an increasing order of event number & end to the final event, say N. At each event we calculate earliest occurrence event time (E) & earliest start & finish time for each activity that begins at that event. When calculations end at the final event N, its earliest occurrence time gives the earliest possible completion time of the entire project.

The method may be summarized as follows:

Set the earliest occurrence time of initial event 1 to zero, i.e. $E_1 = 0, i = 1$

Calculate earliest start time for each activity that begins at event i (i=1) This is equal to earliest occurrence time of event, i (tail event). That is $ES_{ij} = E_i$, for all activities (i, j) starting at event i.

3) Calculate the earliest finish time of each activity that begins at event i. This is equal to the earliest start time of the activity plus the duration of the activity. That is,

$$EF_{ij} = ES_{ij} + t_{ij} = E_i + t_{ij} \text{ for all activities (i, j) beginning at event } i$$

4) Proceed to the next event, say j; j > i

Calculate the earliest occurrence time for the event j. This is the maximum of the earliest finish times of all activities ending into that event, i.e.

$$E_j = \max\{EF_{ij}\} = \max\{E_i + t_{ij}\}, \text{ for all immediate predecessor activities.}$$

If $j = N$ (final event), then earliest finish time for the project, i.e. the earliest occurrence time E_N for the final event is given by

$$E_N = \max\{EF_{ij}\}, \text{ for all terminal activities.}$$

$$\max\{E_N - 1 + t_{ij}\} \text{ \& stop the method.}$$

Backward Pass Method (For Latest Allowable Event Time)

In this method calculations begin from final event N, proceed through the network visiting events in the decreasing order of event number & end at the initial event 1. At each event, we calculate the *latest occurrence event time (L)* for the corresponding event, latest finish start time for each activity that is terminating at the event, such that the earliest finish time for the project remains the same. The method may be summarized as follows.

Set the latest occurrence of last event N equal to its earliest occurrence time (known from forward pass method), that is

$$L_N = E_N$$

Calculate latest finish time of each activity which ends at event j. This is equal to latest occurrence time of final event N. That is,

$$LF_{ij} = L_j \text{ for all activities (i, j) ending at event j.}$$

Calculate the latest start times of all activities ending at j. It is obtained by subtracting the duration of the activity from the latest finish time of the activity, That is,

$$LS_{ij} = LF_{ij} - t_{ij}, \text{ for each activity (i, j) ending at Event j,}$$

Proceed backward to the event in the sequence that decreases j by 1.

Calculate the latest occurrence time of event i ($i < j$). This is the minimum of the latest start times of all activities starting from that event. That is,

$$L_i = \min\{LS_{ij}\}$$

If $j = 1$ (initial event), then the latest finish time for project, i.e. latest occurrence time L_1 for the initial event is given by.

$$L_1 = L_i \text{ For all immediate successor activities}$$

$$L_1 = \min\{L_i\} \text{ \& stop the method}$$

6.5.1. Float (or slack) of an Activity and Event

The *float (slack)* or *free time* is the length of time to which a non critical activity and/or an event can be delayed the total project completion time. The *float* of an activity is the

amount of time by which it is possible to delay its completion time without affecting the total project completion time.

1. Event Float

The **float** (also sometimes called '**slack**') of an event is the difference between its latest time (L_i) and its earliest time (E_i). That is

$$\text{Event Float} = L_i - E_i$$

It is a measure of how much later than expected a particular event could occur without delaying the completion of the entire project.

2. Activity Float

It is the float (or slack) in the activity time estimates.

Types of activity floats:

a. Total Float

It is the difference between the time available to perform the activity (**measured from the earliest start time to the latest finish time**) and the expected completion time of the activity. That is, for activity (i, j), the total float is given by

$$\begin{aligned} \text{Total Float (TF}_{ij}) &= (L_j - E_i) - t_{ij} \\ &= LS_{ij} - ES_{ij} \\ &= LF_{ij} - EF_{ij} \end{aligned}$$

b. Free Float

For calculating the total float, only a particular activity was considered with respect to its tail and head event times or by considering latest start and finish time of an activity with respect to its earliest start and finish time. However, it may be needed to know that how much an activity's scheduling time can be increased or decreased without changing its immediate successor activities.

The time by which the completion of an activity can be delayed beyond its **earliest finish time** without affecting the earliest start time of a succeeding activity is called **free float**.

$$\begin{aligned} \text{Free Float (FF}_{ij}) &= (E_j - E_i) - t_{ij} \\ &= \min_{\text{all immediate successors of } (i,j)} ES_{ij} - EF_{ij} \end{aligned}$$

Independent float: it is the amount of acceptable delay in the completion of an activity so that it neither affects its predecessor nor the successor activities. Thus, independent float is the amount of time available when preceding activities take place at their latest permissible time and all the following activities can still take place at their earliest possible times.

$$\text{Independent float } (IF_{ij}) = (E_j - L_i) - t_{ij} = \{ES_{ij} - LS_{ij}\} - t_{ij}$$

The negative value of independent float is considered zero.
Since latest event time is always greater than or equal to the earliest event time (i.e.

$$L_i \geq E_i), (TF_{ij}) \geq (FF_{ij}). \text{Remarks:}$$

This implies the value of free float may range from zero to total float and can never exceed total float value.

The calculation of various floats can help the decision-maker in identifying the underutilized resources, flexibility in the total schedule and possibilities of redeployment of resources.

3. Once the float of an activity is disturbed, float of all other activities of the project is changed and should be recalculated.

If total float value is

a. Negative (i.e. $L-E < 0$),

-resources are not adequate and activity may not finish in time

-Induce extra resources or certain activities (also called critical activities) need crashing to reduce negative float value.

b. Zero (i.e. $L-E=0$)

-Resources are just sufficient to complete the activity. Any delay in activity execution will necessarily increase the project cost.

c. Positive (i.e. $L-E > 0$)

-Resources are surplus

-Surplus resources can be deployed elsewhere or execution of the activity can be delayed.

Critical Path

Path: A path is unbroken chain of activities that connects form the start node to the end node (terminal node).

Certain activities in a network diagram of a project are called **critical activities** because delay in their execution will cause further delay in the project completion time. Thus, all activities having zero total float value are identified as **critical activities**.

The critical path is the continuous chain of critical activities in a network diagram. It is the **longest path** starting from first to the last event & is shown by the thick line or double lines in the network diagram most of the time.

The length of the critical path is the sum of the individual times of the critical activities living on it & defines the minimum time required to complete the project.

The critical path on a network diagram can be identified as:

For all activities (i, j) lying on the critical path the E –values & L-values for tail & head events are equal That is, $E_i = L_j$

On critical path $E_i = L_i = E_j = L_j$

Finding the critical path is important for directing decision-makers attention & effort to critical activities where improvement will pay the largest dividend.

Example

An established company has decided to add a new product to its line. It will buy the product form manufacturing concern, package is & sell it to a number of distributes selected on a geographical basis. Market research has identified the volume expected & the size of sales force required. The steps

<u>Activity</u>	<u>Description</u>	<u>Time (Week)</u>
A	Organize sales office	6
B	Hire Salesman	4
C	Train salesman	7
D	Select advertising agency	2
E	Plan advertising campaign	4
F	Conduct	10
G	Design package	2
H	Setup packaging facilities	10
I	Package initial Stocks	6

Operations Research

	Stock from manufacture	
K	Select distributors	9
L	Sell to distributors	3
M	Ship stocks	5

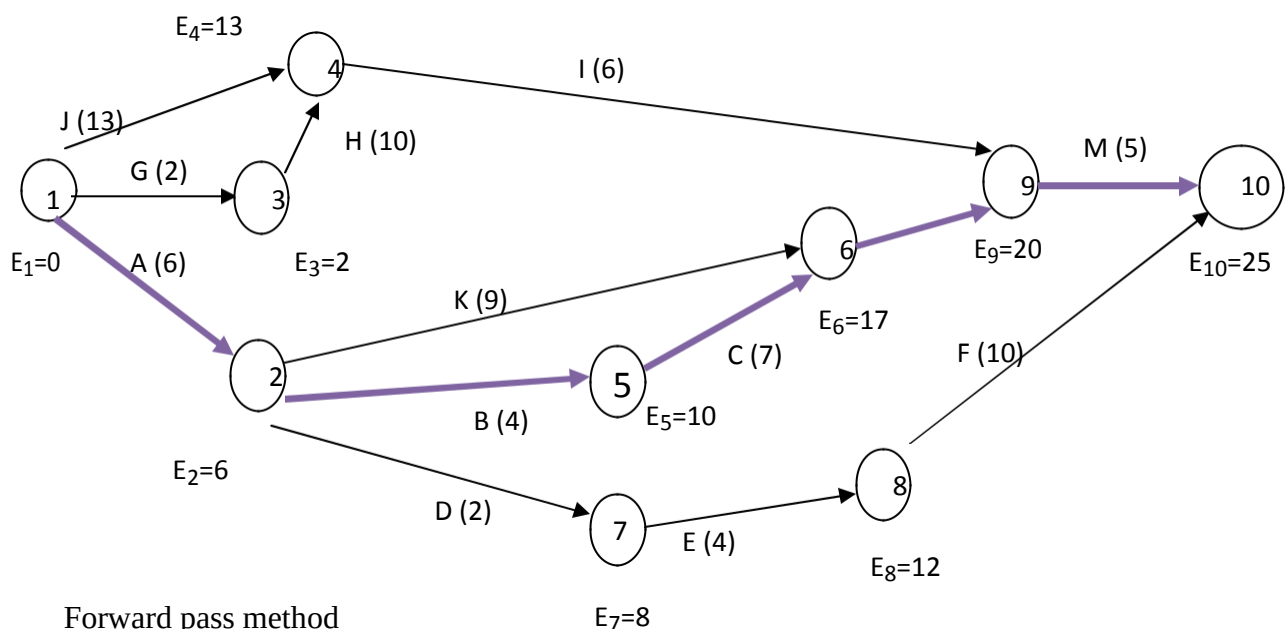
Required:

Draw the an arrow diagram

Indicate the critical path; the critical activities & the total project time

For each non-critical activity, find the total float

Solution: (a). the arrow diagram for the given project along with E-values and L-values is shown in the following diagram.



Forward pass method

Determine the earliest start time E_i and latest finish L_j for each event by processing as follows.

$$E_1=0$$

$$E_2= E_1+t_{1,2}=0+6$$

$$E_3= E_1+t_{1,3}=0+2=2$$

$$E_4=\text{Max} \{E_1+t_{1,4}; E_3+t_{3,4}\} \quad i=1,3$$

$E_5 = E_2 + t_{2,5} = 6 + 4 = 10$

Operations Research

$E_6 = \text{Max} \{E_i + t_{i,6}\} = \text{Max} \{E_2 + t_{2,6}, E_5 + t_{5,6}\}$

$\text{Max} \{0 + 13, 2 + 10\} = 13$

$= 2, 5$

$E_8 = E_7 + t_{7,8} = 8 + 4 = 12$

$\text{Max} \{6 + 9; 10 + 7\} = 17$

$E_{10} = \text{Max} \{E_i + t_{i,10}\}$

$E_7 = E_2 + t_{2,7} = 6 + 2 = 8$

$i = 8, 9$

$E_9 = \text{Max} \{E_i + t_{i,9}\} = \text{Max} \{E_4 + t_{4,9}, E_6 + t_{6,9}\}$

$= \text{Max} \{E_8 + t_{8,10}; E_9 + t_{9,10}\}$

$= 4, 6$

$= \text{Max} \{12 + 10; 20 + 5\} = 25$

$= \text{Max} \{13 + 6; 17 + 3\} = 20$

Backward pass method

$L_{10} = E_{10} = 25$

$L_9 = L_{10} - t_{9,10} = 25 - 5 = 20$

$L_8 = L_{10} - t_{8,10} = 25 - 10 = 15$

$L_7 = L_8 - t_{7,8} = 15 - 4 = 11$

$L_6 = L_9 - t_{6,9} = 20 - 3 = 17$

$L_5 = L_6 - t_{5,6} = 17 - 7 = 10$

$L_4 = L_9 - t_{4,9} = 20 - 6 = 14$

$L_3 = L_4 - t_{3,4} = 14 - 10 = 4$

$L_2 = \text{min} \{L_j - t_{2,j}\}$

$L_2 = \text{min} \{L_j - t_{1,j}\}$

$J = 5, 6, 7$

$j = 2, 3, 4$

$= \text{min} \{L_5 - t_{2,5}; L_6 - t_{2,6}; L_7 - t_{2,7}\}$

$= \text{min} \{L_2 - t_{1,2}; L_3 - t_{1,3}; L_4 - t_{1,4}\}$

$= \text{min} \{10 - 4; 17 - 9; 11 - 2\} = 6$

$= \text{min} \{6 - 6; 5; 4 - 2; 14 - 13\} = 0$

The vertical path in the network diagram above has been shown by thick lines by joining all those events where two values E_i & L_j are equal. The critical path of the project is: 1-2-5-6-9-10 the critical activities are A, B, C, L&M. the total project time is 25 weeks

For each non-critical activity, the total float is:

Activity (i,j)	Duration (t _{ij})	Earliest time		Last time		Float	
		Start (E _i)	Finish (E _i +t _{ij})	Start (L _j -t _{ij})	Finish (L _j)	Total [(L _j -t _{ij})-E _i]	Free (E _j -E _i)-t _{ij}
1-3	2	0	2	2	4	2	0
1-4	13	0	13	1	14	1	0
2-6	9	6	15	8	17	2	2
2-7	2	6	8	9	11	3	0

Operations				Research			
3-4	10	2	12	4	14	2	1
4-9	6	13	19	14	20	1	1
7-8	4	8	13	11	15	3	0
8-10	10	12	22	15	25	3	3

Remark:

What does the critical path really mean?

