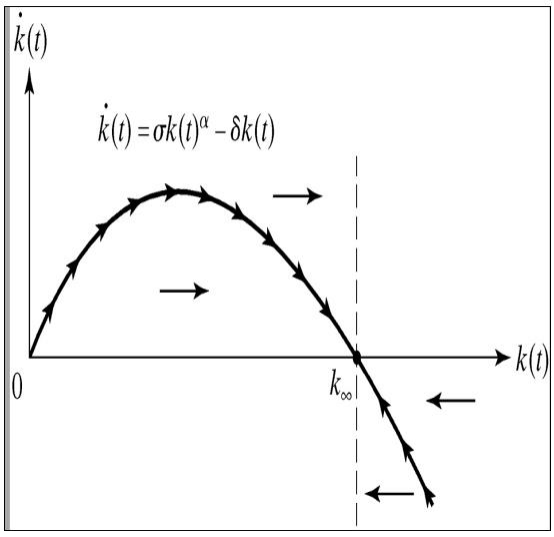
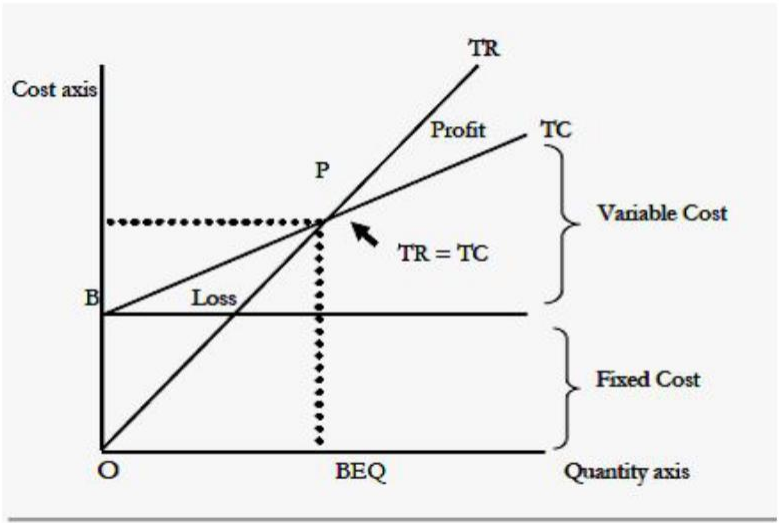
MATHEMATICAL ECONOMICS

(ECON 3011)





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Preface

Dear Distance learners! *Welcome to the course Mathematical Economics.*

Mathematical economics *is one of the most important instruments of economics used to describe the observed economic phenomenon. Starting from neoclassical economic thinking, mathematics and statistics have been given important role in explaining economic and different social problems. The objective of this module is to familiarize Distance students with the basic mathematical tools & symbols which help them to solve economic problems so as to develop the ability of quantitative skills in economic application and provide opportunity for application of concepts, tools, principles and models in the study of socio-economic problems.*

In order to understand the course (module), your knowledge on the course Calculus for Economists, Linear Algebra for Economists, Microeconomics and Macroeconomics is important. Therefore, students are expected to refresh basic concepts on differential and integral calculus, matrix algebra and linear programming.

The module is organized in seven chapters. The first unit deals about the introduction to the course Mathematical Economics. The second unit is more of a revision on Calculus and Linear Algebra for Economists, which is more of supposed to be undertaken as reading assignment. The third and fourth unit focuses on derivative in use and unconstrained optimization. The last three units deal about constrained optimization, Comparative static analysis and dynamic optimization, respectively.

The module is rich in questions aimed at various exercises or activities, self test exercises and model examinations in each chapter in detail. The activities and self test exercises included in the module are advised to be attempted by the distance learner and then crosschecked against check progress in each chapter and answer keys at the last page of the module, respectively.

Dear learners and other interested readers! We hope that you will find this material worth reading. And perhaps, if any editorial or technical errors are committed, they are all ours.

Mesfin Menza & Sisay Debebe

CHAPTER ONE

INTRODUCTION TO MATHEMATICAL ECONOMICS

Chapter contents

Dear students, this chapter consists of the following major topics:

- The Concept, Nature and Application of Mathematical Economics
- Mathematical Versus Non-mathematical Economics
- Advantages of the Mathematical Economics
- Review of Economic Model and Analysis
- Static (Equilibrium) Analysis in Economics

Learning Objectives

After completion of this chapter, students will be able to:

- ✓ recognize the basic principles of mathematics and its application in the area of analytical and applied economics.
- ✓ differentiate mathematical economics from non-mathematical economics.
- ✓ realize the advantages of using mathematical economics in economics analysis.
- ✓ employ equilibrium analysis in economics using methods introduced.