The critical path shows the shortest time in which a project can be completed. Even though the critical path is the longest path, it represents the shortest time it takes to complete a project.

There can be more than one critical path on a project. Project managers should closely monitor performance of activities on the critical path to avoid late project completion. If there is more than one critical path, project managers must keep their eyes on all of them

Example2. The following table gives the activities of a construction project and duration.

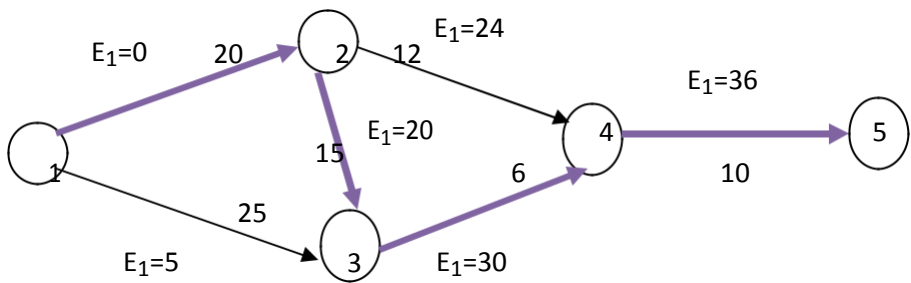
Activity	1-2	1-3	2-3	2-4	3-4	4-5
Duration (days)	20	25	10	12	6	10

Draw the network for the project.

Find the critical path.

Find the total, free and independent floats of each activity.

Solution: The first step is to draw the network and fix early start and early finish schedule and then late start-late finish schedule.



Activity	Total Slack	Slack	Independent Slack
1-2	0	0	0
1-3	5	5	5
2-3	0	0	0
2-4	4	4	4
3-4	0	0	0
4-5	0	0	0

To find the critical path, connect activities with 0 total slack and we get 1-2-3-4-5 as the critical path.

Check with alternate paths.

$1-2-4-5 = 42$

$1-2-3-4-5 = 46^*$ (critical path)
 $1-3-4-5 = 41$

6.6. PERT Networks

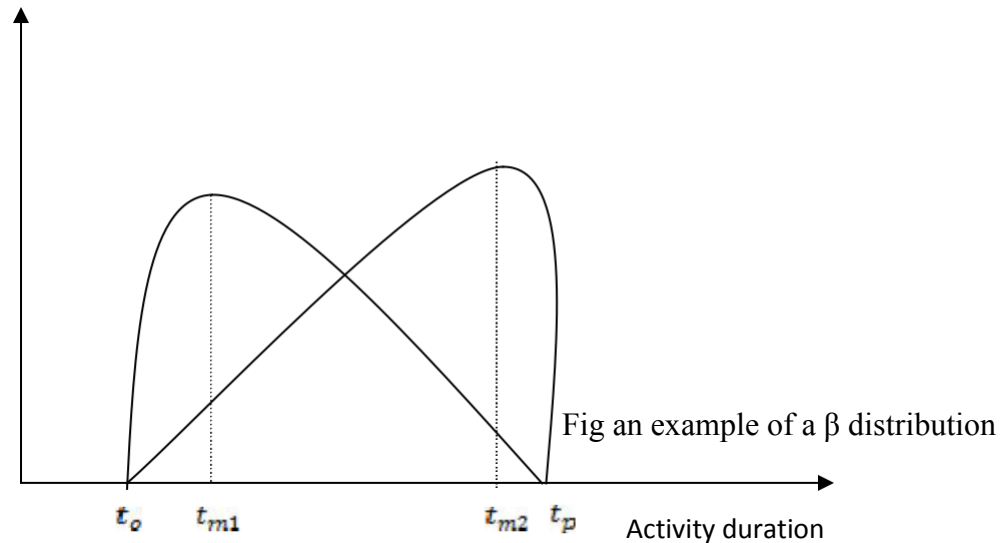
Program Evaluation and Review Technique (PERT) - is a network analysis technique used to estimate project duration when there is a high degree of uncertainty about the individual activity duration estimates. PERT applies the critical path method to a weighted average duration estimate.

If the duration of activities in a project is uncertain than activity scheduling calculations are done by using the expected value of the durations. However, such expected duration estimated may not be given an accurate answer. Thus, rather than estimating directly the expected completion time of an activity, three values are considered in PERT.

The three time estimates are:

- Optimistic time (t_o or a): This is the shortest possible time to perform an activity, assuming that everything goes well.
 - Pessimistic time (t_p or b): This is the maximum (longest) time that is required to perform an activity, under extremely bad conditions. However, such conditions do not include acts of God like earthquakes, flood, etc.
 - Most likely time (t_m or m): This is the most realistic time to complete the activity. Statistically, it is the modal values of duration of the activity i.e. consider the time estimate under the normal situation.
- The β distribution is not necessarily systematic; the degree of skewness depends on the location of and. Thus the range specified by the optimistic time (t_o) and pessimistic time (t_p) estimates is assumed to enclose every possible estimates of the duration of the AMU, CBE,

activity. The most likely time (t_m) estimate may not coincide with the midpoint ($(t_o + t_p)/2$) and may occur to its left or its right as shown in the following figure.



In beta distribution the midpoint ($(t_o + t_p)/2$) is assumed to weight half as much as the most likely point (t_m). Thus the expected or mean (t_e or μ) value of the activity duration can be approximated as the arithmetic mean of $(t_o + t_p)/2$ and $2t_m$. That is

$$\text{Expected activity time } (t_e) = ((t_o + t_p)/2 + 2t_m)/3 = ((t_o + 4t_m) + t_p)/6$$

Estimating the variance is apparently based on an analogy to the normal distribution where 99% of the area under normal curve is within $\pm 3\sigma$ from the mean or fall within the range approximately 6 standard deviation in length, therefore t_o , t_p or range $t_p - t_o$ enclose about 6 standard deviation of a symmetric distribution, thus, if σ denotes standard deviation, then

$$6\sigma \equiv t_p - t_o \text{ or } \sigma = (t_p - t_o)/6$$

$$\text{Variance of activity time, } \sigma^2 = [1/6(t_p - t_o)]^2$$

$$\text{Standard deviation, } \sigma = \sqrt{\text{Variance}}$$

Estimation of project completion time

As we are expecting variability in the activity duration, the total project may not be completed exactly in time. Thus, it is necessary to calculate the probability of actually meeting the schedule time to the project as well as activities.

The probability distribution of time for completing an event can be approximated by the normal distribution due to central limit theorem. Thus, the probability of completing the project by schedule time (t_s) is given by

$$\text{Prob}(Z = t_e/\sigma_e)$$

Where t_e = expected completion time of the project

Z= number so standard deviations the scheduled time or target date lies away from the mean or expected date.

In order to find out the probability of completing the project in some given time, we shall consider only the expected length of the critical path and its variance. The expected time of the project can be calculated by adding the expected time of each activity lying on the critical path can be known by adding variance of critical activity.

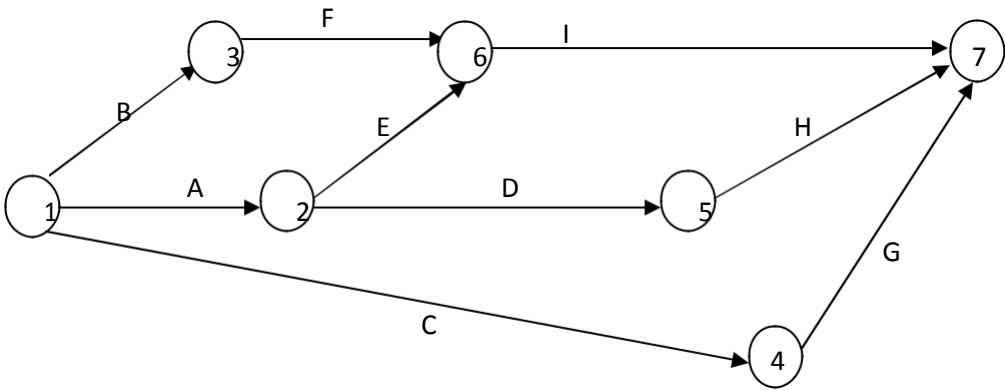
Example1.

- 1. A project is represented by a network shown below & has the following data

Task:	A	B	C	D	E	F	G	H	I
Optimistic time	5	18	26	16	15	6	7	7	3
Pessimistic time:	10	22	40	20	25	12	12	9	5
Most Likely time:	8	20	33	18	20	9	10	8	4

Determine the following

- Expected task time & their variance
- The earliest & latest expected times to reach each event
- The critical path



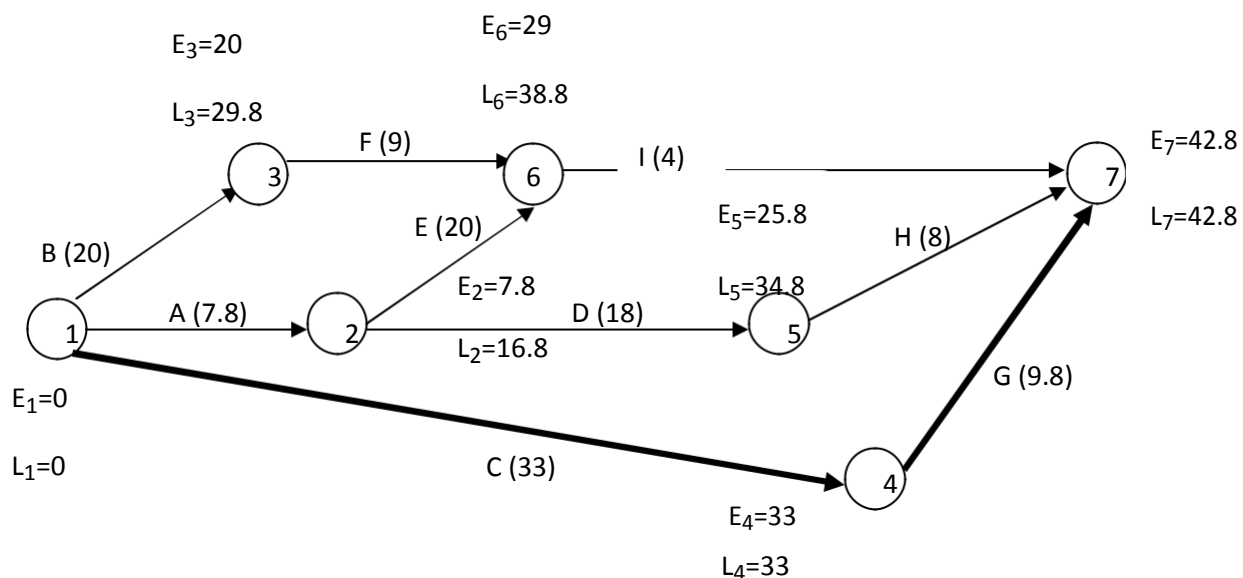
- The probability of an event occurring at the expected completion date if the original scheduled time of completing the project is 41.5 weeks.
- The duration of the project that will have 95% chance of being completed.
- Solution (a) using the following formula, the expected activity time (te) or μ and variance (σ^2) is given in the following table.

$$((to+ 4tm)+tp)/6 \text{ and } \sigma^2=[1/6(tp- to)]^2$$

Operations Research

Activity	to	tp	tm	te= ((to+ 4tm)+tp)/6	$\sigma^2=[1/6(tp - to)]^2$
1-2	5	10	8	7.8	0.694
1-3	18	22	20	20.0	0.444
1-4	26	40	33	33.0	5.444
2-5	16	20	18	18.0	0.444
2-6	15	25	20	20.0	2.778
3-6	6	12	9	9.0	1.000
4-7	7	12	10	9.8	0.694
5-7	7	9	8	8.0	0.111
6-7	3	5	4	4.0	0.111

(b) The earliest and latest estimated time presented in the following graph.