1.1. The Concept, Nature and Application of Mathematical Economics

Mathematics evolved through the use of abstraction and logical reasoning, from counting, calculation, measurement, and the systematic study of positions, shapes and motions of physical objects. Mathematicians explore such concepts, aiming to formulate new conjectures and establish their truth by rigorous deduction from appropriately chosen axioms and definitions.

Dear students, this chapter is an introduction to the course mathematical economics. So we start our discussion by defining mathematical economics? Use the following space to define the term & compare your answer with the following one. _____

_____

_____

Good! Mathematical economics is not a distinct branch of economics in the sense that macroeconomics, development economics, international economics is. Rather, it is an approach to economic analysis in which the economists make use of mathematical symbols in the statement of the problem and also draw upon known mathematical theorems to aid in reasoning. Thus, it is the subject used to describe case employing mathematical techniques beyond simple algebra and geometry. In mathematical economics the assumptions and conclusions are stated in mathematical symbols and equations rather than words and sentences which are commonly used in most theoretical branches of economics. This course is an applied course designed to acquaint students of economics with the basic principles of mathematics and its application in the area of analytical and applied economics. So it requires that the student be acquainted with the basic principles of economics such as the theory of price determination [demand and supply], the theory of consumer's behavior [total, and marginal utility, indifference curves, marginal rate of substitution, etc], the supply side of the economy [such as the theory of production and costs at least in the short run, profit maximizing firm under different market structures], basics of macro economics such as national income accounting. Thus, the course tries to tie the basics of mathematics to the economic theories that economists use in micro and macroeconomic studies.

The purpose of this module is to introduce the most fundamental aspects of the mathematical methods and their application in optimization theory (both unconstrained and constrained), comparative statistics and dynamics analysis that encountered in our day to day activities.

1.2. Mathematical Versus Non-mathematical Economics

Since mathematical economics is merely an approach to economic analysis, it should not and does not differ from the nonmathematical approach to economic analysis in any fundamental way. The purpose of any theoretical analysis, regardless of the approach, is always to derive a set of conclusions or theorems from a given set of assumptions or postulates via a process of reasoning.

The major difference between these two approaches is that in the former, the assumptions and conclusions are stated in mathematical symbols rather than words and in equations rather than sentences. Symbols are more convenient to use in deductive reasoning, and certainly are more conducive to conciseness and preciseness of statement. The choice between literary logic and mathematical logic, is a matter of little important, but mathematics has the advantage of forcing analysts to make their assumptions explicit at every stage of reasoning.

Moreover, mathematical economics draws upon mathematical logics unlike the literary logic, which is drawn by literary economics. At this juncture, you may ask your self "If the two approaches' ends are the same, establish valid hypotheses, then why do economists choose mathematical approaches with which many literary people are in *phobia*?" In fact, it is amazing to see people in horror with mathematics though mathematics is making things simpler and our life less costly. The choice between literary approach and mathematics approach to economic reasoning is like the choice between horse cart and plane for travel from Arba Minch to Addis. The time and cost savings associated with plane or cars has driven out the horse cart from the market of travel from Arba Minch to Addis. An economist without the tools of mathematics is like a blind person swimming in the middle of an ocean. Till this person gives up, he will continue to struggle till he reaches an island; but, his blindness has left him as incapable to identify whether swimming to the north, south, east or west is the shortest distance to an island.

But an economist equipped with the tools of mathematics is like a normal person with motorboat or ship depending upon his personal inclination to each. As a result, most economic researchers are extensively using the tools of *mathematics to economic reasoning*.

The term mathematical economics is also different from econometrics in that the former is concerned with the application of mathematics to the purely theoretical aspects of economics analysis, with little or no concern about statistical problems as the error of measurement of the variables under study, while the later focuses on the measurements and analysis of economic data; hence it deals with the study of empirical observations using statistical methods of estimation and hypothesis testing.

1.3. Advantages of the Mathematical Economics

Mathematical economics is of paramount importance for economic analyses. Specifically the approach has the following advantages:

a) The language used is more precise and concise;

b) There are wealth of mathematical assumptions that make things simple and life less costly ;hence the analysis is more rigorous;

c) By forcing us to state explicitly all our assumptions as a prerequisite to the use of mathematical theorems, it keeps us from the pitfall of an unintentional adoption of unwanted implicit assumptions

d) It helps us to understand relationships among two or more economic variables simply and neatly with which the geometric and literary approaches are at high probability of risk of committing mistakes.

e) it allows us to treat the general n-variable case.

1.4. Review of Economic Model and Analysis

As you have studied in the course introduction to economics, an economic theory is necessarily generalization or abstraction from the real world. The complexity of the real economy does not make it possible for us to understand all the interrelationships at once nor, for that matter, are all these interrelationships of equal importance for the understanding of the particular economic phenomenon under study. The wise procedure is, hence, to pick out what appeal to our reason to be the primary factors and relationships relevant to our problem and to focus our attention on these alone. Such a deliberately simplified analytical framework is called an ***economic model*** since it is only a skeletal and rough representation of the actual economy.