(c). the critical path is shown by thick line in the above figure where E values and L values are the same. The critical path is 1-4-7 and the critical completion time for the project is 42.8 week.

the last event 7 will occur only after 42.8 weeks. For this, we require only the duration of critical activity. This will help us in calculating the standard of the duration of the last event.

Expected length of critical path= 33+9.8=42.8

Variance of critical path length = 5.429+0.694=6.123

It is given that $T_s=41.5$, $T_e=42.8$ and $= \sqrt{6.123} = 2.474$. Therefore probability of meeting

the scheduled time is given by $\text{Prob} \left(\leq \frac{-}{-} \right) = \text{prob} (Z \leq -0.5) = 0.3$ from normal distribution

Thus, the probability that the project can be completed in less than or equal to 41.5 weeks is 0.30. In other words, the probability that the project will be delayed beyond 41.5 weeks is 70%

(e) Given that $\text{Prob}(Z \leq (t_s - t_e)/\sigma_e) = 0.95$

But $Z_{0.95} = 1.46$, from normal distribution table. Thus

$1.64 = ((t_s - 42.8)/2.47)$ or $t_s = 1.64 \times 2.47 + 42.8 = 46.85$ weeks.

6.7. Project Time-Cost Trade-Off

The project completion time can be reduced by reducing (crashing) the normal completion time of critical activities. The reduction in normal time of completion will increase the total budget of the project. However, the decision maker always looks trade-off between total cost of project and total time required to complete it.

Project crashing

Crashing is employed to reduce the project completion time by spending extra resource (cost). However, as shown in the figure beyond point A, cost increases more quickly when time is reduced. Similarly, beyond point B, the time increases while the cost decreases. Since for technical reasons time may not be reduced indefinitely, therefore, we call this limit as crash point. There is also a cost efficient duration called *normal point*. Thus extending the activity duration beyond normal point may increase costs.

For simplicity the relationship between normal time and cost as well as crash time and cost for an activity is assumed to be linear instead of being concave and/or discrete. Thus the crash cost per unit of time can be estimated by computing the relative change in cost (cost slope) per unit change in time.

Remark: crashing an activity means performing it in the shortest technically possible time by allocating to it necessary resources.

AMU, CBE, Department of Management

Time cost trade-off procedure

The method of establishing Time cost trade-off procedure for the completion of a project can be summarized as follows:

Step1 determine the normal project completion time and associated critical path for the following two cases.

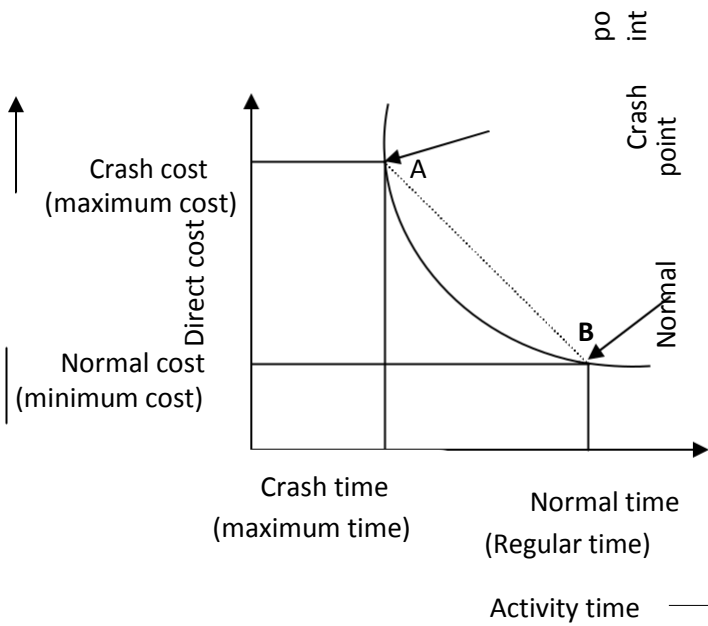
- When all critical activities are completed with their normal time. This provides the starting point for crashing analysis.

When all critical activities are crashed. This provides the stopping point for crashing analysis.

Step2. Identify critical activity and compute the cost slope for each of these by using the relationship

$\text{Cost slope} = (\text{crash cost} - \text{normal cost}) / (\text{crash time} - \text{normal time})$

The values of cost slope for critical activities indicate the direct extra cost required to execute an activity per unit of time.



Step3. For reducing total project completion time, identify and crash an activity time on the critical path with lower cost slope value to the point where

Another path in the network become critical, or

The activity has been crash to its lowest time.

Step4. If the critical path under crashing is still critical, return to step three. However, if due to crashing of an activity time in step 3, other path(s) in the network also become critical, then identify and crash the activities(s) on the critical path (s) with the minimum joint cost slope.

Steps 5 terminate the procedure when each critical activity has been crashed to its lowest possible time. Determine total project cost (indirect cost plus direct cost) corresponding to different project durations.

Example1. The following table gives data on normal time, and cost and crash time and cost for a project

Operations Research

Activities	Normal		Crash	
	Time (weeks)	Cost (Birr)	Time (weeks)	Cost (Birr)
1-2	3	300	2	400
2-3	3	30	3	30
2-4	7	420	5	580
2-5	9	720	7	810
3-5	5	250	4	300
4-5	0	0	0	0
5-6	6	320	4	410
6-7	4	400	3	470
6-8	13	780	10	900
7-8	10	1000	9	1200

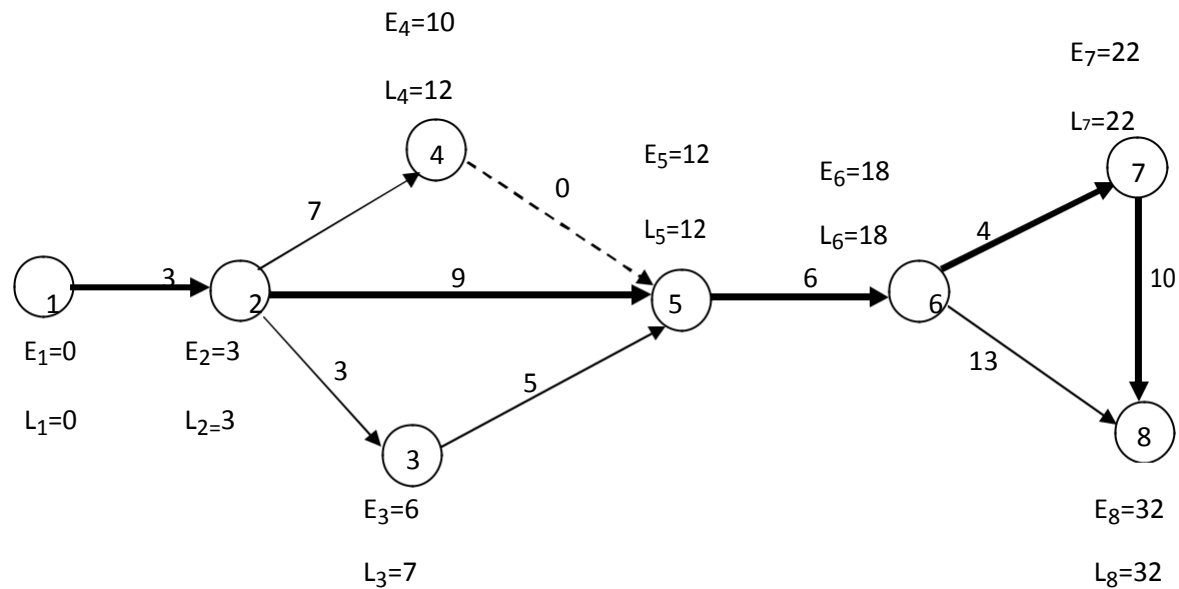
Draw the network and identify the critical path with a double line.

What are the normal project duration and associated cost?

Find out the total float associated with each activity.

Crash the relevant activities systematically and determine the optimal project completion time and cost.

Solution (a) the network for normal activity times indicates a project completion time of 32 weeks with the critical path: 1- 2- 5- 6 – 7 -8, as shown in the following figure.



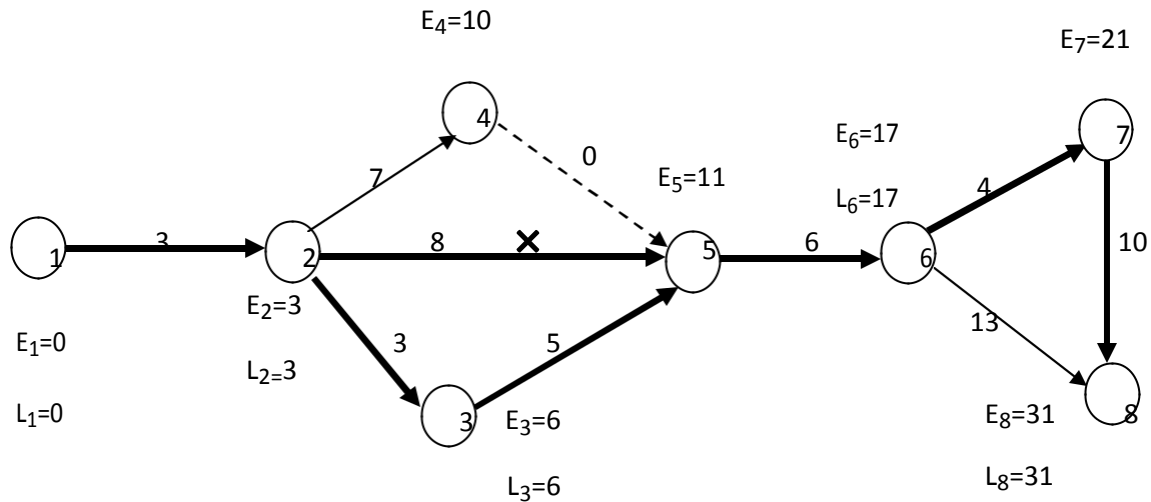
(b). normal project duration is 32 weeks and the associated cost is as follow:

Total cost=Direct normal cost+Indirect cost for 32 week

(c). Calculation for total float associated with each activity are shown in the following table.

Activities	Total float
	$(Li - Ei) - tij$
1-2	$(3 - 0) - 3 = 0$
2-3	$(7 - 3) - 3 = 1$
2-4	$(12 - 3) - 7 = 2$
2-5	$(12 - 3) - 9 = 0$
3-5	$(12 - 6) - 5 = 1$
4-5	$(12 - 10) - 0 = 2$
5-6	$(18 - 12) - 6 = 0$
6-7	$(22 - 18) - 4 = 0$
6-8	$(32 - 18) - 13 = 1$
7-8	$(32 - 22) - 10 = 0$

(d). for critical activities, crash cost slope is given.



The minimum value of crash cost per week is for activity 2-5 and 5-6. Hence, crash activity 2-5 by 2 days from 9 weeks to 7 weeks. But the time should be reduced by 1 week only otherwise another path 1-2-3-4-6-7-8 becomes a parallel path. Net work as shown in the above figure is developed when it is observed that the new project time is 31 weeks and the critical paths are 1-2-5-6-7-8 and 1-2-3-5-6-7-8. The following table shows the crash cost per week (in birr)

Critical activities	Crash cost per week (Birr)
1-2	$\frac{400 - 300}{3 - 2} = 100$
2-5	$\frac{810 - 720}{9 - 7} = 45$
5-6	$\frac{410 - 320}{6 - 4} = 45$
6-7	$\frac{470 - 400}{4 - 3} = 70$

Operations Research

7-8	$\frac{1200 - 1000}{10 - 9} = 200$

With crashing of activity 2-5 the new total cost involved can be calculated as follows;

NewTotalcost=TotalDirectnormalcost+increaseddirectcostduetocrashing of activity

2 - 5 + indirect cost for 31 weeks

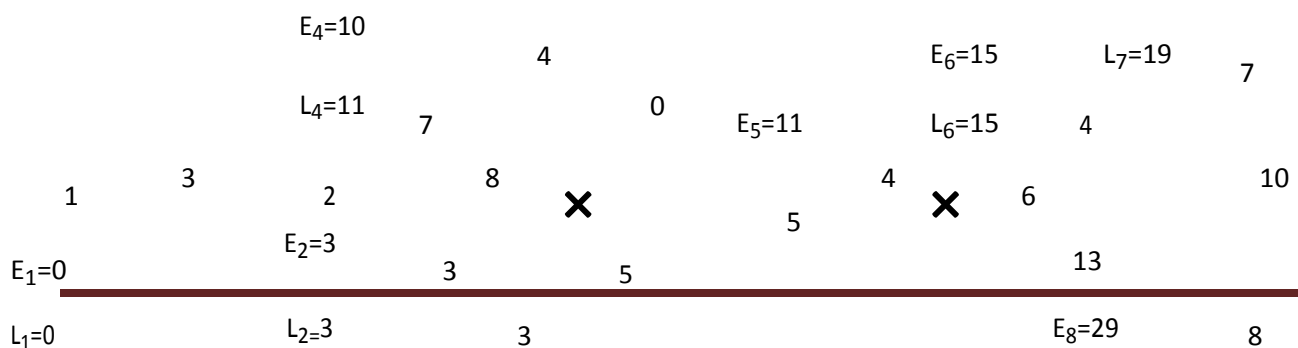
$$= (4,220 + 1 \times 45) + 31 \times 50 = 4,265 + 1,550 = \text{Birr } 5,815$$

Dear learners now with respect to the new network given, the new possibilities for crashing in the critical paths are listed in the following table.

Critical activities	Crash cost per week (Birr)
1-2	100
2-5	× (Crashed)
5-6	0 (no crashing is needed)
6-7	50
7-8	45
1-2	70
2-5	200

The minimum value of crashed cost slope is for activity 5-6. Hence, crashing it by 2 weeks from $E_7=19$

6 weeks to 4 weeks. The new network diagram is presented as follow.



$$E_3=6$$

$$L_3=6$$

As you see in the above figure the critical paths are not changed because activity 5-6 is common for the two critical paths. But with crashing of activity 5-6 by 2 weeks, the new cost involved is:

$$\begin{aligned} \text{New Total cost} &= \text{Total Direct normal cost} + \text{increased direct cost due to} \\ &\text{crashing of } 5-6 + \text{indirect cost for 29 weeks} \\ &= (4,220 + 1 \times 45 + 2 \times 45) + 29 \times 50 = \text{birr } 5805 \end{aligned}$$

With respect to the new network depicted above the new possibility for crashing in the critical paths are listed below.

Critical activities	Crash cost per week (Birr)
1-2	100
2-3	0 (no crashing is needed)
2-5	× (Crashed)
5-6	× (Crashed)
6-7	70
7-8	200

The further crashing of 6-7 activity time from 4 weeks to 3 weeks will results in increased direct cost than the gain due to reduction in project time. Consequently, here we must stop further crashing. The optimal project duration is 29 weeks with associated cost of birr 5,805 as shown in the table below.

<i>Project duration (weeks)</i>	<i>Crashing Activity And Weeks</i>	<i>Direct cost (Birr)</i>			<i>Indirect cost (Birr)</i>	<i>Total cost (Birr)</i>
		<i>Normal</i>	<i>Crashing</i>	<i>Total</i>		
32	--	4,220	--	4,220	$32 \times 50 = 1,600$	5,820
31	2-5(1)	4,220	$1 \times 45 = 45$	4,265	$31 \times 50 = 1,550$	5,815
29	5-6(2)	4,220	$45 + 45 \times 2 = 135$	4,355	$29 \times 50 = 1,450$	5,805
28	6-7(1)	4,220	$135 + 1 \times 70 = 205$	4,425	$28 \times 50 = 1,400$	5,825

Summery

Network model is a technique used for planning, and scheduling large projects in the field of construction, maintenance, fabrication, purchasing computer system, distribution, transportation, facility allocation, and many others.

A project is a set of proposals for the investment of resources in to a clearly identified set of actions that are expected to produce future benefits that exceeds costs.

There are two basic planning and control technique that utilize network to complete predetermined Project or schedule. These are Program Evaluation and Review Technique (PERT) and Critical Path Method.

Critical path analysis helps to estimate the total project duration & to assign starting & finishing times to all activities involved in the project. This helps in checking actual progress against the scheduled duration of the project.

List cost approach, also called the minimum –cost method, uses lowest cell as the basis for selecting route.

Vogel's approximation (penalty or regret) method is a heuristic method and is preferred to the other two methods described above.

The assignment problem is a special case of Transportation problem in which the number of sources and destinations are the same, and the objective is to assign the given job (tasks) to most appropriate machine (person) so as to optimize the objective function like minimizing cost or maximizing profit.

Solution of assignment problem is calculated based on the principles of reducing the given cost matrix to a matrix of opportunity costs. Opportunity cost shows the relative penalties associated with assigning resources to an activity as opposed to making the best (or least assignment).

Review Exercise

Choose the best answer

The objective of network analysis is to

- Minimize total project duration
- Minimize total project cost
- Minimize production delays, interruption and conflicts
- All of the above

In time cost tradeoff analysis

- Cost decrease linearly as time increases
- Cost at normal time is zero
- Cost increases linearly as time increases
- None

In PERT network each activity time assume a Beta distribution because

- It is uni-modal distribution that provides information regarding the uncertainty of time estimates of activities.
- It has got finite non negative error
- It need not be symmetrical about model value
- All

The activity which can be delayed without affecting the execution of the immediate succeeding activity is determined by

- Total float
- Free float
- Independent float
- A and B
- None

Generally pert technique deals with the project of a.

- Reiterative nature
- Non Reiterative nature

Deterministic nature

None

Part II Solve the following question

Listed in the table are the activities & sequencing necessary for the completion of a recruitment procedure for management trainees (MT) in a firm.

<u>Activity</u>	<u>Predecessor Activity</u>
A	-
B	A
C	A
D	C
E	B
F	C
G	D, E
H	G
I	H
J	F
K	I, J
L	K
M	L
N	M, N.

Draw a network diagram for the project Answer:

The owner of a chain of fast food restaurants is considering a new computer system for accounting and inventory control. A computer company sent the following information about the computer system installation.

Activity			Expected time (days)		
			Optimistic	Most likely	Pessimistic
A	Select the computer model	-	4	6	8
B	Design input output system	A	5	7	15
C	Deign monitoring system	A	4	8	12
D	Assemble computer hard ware	B	15	20	25
E	Develop the main programs	B	10	18	26
F	Develop input-output routines	C	8	9	16
G	Create database	E	4	8	12
H	Install the system	D,F	1	2	3
I	Test and implement	G,H	6	7	8

Required:

Construct an arrow diagram for this problem

Determine the critical path and compute the expected completion time

Determine the probability of completing the project in 55 days

Chapter Six

Queuing Theory

Objectives:

On successful completion of this chapter, you should be able to:

- Define the term queuing theory
- Understand the essential features of queuing system
- Explain the queuing discipline

6.1. Introduction

The queuing theory also called the waiting line theory, owes its development to A K Erlang's effort to analyze telephone traffic congestion with a view to satisfying the randomly arising demand for the services of the Copenhagen automatic telephone system, in year 1909. The theory is applicable to situations where the "customers" arrive at some service stations for some service, wait (occasionally not): and then leave the system after getting the service.

Queuing Theory is a collection of mathematical models of various queuing systems. It is used extensively to analyze production and service processes exhibiting random variability in market demand (arrival times) and service times.

Queues or waiting lines arise when the demand for a service facility exceeds the capacity of that facility, that is, the customers do not get service immediately upon request but must wait, or the service facilities stand idle and wait for customers.

Some customers wait when the total number of customers requiring service exceeds the number of service facilities, some service facilities stand idle when the total number of service facilities exceeds the number of customers requiring service.

A common situation occurring in everybody life is that of queuing or waiting in a line. Registration for school term, red light at a traffic signal, thicket at a movie theater, checkout counter at a supermarket, buy signal when making a phone call, teller window at a bank, line or table at a dining hall or restaurant, and so on are examples waiting lines.

Queuing theory can be applied to a wide variety of operational situations where imperfect matching between the customers and the service facilities is caused by one's inability to predict accurately the arrival and service time of customers. In particular, it can be used to determine the level of service (either the service rate or the number of service facilities) that balances the following two conflicting costs:

Cost of offering the service

Cost incurred due to delay in offering service

An increase in the existing service facilities would reduce the customer's waiting time. Conversely, decreasing the level of service should result in long queue(s). This means an increase (decrease) in the level of service increases (decreases) the cost of operating service facilities but decreases (increases) the cost of waiting time.

Figure1.1. Illustrates both types of costs as a function of level of service. The optimum service level is one that minimizes the sum of the two costs.

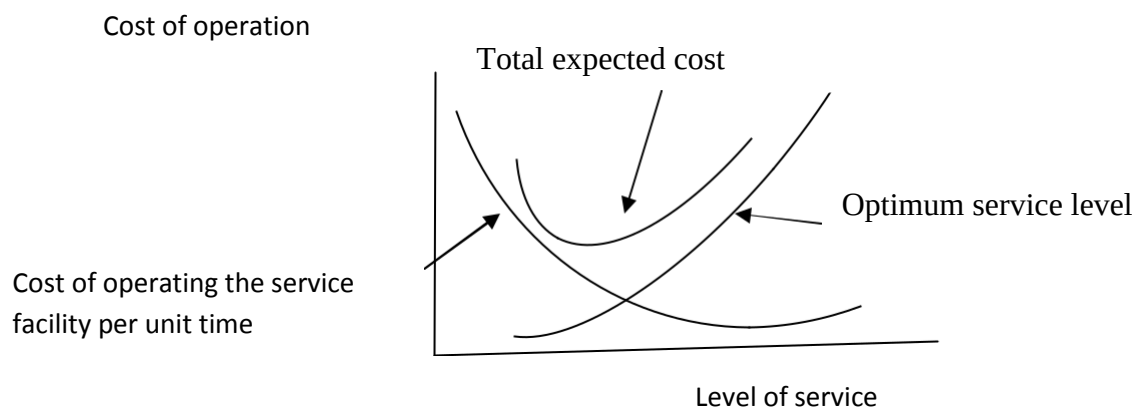


Fig.6.1. Queuing costs vs. level of service

Of the two types of costs mentioned above, the cost of waiting is difficult to estimate. However, it is measured in terms of loss of sales or goodwill when the customer is a human being who has no sympathy with the service. But, if the customer is a machine waiting for repair, then cost of waiting is measured in terms of the cost of lost production.

The waiting line is developed because the service to a customer may not be rendered immediately as the customer reaches to the service facility. Thus lack of adequate service facility would cause waiting lines of customers to be formed. The only way that the service demand can be met with ease is to increase the service capacity (raising the efficiency of the existing capacity if possible) to a higher level. A manager has to decide an appropriate level of service which is neither too low nor too high.

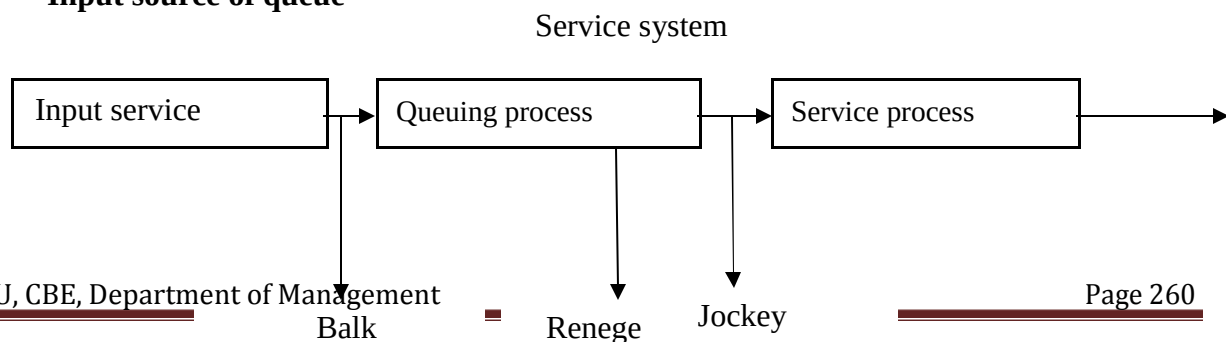
6.2. Essential Features of queuing system

The essential features or characteristics of queuing system are:

1. Input source
2. Queuing process
3. Queue discipline
4. Service process(or mechanism)

Customers requiring service are generated at different times by an input source, commonly known as population. The rate (constant or random) at which customers arrive at the service facility is determined by the arrival process. The customer's entry into the service system depends upon the queue conditions. If at the time of customer's arrival, the server is idle, then the customer is served immediately. Otherwise the customer is asked to join the queues, which can have several configurations.

Input source of queue



An input source is characterized by

- Size of the calling population
- Pattern of arrivals at the system
- Behavior of the arrivals

An input source need not be homogeneous population but may consist of several sub-populations.

Size of calling population: the size of calling population whether it is homogeneous or consists of several sub-populations is considered either to be finite. When the number of potential arrivals is dependent on the number of customers already in the system (those being served plus those in queue), the calling population is limited or finite. Alternatively, if a new customer's arrival is independent of the number of customers already in the system, the calling population is considered unlimited or infinite.

Behavior of arrivals: if a customer, on arriving at the service system stays in the system until served, no matter how much he has to wait for the service is called a patient customer. Whereas the customer, who waits for a certain time in the queue and leaves the service system without getting service due to certain reasons such as a long queue in front of him is called an impatient customer.

Pattern of arrivals: the arrival process (or pattern) to the service system is classified into two categories: static and dynamic. These two are further classified based on the nature of arrival rate and the control which can be exercised on the arrival process.