If our model is mathematical, it will usually consist of a set of equations designed to describe the structure of the model. By relating a number of variables to one another in certain ways, these equations give mathematical form to the set of analytical assumptions adopted. Then, through the application of relevant mathematical operations to these equations, we may seek to derive a set of conclusions which logically follow from these assumptions. Hence, the ingredients to any mathematical model are ***variables***, ***constants***, ***parameters***, ***equations***, and ***identities***.

The main aim behind building economic models is to describe how the economy works and to obtain some how valid predictions about economic variables. Different models are built for different purposes. Some of the models are designed to investigate the equilibrium value of the variable as an ultimate end while some to investigate the movement of a variable (s) over time. Thus, in economics, we have three types of analyses: ***Static (equilibrium) analysis, Comparative static analysis and Dynamic analysis.***

As the name implies, static/equilibrium analysis is the study about equilibrium. That is, it studies the determination of equilibrium values. Equilibrium is a constellation of selected interrelated variables so adjusted to one another that no inherent tendency to change in the model which they constitute. The mathematics of static will be handled in linear algebra for economists. On the other hand, comparative static studies on the comparison of two or more equilibrium values. This may involve rate of changes and growth rates. Almost all of the differential calculus is related with the comparative static. The third form of economic analyses is dynamic analysis. It involves the time path of different relevant economic variables. The whole of integral calculus deals with the mathematics of dynamic analyses.

1.5. Static (Equilibrium) Analysis in Economics

Dear students, from your knowledge of different courses you have studied so far, define equilibrium. Use the following space._____

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Great! Like any economic term, equilibrium can be defined in various ways. One definition here is that equilibrium is a constellation of selected interrelated variables so adjusted to one another that inherent tendency to change prevails in the model which they constitute.

In essence, equilibrium for a specific model is a situation that is characterized by a lack of tendency to change. It is for this reason that the analysis of equilibrium is referred to as static. The fact that equilibrium implies no tendency to change may tempt one to conclude that equilibrium necessarily constitutes a desirable or ideal state of affairs.

For the present analysis we will focus on two types of the equilibrium. One is the equilibrium attained by a market under given demand and supply conditions. The other is the equilibrium of national income under given conditions of consumption and investment patterns.

1.5.1. Partial Market Equilibrium: A Linear Model

In a static-equilibrium model, the standard problem is that of finding the set of values of the endogenous variables which will satisfy the equilibrium conditions of the model.

Partial equilibrium market model is a model concerned with price determination in an isolated market. Assume the three variables: Qd = the quantity demanded of the commodity, Qs = the quantity supplied of the commodity and P = the price of the commodity. The model is Qd = a - bp (a, b > 0) and Qs = -c + dp (c, d > 0). The Equilibrium Condition: Qd = Qs. The slope of demand function is "-b" (the negative sign is due to inverse relationship between price and quantity demanded, cetrius paribus), the vertical intercept = a. The slope of Qs = d, the vertical intercept = -c.

1.5.2. Partial Market Equilibrium: A Nonlinear Model

The partial market model can be nonlinear. Suppose a model is given by: $Qd = 4 - p^2$; $Qs = 4p - 1$

This system of equations can be reduced to a single equation by substitution: $4 - p^2 = 4p - 1$ or $p^2 + 4p - 5 = 0$ which is a quadratic equation. In general, given a quadratic equation in the form $ax^2 + bx + c = 0$ $(a > 0)$, its two roots can be obtained from the quadratic formula: $-b \pm \sqrt{b^2-4ac}/2a$ where the "+ " part of the "±" sign yields X_1 and the "-" part yields X_2. Thus, by applying the quadratic formulas to $p^2 + 4p - 5 = 0$, we have $P_1 = 1$ and $p_2 = -5$, but only the first is economically admissible, as negative prices are ruled out.

1.5.3. General Market Equilibrium

In the above, we have discussed methods of an isolated market, where in the Qd and Qs of a commodity are functions of the price of that commodity alone. In the real world, there would normally exist many substitutes and complementary goods. Thus a more realistic model for the demand and supply functions of a commodity should take into account the effects not only of the price of the commodity itself but also of the prices of other commodities. As a result, the price and quantity variables of multiple commodities must enter *endogenously* into the model. Thus, when several interdependent commodities are simultaneously considered, equilibrium would require the absence of excess demand, which is the difference between demand and supply, for each and every commodity included in the model.

Consequently, the equilibrium condition of an n-commodity market model will involve n equations, one for each commodity, in the form $E = Q_{di} - Q_{si} = 0$ $(i = l, 2,...,n)$ where Q_{di} & Q_{si} are the demand and supply functions of commodity I respectively, solving **n** equations for P: we obtain the **n** equilibrium prices P_i - if a solution does indeed exist. And then the Q_i may be derived from the demand or supply functions.