In **static arrival process**, the control depends on the nature of arrival rate (random or constant). Random arrivals are either at a constant rate or varying with time. Thus to analyze the queuing system, it is necessary to attempt to describe the probability distribution of arrivals. From such distributions we obtain average time between successive arrivals, also called inter-arrival time (time between two consecutive arrivals),

and the average arrival rate (i.e. number of customers arriving per unit of time at the service system).

The **dynamic arrival process** is controlled by both service facility and customers. The service facility adjusts its capacity to match changes in the demand intensity, by either varying the staffing levels at different timings of service, varying service charges (such as telephone call charges at different hours of the day or week) at different timings, or allowing entry with appointments.

The variation in the demand intensity also affects the customer's behavior. They either balk or renege from the service system when confronted with a long slow moving waiting line.

The arrival time distribution can be approximated by one of the following probability distributions;

Poisson distribution

Exponential distribution

Erlang distribution

Frequently in queuing problems, the number of arrivals per unit of time can be estimated by a probability distribution known as the Poisson distribution, as it adequately supports many real world situations.

The Poisson distribution is a discrete probability distribution of the number of customers (events) arriving in some time interval. The Poisson probability mass function is

$$P(X=n/P_n=\lambda t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, n=0, 1, 2, \dots$$

Now, if we are interested in the inter arrival time probability distribution, then the time interval from 0 to t, the probability of no arrival is given by

$$P(X=0/P_n=\lambda t) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

Let us define the random variable T as the time between successive arrivals. Since a customer can arrive at any time, T must be a continuous random variable. The probability of no arrival in the time interval from 0 to t will be equal to the probability that T exceeds t , so we have

$$P(T > t) = p(x = p_n \lambda t) = e^{-\lambda t}$$

The probability that there is an arrival in the time interval from 0 to t is given by

$$P(T \leq t) = 1 - p(T > t) = 1 - e^{-\lambda t}; t \geq 0$$

This probability is also called the cumulative probability distribution function, $F(t)$ of T . Also the distribution of the random variable T is referred to as the exponential distribution. Its probability density function (pdf) is

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$

In many practical situations, the inter-arrival time is approximated by an exponential distribution (sometimes called the negative exponential distribution).

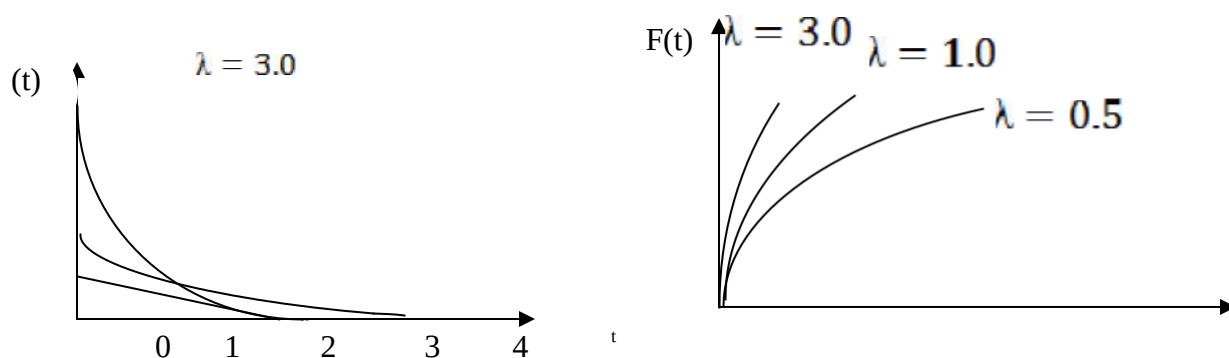


Fig 6.2. _ Exponential probability distribution function fig---Exponential cumulative probability density function

The mean of exponential distribution is the expected or average time between arrivals E (T). Thus with λ arrivals per unit time, $E(T) = \frac{1}{\lambda}$

Hence from the above **Operations Research**

discussion we may conclude that the Poisson distribution of arrivals with arrival rate λ is equal to the negative exponential distribution of inter-arrival times with mean inter arrival times $\frac{1}{\lambda}$.

The probability density function of exponential distribution can also be used to compute the probability that the next customer arrives within time, T of the previous arrival. This means that if a customer has already arrived at the service system, then probability of arriving for the next customer can be determined by using this equation.

$$(1.5) = P(T \leq 1.5) = 1 - (x=0/p_n=1.8)$$

$$1 - 0.165 = 0.835$$

Also, if customers arrive at the service system at an average rate of 24 customers per hour, and a customer has already arrived, then probability of another customer arriving in the next 5 minutes (i.e. $t=1/12$ hr) is

$$\begin{aligned} P(\text{inter-arrival time} \leq 1/12) &= 1 - e^{-24(1/12)} \\ &= 1 - e^{-2} = 1 - 0.1353 = 0.8647 \end{aligned}$$

Further if $\lambda=24$ customers per hour, $t=1/12$ hour and $\lambda t=24 \times (1/12)=2$, the probability of n customers arriving within the next 10 minutes is:

$$P(x = 2/p_n = 2) = \frac{e^{-24(\frac{1}{12})} \left(\frac{24}{12}\right)^2}{2!} = 0.27$$

Queuing process

The queuing process refers to the number of queues, and their respective lengths. The number of queues depends upon the layout of the service system. Thus there may be single or multiple queues. Certain service systems adopt “take a number” policy to avoid a formal queue to form; the length (or size) of the queue depends upon the operational

In certain cases, a service system is unable to accommodate more than the required number of customers at a time. No further customers are allowed to enter until space becomes available to accommodate new customers. Such type of situations are referred to as finite (or limited) source queue. On the other hand, if the service system is able to accommodate any number of customers at a time, then it is referred to as infinite (or unlimited) source queue.

In many other situations, when arriving customers experience long queue(s) in front of a service facility, they often do not enter the service system even though additional waiting space is available. The queue length in such cases depends upon the attitude of the customers. In some finite source queue systems, the maximum permissible queue is of zero length.

A multiple queue system at a service system can also be finite or infinite. But this has certain advantages such as:

- The service provided to the customers can be differentiated by adopting different service rules called queuing discipline

- II. Division of manpower is possible

- III. Customers have the option of joining any queue and can also switch to the end of any other queues

- IV. Balking behavior of the customers can be controlled

Queue discipline

In the queue structure, the important thing to know is the queue discipline. The queue discipline is the order or manner in which customers from the queue are selected for service.

There are a number of ways in which customers in the queue are served. Some of these are:

- a) **Static queue disciplines:** are based on the individual customer's status in the queue. Few

If the customers are served in the order of their arrival, then this is known as the **first-come, first-served (FCFS)** service discipline. Prepaid taxi queue at airports where a taxi is engaged on a first-come, first-served basis is an example of this discipline.

Last-come-first-served (LCFS) sometimes, the customers are serviced in the reverse order of their entry so that the ones who join the last are served first. For example, assume that letters to be typed, or order forms to be processed accumulate in a pile, each new addition being put on the top of them. The typist or the clerk might process these letters or orders by taking each new task from the top of the pile. Thus, a just arriving task would be the next to be serviced provided that no fresh task arrives before it is picked up. Similarly, the people who join an elevator last are the first ones to leave it.

Dynamic queue disciplines are based on the individual customer attributes in the queue. Few of such disciplines are:

Service in Random Order (SIRO) -- Under this rule customers are selected for service at random, irrespective of their arrivals in the service system. In this every customer in the queue is equally likely to be selected. The time of arrival of the customers is, therefore, of no relevance in such a case.

Priority Service-- Under this rule customers are grouped in priority classes on the basis of some attributes such as service time or urgency or according to some identifiable characteristic, and FCFS rule is used within each class to provide service. Treatment of VIPs in preference to other patients in a hospital is an example of priority service.

Pri-emptive priority- under this rule, the highest priority customer is allowed to enter into the service immediately after entering into the system even if a customer with lower priority is already in service. That is lower priority customer's service is interpreted (pre-empted) to start service for a special customer. This interrupted service resumed again after the highest priority customer is served or according to the pre-emptive rules.

Non-pre-emptive priority- in this case highest priority customer goes ahead in the AMU,

For the queuing models that we shall consider, the assumption would be that the **customers are serviced on the first-come-first-served basis.**

Service process (mechanism)

The service process is concerned with the manner in which customers are serviced and leave the service system. It is characterized by

The arrangement (or capacity) of the service system

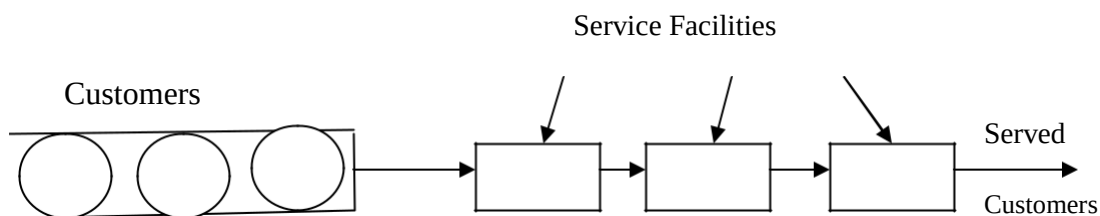
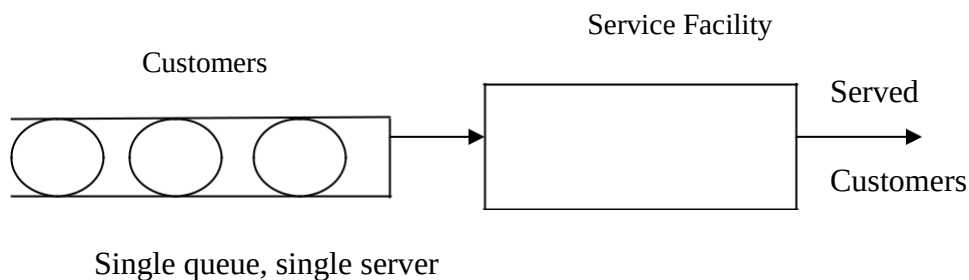
The distribution of the service lines

Server's behavior

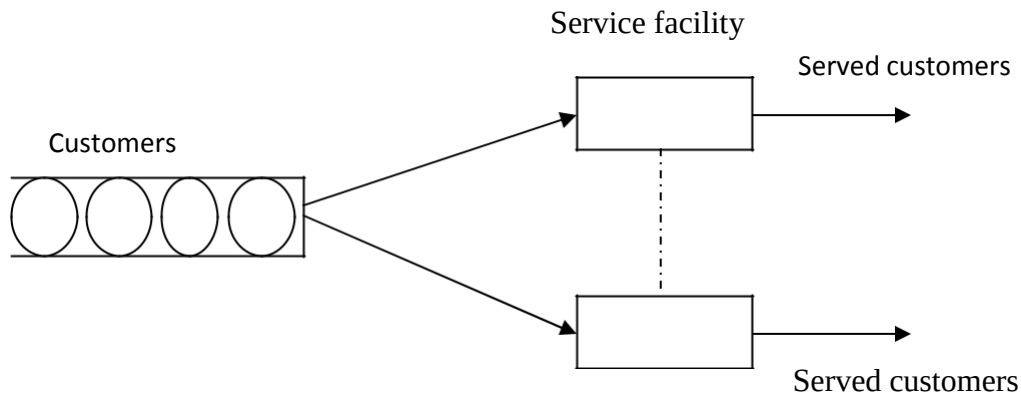
Management policies

Arrangement of service facilities the capacity of the service mechanism is measured in terms of customers that can be served simultaneously and/or effectively. The service facilities (or servers), commonly known as service channels may be series or in parallel or mixed (partly in series and partly in parallel).

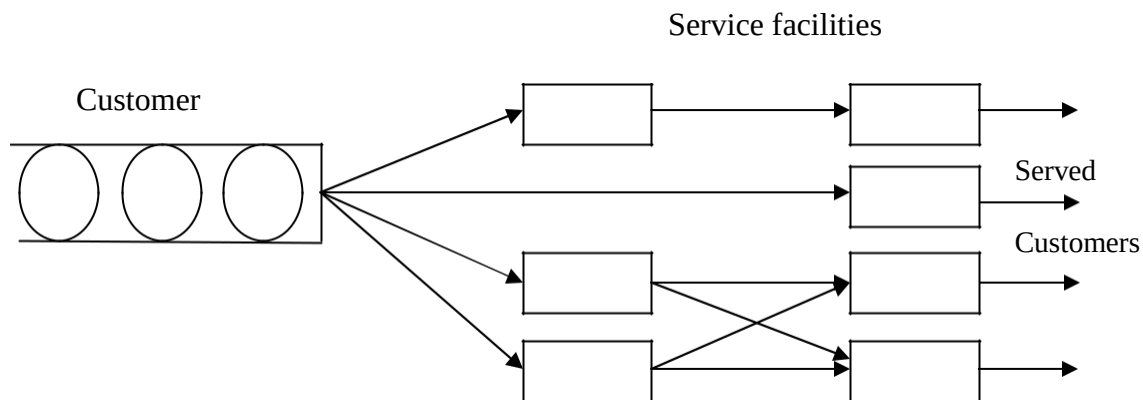
The series arrangement consists of a sequence of a number of service facilities such that a customer must go through one facility after another in a particular sequence before the whole service is completed. Each service facility may however, work independently of the others, having its own rules of service.



In the parallel arrangement the incoming customer may join the queue of his choice in front of service facilities or may be served by any service facilities.



Arrangement of service facilities in parallel



Mixed Arrangement of Service Facilities

Distribution of service time-the time taken by the server from the commencement of service to the completion of service for a customer is known as the **service time**.

If μ is the average service rate, then the expected number of customers served during time interval t will be μt . Thus, if we consider zero time at the start of service, the probability that services is not completed by time t is given by

$$p(x = 0/p_n = \mu t) e^{-\mu t}$$

If the random variable T represents the service time, then the probability of service completion within time t is given by

$$p(T \leq t) = 1 - e^{-\mu t}, t \geq 0$$

6.3. Probability distribution in queuing system

It is assumed that customers joining a queuing system arrive in random manner and follow a Poisson distribution or equivalently the inter-arrival times follow exponential distribution.

In most of the cases, service times are also assumed to be exponentially distributed. It implies that the probability of service completion in any short-time period is constant and independent of the length of time that the service has been in progress. The basic reason for assuming exponential service is that it helps in formulating simple mathematical models which ultimately help in analyzing a number of aspects of queuing problems.

The number of arrivals and departures (those served) during an interval of time in a queuing system is controlled by the following assumptions (also called axioms).

The probability of an event (arrival or departure) occurring during the time interval (t, t+Δt) depends on the length of time interval Δt. That is, probability of the event does not depend either on number of events that occur up to time t or the specific value of t, meaning that the events that occur in non-overlapping time are statistically independent.

The probability of more than one event occurring during the time interval (t, t+ Δt) is negligible. It is denoted by 0(Δt).

At most one event (arrival or departure) can occur during a small time interval Δt. the probability of an arrival during the time interval (t, t + Δt) is given by

$$p_1(\Delta t) = \lambda \Delta t + 0(\Delta t)$$

Where λ is a constant and independent of the total number of arrivals up to time t; Δt is a small time interval and 0(Δt) represents the quantity that becomes negligible when compared to Δt as Δt→0, i.e.

$$\lim_{\Delta t \rightarrow 0} \{ 0(\Delta t) / \Delta t \} = 0$$

Distribution of arrivals (pure birth process)

The arrival process assumes that the customers arrive at the queuing system and never leave it. Such a process is called pure birth process. The aim is to derive an expression for the probability $P_n(t)$ of n arrivals during time interval $(t, t + \Delta t)$. The terms commonly used in the development of various queuing models are the following:

Δt = a time interval so small that the probability of more than one customer's arrival is negligible, i.e. during any given small interval of time Δt only one customer can arrive.

$\lambda \Delta t$ = probability that a customer will arrive in the system during time Δt .

$1 - \lambda \Delta t$ = probability that no customer will arrive in the system during time Δt .

If the arrivals are completely random, then the probability distribution of a number of arrivals in a fixed time interval follows Poisson distribution.

Distribution of inter-arrival times (exponential process)

If the number of arrivals, n , in time t follows the Poisson distribution, then

$$p_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, n = 0, 1, 2, \dots$$

Is an associated random variable defined as the inter-arrival T follows the exponential distribution $f(t) = \lambda e^{-\lambda t}$ and vice versa.

Markovian property of inter-arrival times

The Markovian property of inter arrival times states that the probability that a customer currently in service is completed at some time t is independent of how long he has already been in service. That is

$$\text{prob.}\{T \geq t_1 / T \geq t_0\} = \text{Prob.} = \{T \geq t_1\} / \{T \geq t_0\} = \text{prob.} \{0 \leq T \leq t_1 - t_0\}$$

Where T is the time between successive arrivals

Distribution of departures (pure death process)

The departure process assumes that no customer joins the system while service is continued for those who are already in the system. Let, at time $t=0$ (starting time) there be $N \geq 1$ customers in the system. Since service is being provided at the rate of μ . Therefore, customers leave the system at the rate μ after being serviced. Such a process is called pure death process

Basic axioms

- i. Probability of the departure during time t is μt .
Probability of more than one departure between time t
and $t + t$ is negligible.

The number of departures in non- overlapping intervals are statistically independent.

The following terms are used in the development of various queuing models.

$t =$ probability that a customer in-service at time t will complete service during time t .

$1 - \mu t =$ probability that the customer in-service at time t will not complete service during time t .

The probability density function $s(t)$ of service time is given by

$$s(t) = \begin{cases} \mu e^{-\mu t}; & 0 \leq t \leq \infty \\ 0 & ; t < 0 \end{cases}$$

This shows that service times follows negative exponential distribution with mean $1 / \mu$ and variance $1 / \mu^2$.

Distribution of service times

The service time of short duration have the highest probability of occurrence. As the service time increases, the probability of occurrence tails off exponentially towards zero probability.

The area under the negative exponential distribution curve is determined as:

$$F(T) = \int_0^T \mu e^{-\mu t} dt = [-\mu e^{-\mu t}]_0^T$$

$$= -e^{-\mu T} + e^0 = 1 - e^{-\mu T}$$

It is also described as:

$$F(T) = P(t \leq T) = 1 - e^{-\mu T}$$

Where $F(T)$ is the area under the curve to the left of T . Thus

$$1 - F(T) = P(t > T) = e^{-\mu T} \text{ is the area under the curve to the right of } T.$$

6.4. Single Server Queuing Models

Model I {M/M1}: (∞ /FCFS)} Exponential service-unlimited Queue

The derivation of this model is based on certain assumptions about the queuing system:

Exponential distribution of inter-arrival time or Poisson distribution of arrivals

Single waiting line with no restriction on length of queue (i.e infinite capacity) and no balking or reneging

Queue discipline is “first-come, first served”

IV. Single server with exponential distribution of service time

The following events (possibilities) may occur during a small interval of time, Δt just before time t . it is assumed that the system is in state n (number of customers) at time t .

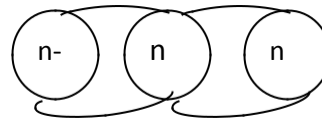
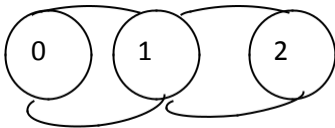
The system is in state n (number of customers) and no arrival and no departure, leaving the total to n customer.

The system is in state $n+1$ (number of customers) and no arrival and one departure, reducing the total to n customers.

The system is in state $n-1$ (number of customers) and one arrival and no departure, bringing the total to n customers.

We will begin with determining probability, P_n of n customers in the system at time t and value of its various operating characteristics is summarized as follows:

Figure XX illustrates the process of determining P_n by considering each possible number of customers either waiting or receiving service at each state which may be entered by the arrival of new customer or left by the completion of the leading customer's service.



Step 1 obtain system of differential difference equation

If $p_n(t)$ is the probability of a customers at time t in the system, then the probability that the system will contain n customers at time $(t + \Delta t)$ can be expressed as the sum of the joint probabilities of the three mutually exclusive and collectively exhaustive cases as mentioned above. That is

For $n \geq 1$ and $t \geq 0$.

$$\begin{aligned}
 p_n(t + \Delta t) = & P_n(t) \cdot \text{prob. (no arrival in } \Delta t \text{ and no departure in } \Delta t) \\
 & + P_{n+1}(t) \cdot \text{Prob. (no arrival in } \Delta t \text{ and one departure in } \Delta t) \\
 & + P_{n-1}(t) \cdot \text{Prob. (no arrival in } \Delta t \text{ and no departure in } \Delta t) \\
 & p_n(t) \{ \text{prob. (no arrival in } \Delta t) \times \text{prob. (no departure in } \Delta t) \} \\
 & + P_{n+1}(t) \cdot \text{Prob. (no arrival in } \Delta t) \times \text{prob. (one departure in } \Delta t) \\
 & + P_{n-1}(t) \cdot \text{Prob. (one arrival in } \Delta t) \times \text{prob. (one departure in } \Delta t)
 \end{aligned}$$

$$p_n(t)\{1 - \lambda \Delta t\} (1 - \mu \Delta t) + p_{n+1}(t) \{1 - \lambda \Delta t\} \mu \Delta t + p_{n-1}(t) \{\lambda \Delta t\} \{1 - \mu \Delta t\}$$

$p_n(t)\{1 - (\lambda + \mu)\Delta t\} + p_{n+1}(t)\mu \Delta t + p_{n-1}(t) \lambda \Delta t$ Terms involving $(\Delta t)^2$ since Δt is very small, therefore terms involving $(\Delta t)^2$ can be neglected. Subtracting $p_n(t)$ from both sides and dividing by Δt , we got

$$\frac{p_n(t + \Delta t) - p_n(t)}{\Delta t} = \lambda p_{n+1}(t) + \mu p_{n+1}(t) - (\lambda + \mu) p_n(t)$$

Taking limit on both sides as $\Delta t \rightarrow 0$, then above equation reduces to

$$p'_{n(t) = \frac{d}{dt}} \{ \} = \lambda p_{n-1}(t) + \mu p_{n+1}(t) - (\lambda + \mu) p_n(t); n \geq 1 \quad (1)$$

Similarly, if there is no customer in the system at time $(t + \Delta t)$, when there will be no service completion during Δt . Thus for $n=0$ and $t \geq 0$, we have only two probabilities instead of three. The resulting equation is

$$\frac{p_n(t + \Delta t) - p_0(t)}{\Delta t} = \mu p_1(t) - \lambda p_0(t) + \frac{0(\Delta t)}{\Delta t}$$

Taking limit on both sides as $\Delta t \rightarrow 0$, we get

$$p'_{0(t) = \frac{d}{dt}} \{ \} = \mu p_1(t) - \lambda p_0(t); n = 0 \quad (2)$$

Step 2 obtaining the system of steady state equations

In the steady-state, $p_n(t)$ is independent of time t , and the rate of change $p'_n(t)$ can be considered to be zero. That is

$$\lim_{n \rightarrow \infty} p_n(t) = p_n$$

$$\text{And } \lim_{n \rightarrow \infty} \frac{d}{dt} \{ p_n(t) \} = 0; n = 0, 1, 2, \dots$$

Consequently, equations (1) and (2) may be written as:

$$\lambda p_{n-1} + \mu p_{n+1} - (\lambda + \mu) p_n = 0; n \geq 1 \quad (3)$$

$$\mu p_1 - \lambda p_0 = 0; n = 0 \quad (4)$$

Thus these equations constitute the system of steady-state difference equations. The solution of these equations can be obtained either by (i) iterative method, (ii) generating functions method, or (iii) linear operator method. Here we shall be using the iterative method to find the values of $P_1, P_2, \dots$ in terms of P_0, λ and μ .

Step 3 Solve the system of difference equations

From equation 4 we get

$$\lambda p_0 = \mu p_1 \text{ or } p_1 = \left(\frac{\lambda}{\mu}\right) p_0; n = 0$$

Putting $n=1$ in equation (3), we get

$$0 = -(\lambda + \mu)p_1 + \lambda p_0 + \mu p_2$$

$$\text{Or } p_2 = \left(\frac{\lambda + \mu}{\mu}\right) p_1 - \frac{\lambda}{\mu} p_0 = \left(\frac{\lambda + \mu}{\mu}\right) \frac{\lambda}{\mu} p_0 - \frac{\lambda}{\mu} p_0 = \left(\frac{\lambda}{\mu}\right)^2 p_0$$

$$p_3 = \left(\frac{\lambda + \mu}{\mu}\right) p_2 - \left(\frac{\lambda}{\mu}\right) p_1$$

$$= \left(\frac{\lambda + \mu}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^2 p_0 - \left(\frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right) p_0 = \left(\frac{\lambda}{\mu}\right)^3 p_0; n = 2$$

In general, by using the inductive principle, we get

P_n = probability of being in state n (i.e. n customers in the system)

$$= \left(\frac{\lambda}{\mu}\right)^n p_0; n = 1, 2, \dots$$

To obtain the value of p_0 , we make use of the fact that (sum of all possibilities)

$$\sum_{n=0}^{\infty} p_n = p_0 + p_1 + \dots + p_n + \dots = 1$$

Or

$$\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n p_0 = 1 \text{ or } p_0 = \frac{1}{\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n} = \frac{\lambda}{\mu} < 1$$

The denominator of this expression is an infinite geometric series, whose sum is:

$$\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = \frac{1}{1 - (\lambda/\mu)}$$

And hence $p_0 = 1 - \frac{\lambda}{\mu} = 1 - p$

And $p_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) = p^n (1-p); p < 1, n=0, 1, 2, \dots$

$p_w =$ Probability that an arriving customer has to wait $= 1 - p_0 = \lambda/\mu$

This expression gives the required probability distribution for exactly n customers in the queuing system. Hence p_0 denotes the probability of the system being empty (no customer).