1.5.4. Equilibrium in National-Income Analysis

The equilibrium analysis can be also applied to macroeconomic areas of Economics. As a simple example, we may cite the familiar Keynesian national income model:

$Y= C + I_0 + G_0$ (equilibrium condition- national income identity);

$C = Ca + bY$ (the consumption function),

Where, Y and C stand for the endogenous variables national income and consumption expenditure, respectively; Ca stands for autonomous consumption, and I_0 and G_0 represent the exogenously determined investment and government expenditures, respectively. Solving these two linear equations, we obtain the equilibrium national income and consumption expenditure: $Y=Ca + bY + I_0 + G_0$, by collecting like terms we have;

$Y=Ca + I_0 + G_0/1-b$. This equivalent to $C=Ca + b (Ca + I_0 + G_0/1-b)$

Example: Given the consumption function C=200+0.5Y, I=200, G=300 for closed economy, calculate the equilibrium level of income and consumption.

Solution At equilibrium, total receipts=total spending

$Y=C+I+G$

$Y=200+0.5Y+200+300$, $0.5Y=700$, $Y=1400$, $C=900$

Model Examination

1. What is Mathematical Economics?
2. Differentiate mathematical economics from non-mathematical economics?
3. Discuss the advantages and disadvantages of Mathematical Economics as compared to other field?
4. Discuss different types of Economics Model and their application in mathematical economic analysis?
5. What is static analysis, Comparative static analysis, and Dynamic analysis?

CHAPTER TWO

REVISION ON CALCULUS AND LINEAR ALGEBRA

Chapter contents

- Differential calculus
- Integral calculus
- Matrix algebra
- Matrix operations
- Determinants and Inverse of Matrix
- Matrix representation of Linear Equations
- Leontief model or Input-Output analysis
- Linear programming

Learning Objectives

After successful completion of this chapter, students will be able to:

- ✓ apply the concept of derivatives for economic analysis
- ✓ use the concepts and principles of integral calculus in economics analysis
- ✓ realize the use of differentials and difference equations
- ✓ analyze matrix algebra in detail
- ✓ know the application of Leontief model and linear programming in economics.

Introduction

Dear students!, this chapter is more or less a revision of your first year two consecutive courses: *Calculus for economists and Linear algebra for economists*. Since it is the revision, it needs your efforts in revising the basic concepts, principles and theorems. But in order to refresh your memory and to use in forthcoming chapters, we will focus on the most important concepts which are relevant from the view point of mathematical economics. Thus, in this chapter, students are highly advised to revise the basic concepts from both calculus (rules of derivative, rules of integration and their application in economics) and linear algebra (determinants and inverse of a matrix; solution of simultaneous linear equation, Leontief input-output model and linear programming) in detail, since here we simply highlight the issues very roughly so as to show you directions for your worth reading.

2.1. Differential Calculus: The Concept of the Derivative

Derivative is defined as a derived function from an original function , say y=f(x) that shows the rate of change in dependent variable(y) due to very small change in independent variable(x).

If f is a function defined by y=f (x), then the derivative of f (x) for any value of x, is denoted by dy/dx , y' or f '(x) is the instantaneous rate of change of f (X) which is given by:

$$\frac{dy}{dx} = \lim_{\sigma x \to 0} \frac{f(x+\sigma x) - f(x)}{\sigma x}$$

As you might remember from the previous discussion, the derivative is also the slope of the tangent line to f (x) at a point. Before we directly go to the discussion of the rules of derivatives, let's see some examples.

Example: Find the derivative of, f (x) = X^2

$$\frac{dy}{dx} = \lim_{\sigma x \to 0} \frac{f(x+\sigma x) - f(x)}{\sigma x}$$

$$= \lim_{\sigma x \to 0} \frac{(x+\sigma x)^2 - x^2}{\sigma x}$$

$$= \lim_{\sigma x \to 0} \frac{(x^2 + 2\sigma x.x + \sigma x^2 - x^2)}{\sigma x}$$

$$= \lim_{\sigma x \to 0} (2x + \sigma x)$$

$$= \underline{\underline{2x}}$$

Note that,

i. The derivative, which is one of the most fundamental concepts in calculus, is the same as the following two concepts.
   a. Slope of a line tangent to a curve at x
   b. Instantaneous rate of change of f(x) at x
ii. The process of obtaining f'(x) from f (x) is known as differentiation and if the derivative exists, then the function is differentiable at a point or over an interval.

In the following section, we will present the rules of derivatives, which will largely simplify our calculation of the derivative and commonly applied for economic functions.