Step 4 obtain probability density function of waiting time excluding service time distribution

The waiting time distribution of each customer in the steady-state is same, and it is a continuous random variable except that there is a non-zero probability that the delay will be zero probability that the delay will be zero, i.e. waiting time is zero. Let w be the time required by the server to serve all the customers present in the system at a particular time in the steady state. Let $\phi_w(t)$ be the probability distribution function of w, i.e. $\phi_w(t) = p(w < t), 0 < t < \infty$

If an arriving customer find $n(\geq 1)$ customers already in the system, then in order for a customer to get service at time between 0 and t, all the customers must have been

served by time t. let $s_1, s_2, \dots, s_n$ denote service of n customers, respectively.

$$\begin{cases} 0 & ; n = 0 \\ \sum_{i=1}^n S_i & ; n \geq 1 \end{cases}$$

The distribution function of waiting time w for a customer who has to wait is given by

$$p\{w \leq t\} = \begin{cases} p_0 = 1-p & ; n=0, t=0 \\ p \left\{ \sum_{i=1}^n S_i \leq t \right\} & ; n \geq 1, t > 0 \end{cases}$$

Since the service time for each customer is independent and identically distributed, therefore its probability function density is $\varphi_S = \frac{-\mu t}{\sqrt{t^2 + \mu^2}}$ where μ the mean service rate is. Thus

$$\phi_n(t) = \sum_{i=1}^n [p_n \text{ prob}\{(n-1) \text{ customers got service at time } t\}]$$

$$\times \text{prob}\{\text{one customer is under service during time } \Delta t\}$$

$$= \sum_{i=1}^n \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n \left[\frac{(\mu t)^{n-1} e^{-\mu t}}{(n-1)!} \right] \mu \Delta t$$

Thus the expression for $\phi_w(t)$ may be written as

$$\begin{aligned} \phi_w(t) = p\{w \leq t\} &= \begin{cases} 1-p & ; t=0 \\ \sum_{n=1}^{\infty} p^n \int_0^t \phi_n(t) dt \end{cases} \\ &= \begin{cases} 1-p & ; t=0 \\ \sum_{n=1}^{\infty} p^n (1-p) \int_0^t \frac{(\mu t)^{n-1} \mu e^{-\mu t}}{(n-1)!} dt & ; t > 0 \end{cases} \\ &= \begin{cases} 1-p & ; t=0 \\ \int_0^t p(1-p) \mu e^{-\mu t} dt \sum_{n=1}^{\infty} \frac{(\mu p t)^{n-1}}{(n-1)!} & ; t > 0 \end{cases} \\ &= \begin{cases} 1-p & ; t=0 \\ p(1-p) \int_0^t \mu e^{-\mu(1-p)t} dt & \end{cases} \end{aligned}$$

This shows that the waiting time distribution is discontinuous at $t=0$ and continuous in the range $0 < t < \infty$. thus expression for $\phi_w(t)$ may also be written as

$$\phi_w(0) = 1 - p; t = 0$$

$$\begin{aligned}\phi'_w(t) &= \lambda(1-p)e^{-\mu(1-p)t} dt \\ &= \lambda\left(1 - \frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t} dt; t > 0\end{aligned}$$

Step 5 calculate busy period distribution

For the busy period distribution, let the random variable w denote the total time (waiting and service) that a customer spends in the system. Then the probability density function for the distribution is given by

$$\phi(w; w > 0) = \frac{\phi(w)}{\text{prob}(w > 0)}; \phi(w) = \frac{d}{dt}\{\phi_w(t)\}$$

$$= \frac{\phi(w)}{\int_0^\infty \phi(t) dt} = \frac{\lambda\left(1 - \frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t}}{\int_0^\infty \lambda\left(1 - \frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t} dt}$$

$$\frac{\lambda\left(1 - \frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)t}}{\frac{\lambda}{\mu}} = (\mu - \lambda)e^{-(\mu-\lambda)t}; t > 0$$

Now $\int_0^\infty \phi(w; w > 0) dt = \int_0^\infty (\mu - \lambda) e^{-(\mu-\lambda)t} dt = 1$

Which is required distribution of busy period

Remark if λ is viewed as a measurement of demand for service, and μ as the capacity of the service facility, then $\mu - \lambda$ represents the 'excess capacity' of the system to fill the demand.

Performance measure for Model I

1. a) Expected number of customers in the system (customers in the line plus the customer being served)

$$\begin{aligned}
 L_s &= \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n(1-p)p^n, 0 < p < 1 \\
 &= (1-p) \sum_{n=0}^{\infty} n p^n = p(1-p) \sum_{n=1}^{\infty} n p^{n-1} \\
 &= p(1-p) \{1 + 2p + 3p^2 + \dots\} \\
 &= (1-p) \left\{ \frac{p}{(1-p)^2} \right\} \quad [\text{Arithmetic - geometric series}]
 \end{aligned}$$

Or $L_s = \frac{p}{1-p} = \frac{\lambda}{\mu - \lambda}$
 b) Expected number of customers waiting in the queue (i.e. queue length)

$$L_q = \sum_{n=1}^{\infty} (n-1) p_n = \sum_{n=1}^{\infty} n P_n - \sum_{n=1}^{\infty} p_n$$

$$= \sum_{n=0}^{\infty} n P_n - \left[\sum_{n=0}^{\infty} p_n - p_0 \right]$$

$$L_s - (1 - p_0) = \frac{\lambda}{\lambda - \mu} - \frac{\lambda}{\mu}; 1 - p_0 = \frac{\lambda}{\mu}$$

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

2. a) expected waiting time for a customer in the queue

$$\begin{aligned}
 W_q &= \int_0^{\infty} t \cdot \left\{ \frac{d}{dt} \phi_w(t) \right\} dt \\
 &= \int_0^{\infty} t \cdot \lambda (1-p) e^{-\mu(1-p)t} dt
 \end{aligned}$$

$$\text{Integrating by parts } W_q = (1-p) \left[\frac{t e^{-\mu(1-p)t}}{-\mu(1-p)} - \frac{e^{-\mu(1-p)t}}{\mu^2(1-p)^2} \right]_0^{\infty}$$

$$= \lambda \left(1 - \frac{\lambda}{\mu} \right) \frac{1}{(\mu - \lambda)^2}$$

$$\text{Or } W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{L_q}{\lambda}$$

b) Expected waiting time for a customer in the system (waiting and service)

$$W_s = \text{Expected waiting time in queue} + \text{Expected service time}$$

$$= W_q + \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu}$$

$$\text{Or } = \frac{1}{\mu-\lambda} = \frac{L_s}{\lambda}$$

probability that the number of customers in the system is greater than or equal to k

$$p(n \geq k) = \left(\frac{\lambda}{\mu}\right)^k; = \left(\frac{\lambda}{\mu}\right)^{k+1}$$

the variance (fluctuation) of queue length

$$\begin{aligned} \text{Var}(n) &= \sum_{n=1}^{\infty} n^2 p_n - \left(\sum_{n=1}^{\infty} n p_n\right)^2 \\ &= \sum_{n=1}^{\infty} n^2 p_n - (L_s)^2 = \sum_{n=1}^{\infty} n^2 (1-p)p^n - \left(\frac{p}{1-p}\right)^2 \\ &= (1-p)[1 \cdot p^2 + 2^2 \cdot p^2 + 3^2 \cdot p^3 + \dots] - \left(\frac{p}{1-p}\right)^2 \\ \text{Var}(n) &= \frac{p}{(1-p)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2} \end{aligned}$$

probability that the queue is non-empty

$$\begin{aligned} p(n > 1) &= 1 - p_0 - p_1 \\ &= 1 - \left(1 - \frac{\lambda}{\mu}\right) - \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right) = \left(\frac{\lambda}{\mu}\right)^2 \end{aligned}$$

probability of k or more customers in the system

$$\begin{aligned} p(n \geq k) &= \sum_{n=k}^{\infty} p_k = \sum_{n=k}^{\infty} (1-p)p^k \\ &= (1-p)p^k \sum_{n=k}^{\infty} p^{n-k} \\ &= (1-p)p^k [1 + p + p^2 + \dots] = \frac{(1-p)p^k}{(1-p)} = p^k \end{aligned}$$

$$\text{Or } p(n \geq k) = \left(\frac{\lambda}{\mu}\right)^k; p(n > k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$$

7. expected length of non-empty queue

$$L_b = \frac{\text{Expected length of waiting line}}{p(n > 1)}$$

$$= \frac{L_b}{p(n > 1)} = \frac{\frac{\lambda^2}{\mu} (\mu - \lambda)}{(\lambda/\mu)^2} = \frac{\mu}{\mu - \lambda}$$

probability of an arrival during the service time when system contains r customers

$$\begin{aligned} p(n=r) &= \int_0^{\infty} p_r(t) s(t) dt = \int_0^{\infty} \frac{(\lambda t)^r}{r!} \mu e^{-\mu t} dt \\ &= \frac{\lambda^r \mu}{r!} \int_0^{\infty} e^{-(\lambda+\mu)t} \cdot t^r dt = \frac{\lambda^r \mu}{r!} \frac{p(r+1)}{(\lambda + \mu)^{r+1}} \\ &= \left(\frac{\lambda}{\lambda+\mu}\right)^r \left(\frac{\mu}{\lambda+\mu}\right); \text{ since } \int_0^{\infty} e^{-xt} t^r dt = \frac{r!}{x^{r+1}}; p(r+1) = r! \end{aligned}$$

Example

A television repairman finds that the time spent on his jobs has an exponential distribution with a mean of 30 minutes. If he repairs sets in the order in which they come in, and if the arrival of sets follows a Poisson distribution approximately with an average rate of 10 per 8-hour day, what is the repairman's expected idle time each day? How many jobs are headed of the average set just brought in?

Solution

From the data of the problem, we have

$$\lambda = 10/8 = 5/4 \text{ sets per hour; and } \mu = (1/30)60 = 2 \text{ sets per hour}$$

Expected idle time of repairman each day

Number of hours for which the repairman remains busy in an 8 hour day (traffic intensity) is given by

$$(8) \left(\frac{\lambda}{\mu}\right) = (8) \left(\frac{5}{8}\right) = 5 \text{ hours}$$

Hence the idle time for a repairman in an 8-hours day will be: (8-5) = 3 hours

Expected (or average) number of TV sets in the system

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{5/4}{2 - (5/4)} = \frac{5}{3} = 2(\text{approximately}) \text{ TV sets}$$

Example 2

Arrivals at telephone booth are considered to be Poisson with an average time of 10 minutes between one arrival and the next. The length of phone call is assumed to be distributed exponentially with mean three minutes.

What is the probability that a person arriving at the booth will have to wait?

The telephone department will install a second booth when convinced that an arrival would expect waiting for at least 3 minutes for a phone call. By how much should the flow of arrivals increase in order to justify a second booth?

What is the average length of queue that forms from time to time?

What is the probability that it will take him more than 10 minutes altogether to wait for the phone and complete his call?

Solution

From the data of the problem we have

$\lambda = 1/10 = 0.10$ person per minute and $\mu = 1/3 = 0.33$ person per minute

Probability that a person has to wait at the booth

$$p(n > 0) = 1 - p_0 = \lambda / \mu \\ = 0.10 / 0.33 = 0.3$$

The installation of second booth will be justified only if the arrival rate is more than the waiting time. Let λ' be the increased arrival rate. Then expected waiting time in the queue will be

$$W_q = \frac{\lambda'}{\mu(\mu - \lambda')} \\ 3 = \frac{\lambda'}{0.33(0.33 - \lambda')} \text{ or } \lambda' = 0.16$$

Where $W_q = 3$ (given) and $\lambda = \lambda'$ (say) for the second booth

Hence, the increase in the arrival rate is $0.60 - 0.10 = 0.06$ arrivals per minute

c) Average length of non-empty queues

$$L_b = \frac{\mu}{\mu - \lambda} - \frac{0.33}{0.23} = 2 \text{ customers approximately}$$

d) Probability of waiting for 10 minutes or more is given by

$$\begin{aligned} p(t \geq 10) &= \int_0^{\infty} \frac{\lambda}{\mu} (\mu - \lambda) e^{-(\mu - \lambda)t} dt \\ &= \int_{10}^{\infty} (0.3)(0.23) e^{0.23t} dt = 0.069 \left[\frac{e^{-0.23t}}{-0.23} \right] \end{aligned}$$

This shows that 3 percent of the arrivals on average will have to wait for 10 minutes or more before they can use the phone.

Model II $\{(M/M/1);(\infty/SIFO)\}$

This model is identical to the model I with a difference only in queue discipline. Since the derivation of P_n is independent of any specific queue discipline, therefore in this model also we have

$$P_n = (1 - p)p^n; n = 1, 2, \dots$$

Consequently, other results will also remain unchanged in any queuing system as long as P_n remain unchanged.

Model III $\{(M/M/1);(N/FCFS)\}$ exponential service, Finite (or limited) Queue

The model is different from model I in respect of the capacity of the system. Suppose that no more than N customers can be accommodated at any time in the system due to certain reasons. Thus any customer arriving when the system is already contains N customers dies not enter the system and is lost.

The difference equations derived in model I will also be same for this model as long as $n \leq N$. The systems of steady-state difference equations for this model are:

$$\begin{aligned} \lambda p_0 &= \mu p_1 & ; n = 0 \\ (\lambda + \mu) p_n &= \lambda p_{n-1} + \mu p_{n+1} & ; n = 1, 2, \dots, N - 1 \end{aligned}$$

$$\lambda p_{N-1} = \mu p_N \quad ; n = N$$

Assumptions of this model are same as that of Model I except that the length of the queue is limited. Moreover, in this case the service rate does not have to exceed arrival rate ($\mu > \lambda$) in order to obtain steady state conditions.

Using the usual procedure from first two difference equations the probability of a customer in the system for $n = 0, 1, 2, \dots, N$ are obtained as follows.

$$P_n = (\lambda/\mu)^n p_0, n \leq N$$

Now in order to obtain the value of p_0 use the fact that

$$\begin{aligned} 1 &= \sum_{n=0}^N p_n = \sum_{n=0}^N (\lambda/\mu)^n p_0 \\ &= p_0 \sum_{n=0}^N (\lambda/\mu)^n = p_0 \sum_{n=0}^N p^n \\ &= p_0 [1 + p + p^2 + \dots + p^N] = p_0 \left[\frac{1-p^{N+1}}{1-p} \right] \end{aligned}$$

And consequently, $p_0 = \frac{1-p^{N+1}}{1-p} ; p \neq 1$ and $p < 1 ; p = \lambda/\mu$

$$\begin{cases} \left(\frac{1-p}{1-p^{N+1}} \right) p^n ; n \leq N ; p \neq 1 (\lambda \neq \mu) \\ \frac{1}{N+1} ; p = 1 (\lambda = \mu) \end{cases}$$

The steady-state solution in this case exists even for $p > 1$. this is due to the limited capacity of the system which controls the arrivals by the queue length ($= N-1$) not by the relative rates of arrivals and departure, λ and μ . if

$\lambda < \mu$ and $N \rightarrow \infty$, then $p_n = \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^n$, which is same as in model I.

Performance Measure for Model III

1. Expected number of customers in the system

$$\begin{aligned}
 L_s &= \sum_{n=1}^N np_n = \sum_{n=1}^N n \left(\frac{1-p}{1-p^{N+1}} \right) p^n \\
 &= \frac{1-p}{1-p^{N+1}} \sum_{n=0}^N np^n = \frac{1-p}{1-p^{N+1}} (p + 2p^2 + 3p^3 + \dots + Np^N) \\
 &\begin{cases} \frac{p}{1-p} - \frac{(N+1)p^{N+1}}{1-p^{N+1}}; & p \neq 1 (\lambda \neq \mu) \\ \frac{N}{2} & ; p = 1 (\lambda = \mu) \end{cases}
 \end{aligned}$$

Expected Queue length or expected number of customers waiting in the system

$$L_q = L_s - \frac{\lambda}{\mu} = L_s - \frac{\lambda(1-p_N)}{N}$$

Expected waiting time of a customer in the system (waiting + service)

$$W_s = \frac{L_s}{\lambda(1-p_N)}$$

Expected waiting time of a customer in the queue

$$W_q = W_s - \frac{1}{\mu} \text{ or } \frac{L_q}{\lambda(1-p_N)}$$

Fraction of potential customers lost (=fraction of time system is full)

$$p_N = p_0 p^N$$

Effective arrival rate, $\lambda_s = \lambda(1-p_N)$

Effective traffic intensity, $\rho_s = \lambda_s / \mu$

Example

Consider a single server queuing system with Poisson input, exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hour and the maximum permissible calling units in the system is two. Derive the steady-state probability distribution of the distribution of the number of calling units in the system, and then calculate the expected number in the system.

Solution

From the data of the problem, we have

$\lambda = 3$ units per hour; $\mu = 4$ per hour, and $N=2$

Then traffic intensity, $\rho = \lambda/\mu = 3/4 = 0.75$

The steady-state probability distribution of the number of n customers (calling units) in the system is

$$p_n = \frac{(1 - \rho)\rho^n}{1 - \rho^{N+1}}; \rho \neq 1$$

$$= \frac{(1 - 0.75)(0.75)^n}{1 - (0.75)^{2+1}} = (0.43)(0.75)^n$$

$$p_0 = \frac{(1-\rho)}{1-\rho^{N+1}} = \frac{1-0.75}{1-(0.75)^{2+1}} = \frac{0.25}{1-(0.75)^3} = 0.431$$

The expected number of calling units in the system is given by

$$L_s = \sum_{n=1}^N np_n = \sum_{n=1}^2 n(0.43)(0.75)^n$$

$$= 0.43 \sum_{n=1}^2 n(0.75)^n = 0.43\{(0.75) + 2(0.75)^2\} = 0.81$$

Chapter summary

The queuing theory also called the waiting line theory, owes its development to A K Erlang's effort to analyze telephone traffic congestion with a view to satisfying the randomly arising demand for the services of the Copenhagen automatic telephone system, in year 1909.

Queuing Theory is a collection of mathematical models of various queuing systems. It is used extensively to analyze production and service processes exhibiting random variability in market demand (arrival times) and service times.

It is assumed that customers joining a queuing system arrive in random manner and follow a Poisson distribution or equivalently the inter-arrival times follow exponential distribution.

Review Question

Solve the following problem

A company distributes its products by trucks loaded at its only loading station. Both, company's trucks and constructor's trucks, are used for this purpose. It was found out that on average every five minutes, one truck arrived and the average loading time was 3 minutes. 50% of the trucks belong to the constructor. Find out

The probability that a truck has to Wait

The waiting time of truck that waits and

The expected waiting time of constructor truck per day, assuming a 24 shift

Customers arrive at a one window drive according to a poison distribution with mean of 10 minutes and service time per customer is Solve exponential with mean of 6 minutes. The space in front of the window can accommodate only 3 vehicles including the serviced one. Other vehicles have to wait outside this space. Calculate

probably that an arriving customer can drive directly to the space in front of the window

Probability that an arriving customer will have to wait outside the directed space.

how long an arriving customer is expected to wait before gating the service

A branch of Dashen Bank has only one typist. Since the typing work varies in length (number of pages to the typed) the typing rate is randomly distributed approximating a poison distribution with mean service rate of 8 letters per hour. The letter arrives at a rate of 5 per hour during the entire 8 hour work day. If the type writer is values at birr 1.50 per hour, determine

Equipment utilization

Average system time

The percent time that an arriving letter has to wait

Average cost due to waiting on the part of type writer, i.e. it remaining idle.

Answer for Review exercises

Chapter one

1. A, 2. A, 3. D, 4.A, 5.A

Chapter 2

1. A	7. D	13. B
2. D	8. F	14. A
3. F	9. A	15. A
4. D	10. B	16. B
5. B	11. B	17. A
6. D	12. C	18. D

Part II

Answer: $X_1=4, X_2=2$

2. Answer: $X_1=1, X_2=5$

1. and Maximize $Z=230$

and Minimize $Z=13$

Answer: 10 units of model X, 2 units of model Y and the maximum profit is \$ 560.

4. Answer: $X_1=187.5, X_2=125$ and Maximize $Z=21,875$

Minimize $18Y_1 + 14Y_2$

Subject to $2Y_1 + 2Y_2 \geq$

$12 \quad 3Y_1 + Y_2 \geq 10$

$Y_1 \geq 0, Y_2 \geq 0$

Chapter three

Initial feasible solution

		W_1	W_2	W_3	W_4	Factory Capacity
Factory	F_1	3	2	7	6	5000
		5000				
	F_2	7	5	2	3	6000

Operations Research

F ₃	2	5	4	5	2500
			1000	1500	
Demand	6000	4000	2000	1500	13500

m= 3, n =4 ==> 3+4 -1 =6 occupied cells (Feasible)

<u>Routes</u>		<u>Units</u>		<u>Unit</u>	<u>Total</u>
<u>From</u>	<u>To</u>	<u>Shipped</u>	X	<u>Cost</u>	<u>=Cost</u>
F ₁	W ₁	5000		3	\$ 15000
F ₂	W ₁	1000		7	7000
F ₂	W ₂	4000		5	20000
F ₂	W ₃	1000		2	2000
F ₃	W ₃	1000		4	4000
	F ₃	1500		5	<u>7500</u>
	W ₄			Total transportation cost = <u><u>\$55,500</u></u>	

b.

<u>Routes</u>		<u>Units</u>	<u>Unit</u>	<u>Total</u>
<u>From</u>	<u>To</u>	<u>Shipped</u>	<u>Cost</u>	<u>=Cost</u>
F ₁	W ₁	X	3	\$ 3000

Operations Research

F	W	1000	2	8000
1	2			
F ₂	W ₁	4000	7	17500
F ₂	W ₃	2500	2	4000
F ₂	W ₄	2000	3	45000
F ₃	W ₁	1500	2	<u>5000</u>
		2500	Total transportation cost	= <u><u>\$42,000</u></u>

m= 3,
n=4

=> 3+4-1 =6 occupied calls (Feasible)

Least- Cost method is better than the NWCM because it considers cost factories

Total profit= \$2,119,000

3.

$A \longrightarrow e$

$B \longrightarrow c$

$C \longrightarrow b$

$D \longrightarrow d$

Minimum distance = 570 km

Chapter 4

1. D

3. B

5. A

2. C

4. D

Answer b, regular share 0.047, b. risky shares 0.050 and property 0.060. c.
since the expected of 6% highest for property, the investor should invest in
this alternative.

Chapter five

1. A

3. A

5. D

2. A

4. B

Chapter six

A. 0.6, b. 7.5 minutes c. 10.8 hours

a. $98/125$ or 0.784, b. 0.216. c. $3/20$ hour or 9 minutes.

a. $5/8$ b, 20 minutes c. 62.5% d. birr 4.50

Assignment

A farmer has a 320 acre farm on which she plants two crops: corn and soybeans. For each acre of corn planted, her expenses are \$50 and for each acre of soybeans planted, her expenses are \$100. Each acre of corn requires 100 bushels of storage and yields a profit of \$60; each acre of soybeans requires 40 bushels of storage and yields a profit of \$90. If the total amount of storage space available is 19,200 bushels and the farmer has only \$20,000 on hand;

Required:

What are the **decision variables** of this problem?

Write the **objective function** of this problem

What are the **constraint of the problem**; Write them

Write **completed linear programming model** for the model

Given the following linear programming model;

$$2. \text{ Maximize profit} = 5x_1 + 4x_2$$

$$\text{Subjected to: } 6x_1 + 4x_2 \leq 24$$

$$x_1 + 2x_2 \leq 6$$

$$x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Required:

- a) **Draw the graph** of the problem;
- b) Identify the **feasible solutions** of the problem;
- c) Determine the values of **x_1 , x_2 and maximum profit**.

Suppose you are given the following LPM and its final tableau:

$$\text{Maximize } Z = 5x_1 + 6x_2 + 4x_3$$

Subjected to:

C1: $3x_1 + 4x_2 + 2x_3 \leq 120$

C2: $x_1 + 2x_2 + 4x_3 \leq 50$

C3: $x_1 + 2x_2 + 3x_3 \geq 30$

$x_1, x_2, x_3 \geq 0$

The corresponding optimal simplex tableau is given as follows

Basis Cj	5	6	4	0	0	0	RHSQ
	x1	x2	x3	s1	s2	s3	
S3	0	4	0	-2	7	1	80
X3	0	2	1	-1	3	0	30
X1	0	0	0	1	-2	0	20
Zj	5	8	4	1	2	0	220
Cj-Zj	0	-2	0	-1	-2	0	

Required:

If constraint-I increased by 2 units by what amount profit will increase?

If constraint-II increases by 5 units profit will increase by how many birr?

Calculate the range of feasibility for the right hand side quantity

Write the dual form of the problem

What are the solutions of dual variables?

4. An assembly is to be made from two parts **X** and **Y**. Both parts must be turned on a lathe and **Y** need not be polished. The sequence of activities together with their predecessors is given below:

Activity	Description	Predecessor Activity
A	Open work order	–
B	Get material for X	A
C	Get material for Y	A
D	Turn X on Lathe	B
E	Turn Y on Lathe	B, C
F	Polish Y	E
G	Assemble X and Y	D, F

H	Pack	G

Required: Draw a network diagram for the project.

4.