2.1.1. The Rules of Derivatives

In this section, we formally revise and state the most commonly applied rules of derivatives that are important in several problems of getting the derivative.

a) The constant rule

If c is constant and if f(x) = c, then f'(x) = 0

Example: If f (x) = 2, then f '(x) = 0

b) **The Simple Power rule**

If where $f(x) = x^n$, n ∈ *R*, then $f'(x) = nx^{n-1}$

c) The coefficient rule

For any constant C and function m(x),

$$\text{If } f(x) = C\,m(x), \text{then } f'(x) = C\,m'(x)$$

Example

$$Y = 3x^2$$

$$\frac{dy}{dx} = 3\frac{dx^2}{dx}$$

$$= 3[2x]$$

$$= \underline{\underline{6x}}$$

d) The sum or difference rule

$$\text{If } f(x) = u(x) \pm v(x), \sim \text{then} f'(x) = u'(x) \pm v'(x)$$

Example If $y = x^3 - 3x^2 + 10$,

then the derivative

$$\frac{dy}{dx}(x^3 - 3x^2 + 10)$$
$$= \frac{d}{dx}x^3 - 3\frac{d}{dx}x^2 + \frac{d}{dx}10$$
$$= \underline{\underline{3x^2 - 6x}}$$

e) The product rule

If $f(x) = u(x).v(x)$, then $f'(x) = u'(x).v(x) + v'(x).u(x)$

Example

1. $Y = (3x - 2x^2)(5 + 4x)$

$$\frac{dy}{dx} = (5 + 4x)(3 - 4x) + (3x - 2x^2)(4)$$
$$= 15 - 20x + 12x - 16x^2 + 12x - 8x^2$$
$$= \underline{\underline{-24x^2 + 4x + 15}}$$

f) The Quotient rule

If u (x) and V (x) are two functions and if f(x) is the quotient,

If $f(x) = \frac{u(x)}{v(x)}$, then $f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}$

Example: If Y $= \frac{(1-2x)(3x+2)}{5x-4}$ find, $\frac{dy}{dx}$

$$\frac{dY}{dx} = \frac{\left[((5x-4)\frac{d(1-2x)(3X+2)}{dx}) - ((1-2x)(3x+2)\frac{d(5x-4)}{dx}\right]}{(5x-4)^2}$$
$$= \frac{(5x-4)[(1-2x)(3) + (3x+2)(-2)] - [5(1-2x)(3x+2)]}{(5x-4)^2}$$
$$= \frac{(5x-4)(-1-12x) - [(5-10x)(3x+2)]}{(5x-4)^2}$$
$$= \underline{\underline{\frac{-30X^2 + 48X - 6}{(5X-4)^2}}}$$

g) Differentiation of a composite function

If the function f is a function of g and g is a function of (x), then df/dx is:

$$\frac{df}{dx}=\frac{df}{dg}.\frac{dg}{dx} \qquad Y=f(g), \quad Z=g(x)$$

Example: $Y = \sqrt{x^2-1}$ $find$ $\frac{dy}{dx}$

Let $x^2-1=u$

$$Y=u^{1/2}$$

$$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=\frac{1}{2\sqrt{u}}.2x=\frac{x}{\sqrt{u}}=\frac{x}{\sqrt{x^2-1}}$$

h) Derivatives of Implicit functions

Implicit functions are those functions whose dependent and independent variables are not explicitly stated. That is, we don't know whether a given variable is a cause or an effect variable without any a priori information when we encounter implicit functions. When the dependent and independent variables are not explicitly stated, one may not easily come up with dy/dx. But using implicit differentiation one can calculate dy/dx, but it is a bit complex. .

Suppose we have an implicit function given by F(x,y)=0

Then, totally differentiating the equation we get

$$F_x dx + F_y dy = 0$$

Rearranging terms gives us:

$$F_y dy = -F_x dx$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example: Given a function $F(x,y) = x^2 + y^2 - 25 = 0$, find $\frac{dy}{dx}$ using implicit differentiation.

First, let's find the first partial derivatives from the given function: $F_x = 2x \;\; F_y = 2y$

Then, using the implicit differentiation rule, $\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$.

i) Derivatives of Logarithmic and Exponential Functions

In this section, we will review point out how to find the derivatives of logarithmic and exponential functions.

A. Derivatives of Logarithmic Functions

If $Y = \log_e^x = \ln x, \; find \; \frac{dy}{dx}$

$$y + \sigma y = \ln(x + \sigma x)$$

$$\begin{aligned}
\sigma y &= \ln(x+\sigma x) - y \\
&= \ln(x+\sigma x) - \ln x \\
&= \ln\left(1+\frac{\sigma x}{x}\right)
\end{aligned}$$

$$\sigma y = \frac{\sigma x}{x} - \frac{\left(\frac{\sigma x}{x}\right)^2}{2} + \frac{\left(\frac{\sigma x}{x}\right)^3}{3} - \frac{\left(\frac{\sigma x}{x}\right)^4}{4} + ...$$

$$\frac{\sigma y}{\sigma x} = \frac{1}{x} - \frac{\sigma x}{2x} + \frac{\sigma x^2}{3x^3} - \frac{(\sigma x)^3}{4x^4} + ...$$

$$Lim_{\sigma x \to 0} \frac{\sigma y}{\sigma x} = \frac{1}{x}$$

Thus, $\frac{d}{dx} \ln x = \frac{1}{x}$

B. Derivatives of exponential functions

If $y = a^x, a > 0 \& \neq 1, \; then \; \frac{dy}{dx}$ can easily be found by changing it to the logarithmic function.

$$\ln y = x \ln a$$

$$\frac{d \ln y}{dy} \frac{dy}{dx} = \frac{dx}{dx} . \ln a$$

$$\frac{1}{y} \frac{dy}{dx} = \ln a$$

$$\frac{dy}{dx} = y \ln a$$

$$\frac{dy}{dx} = a^x \ln a$$

Thus, $\frac{da^x}{dx} = a^x \ln a$

If $Y = a^{U(x)}$, Using chain rule

$$\frac{da^{u(x)}}{dx} = u'(x) a^{u(x)} \ln a .$$

As a result, we have

$$\frac{da^{u(x)}}{dx} = u'(x)a^{u(x)} \ln a$$

The exponential function e^{x} is a unique function with special behavior. That is.

$$\frac{de^{x}}{dx} = e^{x}$$

However, in case we have $e^{f(x)}$, we can use the chain rule to evaluate its derivative. That is let f (x)=u and we will have e^{u}.

If $Y=e^{u}$ and U= f(x)

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
$$= e^{u} * u'(x)$$

, $\text{Thus, } \frac{de^{f(x)}}{dx} = f'(x)e^{f(x)}$

Example: Find dy/dx in the functions given below $y = 4^{x}$

Directly inserting in the formula, we get

$$\frac{dy}{dx} = 4^{x} \ln 4$$

Activity: Dear students please try the following questions and check your progress below.

Find $\frac{du}{dx}$ for each of the functions questions below

1. $y = x^{3} + 3\ln x$
2. $y = \ln x^{4}$
3. $y = \ln\sqrt{3x}$
4. $x \ln y = 4$

Check Your Progress

1. $\frac{dy}{dx} = 3x^2 + 3\frac{d\ln x}{dx} = \underline{\underline{3x^2 + \frac{3}{x}}}$

2. $y = \ln x^4 \quad let \ u = x^4$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= \frac{d\ln u}{du}.\frac{du}{dx}$$
$$= \frac{1}{u}.4x^3$$
$$= \frac{4x^3}{x^4}$$
$$= \underline{\underline{\frac{4}{x}}}$$

3. Let $\sqrt{3x} = u$

$$y = \ln u$$
$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{u}.\sqrt{3}\frac{1}{2\sqrt{x}}$$
$$= \frac{1}{\sqrt{3}.\sqrt{x}}.\frac{\sqrt{3}}{2\sqrt{x}}$$
$$= \underline{\underline{\frac{1}{2x}}}$$

4. $\ln y = \frac{4}{x} \quad \frac{d\ln y}{dy}.\frac{dy}{dx} = \frac{-4}{x^2}$ Differentiating both sides w.r.t. X and applying the chain rule.

$$= \frac{1}{y}\frac{dy}{dx} = \frac{-4}{x^2}$$
$$= \frac{dy}{dx} = \frac{-4y}{x^2}$$
$$= \underline{\underline{\frac{-4e^{4/x}}{x^2}}} \ldots \sin ce \sim y = e^{4/x}$$

Self-Test Exercise 2-1

Exercises on Economics Applications (use rules of differentiation)

1. The total revenue for a commodity is described by $R(x) = 300x - 0.02x^2$. What is the marginal revenue when 50 units are sold and interpret your result?
2. Suppose the total cost function for the production of x units of a product is given by: $TC=2X^2+120$
   a. Find the instantaneous rate of change of average cost with respect to the number of units produced.
   b. Find the level of production at which this rate of change equals zero.
3. The profit from the sale of x stereos per month for a certain company is given by:

   $$\pi(x) = 800 - x^2 + 0.2x^2.$$

   Find the rate of change in the monthly profit for 100 stereos.

Model Examination

1. The total physical output P of workers is a function of the number of workers "x". The function f(x) = p is called physical productivity function and is given by find the marginal physical productivity. $P = 10(3x+1)^3 - 10,$
2. Suppose the revenue function for a certain product is given by:

   $R(x) = 15(2x+1)^{-1} + 30x - 5,$

   where x is in thousands units and R is in thousands.
   a. Find the MR when 200 units are sold
   b. How is revenue changing when 2000 units are sold

2.2. Integral calculus: Techniques and applications of integral calculus

Up until now, we have been concerned with finding the derivative of a function. However, there are some problems in economics that require us to come up with the original function given its rate of change. These problems are common in the area of social welfare, distribution of income of a country etc. The technique of finding out the original function based on its derivative is known as *integration or anti derivative*. More specifically, you will learn that the concept of integration is the exact opposite of differentiation.

In this section, you will be briefly introduced the tools of mathematical economics required to handle dynamic analyses of economic theories and empirics.

2.2.1. The Indefinite Integral

Given the primitive function $F(x)$ which yields the derivative $f(x)$, you have the following definitions.

Definitions

- The process of uncovering the primitive function $F(x)$ back from the derivative function $f(x)$ is known as integration.
- The function $F(x)$ is referred to as an integral (or antiderivative) of the function $f(x)$.

You have the following notation that denotes the integration of $f(x)$ with respect to x,

$\int f(x)dx$ .The symbol $\int$ is called the integral sign. The $f(x)$ part is known as the integrand (that is the function to be integrated). The dx part tells that the operation is to be carried out with respect to the variable x.

One important point that you should note is that while the primitive function $F(x)$ invariably yields a unique derivative $f(x)$, the derived function can have an infinite number of possible primitive functions through integration. Why? Because, if $F(x)$ is an integral of $f(x)$, then so is $F(x)$ plus a constant for the derivative of a constant is zero.

2.2.2. Techniques of Integration & the General Power Rule

In this section, I will briefly discuss the rules of integration which shall refer to increasingly from time to time while we are dealing with the techniques of integration.

a) **The constant rule:** Given constant K & C,

$$\int K dx = Kx + C$$

Example:

$$\int 2dx = 2x + c$$

b) **Constant Multiple Rule**:

$$\int kf(x)dx = k\int f(x)dx$$

Example:

$$\int x^7 dx = 2\int x^7 dx$$

c) Sum or Difference Rule:

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

Example:

$$\int (2x^2 + 7x - 10)dx = \int 2x^2 dx + \int 7x dx - \int 10 dx$$

d) Simple Power Rule:

$$\int x^n dx = \frac{x^n + 1}{n+1} + c \quad \text{for n} \neq -1$$

Activity: Dear student please try activities below check your progress.

a) $\int 3x dx$

b) $\int (3x^4 - 5x^2 + x)dx$

c) $\int \frac{x+1}{\sqrt{x}} dx$

d) Given the marginal cost MC= 32-0.004x and it costs $50 to produce the first unit of output, find the total cost of producing 200 units of output.

Check Your Progress

a) $\frac{3x^{1+1}}{1+1}+c=\frac{3x^2}{2}+c$

b) $\int\left(3x^4-5x^2+x\right)dx=\frac{3x^5}{5}-\frac{5x^3}{3}+\frac{1x^2}{2}+C$

$=\frac{3x^5}{5}-\frac{5x^3}{3}+\frac{x^2}{2}+C$

c) $\int\frac{x+1}{\sqrt{x}}dx=\int\left(\frac{x}{\sqrt{x}}+\frac{1}{\sqrt{x}}\right)dx=\int\sqrt{x}dx+\int\frac{1}{\sqrt{x}}dx$

$=\frac{2x^{1/2+1}}{3}+C,\frac{x^{-1/2+1}}{-1/2+1}+C2$

$=\frac{2}{3}x^{2/3}+C,+2x^{1/2}+C2$

$=\underline{\underline{\frac{2}{3}x^{3/2}+2x^{1/2}+C}}$

d) C = $\int MCdx=\int\left(32-0.04x\right)dx=32x-\frac{0.04X^2}{2}+C$

$=32X-0.02X^2+C$

But, TC (1) = 50

$TC(1)=32-0.02(1)+C=50$

C= 50.02-32

C=18.02 which is Fixed Cost

Thus, TC = $32x-0.02x^2+18.02$

TC (200) = $32(200)-\frac{2}{100}x200^2+18.02$

= 6400-800+18.02

= 5600+18.02

= 5618.02

e) Integration of Logarithmic & Exponential Functions

Like what we have done in relation to the derivative of non-algebraic functions, here we will present the integration of the logarithmic functions first accompanied by the integration of exponential functions. The general power rule of integration applies only when n ≠ -1; i.e.

$$\int U^n \frac{du}{dx} \cdot dx = \frac{U^n + 1}{n+1} + C$$ if an only if $n \neq -1$.

But, the following formula applies when n = -1. That is

$$\int U^{-1} \frac{du}{dx} \cdot dx = \ln |u| + C$$ = ln |u| + C

We use absolute value to protect it from being negative & the whole logarithm undefined. This formula is a direct result of the fact that

$$\frac{d \ln x}{dx} = \frac{1}{x}$$

Thus, $$\int \frac{1}{x} dx = \int x^{-1} dx = \ln |x| + C$$

Similarly, when U is a function of X,

$$\frac{d \ln u}{dx} = \frac{du}{dx} \cdot \frac{1}{u}$$

$$\therefore \int \frac{1}{u} \cdot \frac{du}{dx} dx = \ln |u| + C$$

Example: Evaluate the indefinite integral of the following.

$$\int (4x+1)^{-1} 4dx$$

1) Let 4x + 1 = U

$$\frac{du}{dx} = 4$$

Thus,

$$U^{-1} \frac{du}{dx} \cdot dx = \underline{\underline{\ln |4x+1| + C}}$$

The second type of non-algebraic function we saw was exponential functions. In relation to exponential function, we have seen that

$$\Rightarrow \frac{de^x}{dx} = e^x,$$

Thus, $\int e^x dx = e^x + C$

Moreover, we have seen that when U is a function of X, that is,

$$\frac{de^u}{dx}, = U = f(x), \frac{de^u}{dx} = u'(x)e^u, thus, \int e^u u'(x)dx = e^u + C$$

Activity: Dear student try the following indefinite integral and check your progress.

a) $\int 2xe^{x^2} dx$

b) $\int x^2 e^{x^3} dx$

Check Your Progress

a) Let x^2 =U and $\frac{du}{dx} = 2X$

$$\int U'(x)e^u dx$$
$$= e^u + C$$
$$= e^{x^2} + C$$

b) Let x^3 = U, U'(x) = $3x^2$ and manipulating it little, we get x^2 = $\frac{1}{3}U'(x)$

$$= \frac{1}{3}\int e^u \frac{du}{dx} \cdot dx \frac{1}{3}e^{x^3} + C = \frac{\frac{1}{3}e^{x^3} + C}{}$$

2.2.3. Definite Integrals

Let f(x) be a function and "a" and "b" be real numbers. The definite integral of f(x) over the interval from X=a to X=b denoted by is the net change of an anti-derivative of f(x) over the interval. Thus, if F(x) is an antiderivative of f(x), then we have

$$\int_a^b f(x)dx$$

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Note that it is used to find definite integral and not antiderivative of a function.

$$\int_a^b f(x)dx = F(b) - F(x)\Big|_a^b = F(b) - F(a)$$

Now it is time to look at some examples so that the idea of definite integral and fundamental theorem of calculus will be well planted in your mind.

Example: Evaluate the definite integrals of the following

$$\int_1^3 x^3 dx$$

$$= \left[\frac{x^{3+1}}{3+1} + C\right]_1^3$$

$$= \left[\frac{1}{4}x^4 + C\right]_1^3$$

$$= \left[\frac{1}{4}\cdot 3^4 + C\right] - \left[\frac{1}{4}\cdot 1^4 + C\right]$$

$$= \frac{81}{4} - \frac{1}{4}$$

$$= \underline{\underline{20}}$$

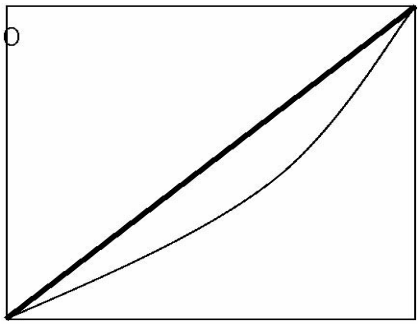
2.2.4. Application of Integral calculus:

2.2.4.1. Distribution of income

While no country has been able to achieve equal distribution of income, a desirable goal in the eyes of many political scientists is to reduce the in-qualities that exist. The table below and the corresponding graph indicate how incomes in the hypothetical economy were distributed? The left column of the table assumes that people have been ranked according to income.

Note in the table that the lowest paid 40% of the population receives only 18% of the income, whereas if income were absolutely equally distributed, they would get 40% of the income. In fact, if income were equally distributed, the data would lie along the 45-degree line. In actuality, the data always lie along a curve below the line, a curve called a ***Lorenz curve***. The further this curve moves from the 45 degree line, the more unequal the income distribution.

Cumulative percentage of people	Cumulative percentage of income
2	0
20	6
40	18
60	35
80	59
100	100



As a measure of inequality, we use the so-called Gini coefficient of inequality "g" defined by

$$g = \frac{\text{Area between the Lorenz curve and the line } y = x}{\text{Area under the line } y = x}$$

The numerator is a definite integral whose value is between 0 and ½; the denominator has a value of exactly ½. Thus, g is between 0 (absolute equality) and 1 (absolute inequality). If y = f(x) is the equation of the Lorenz curve, the formula for g takes the simple form

$$g = 2\int_0^1 [x - f(x)]dx$$

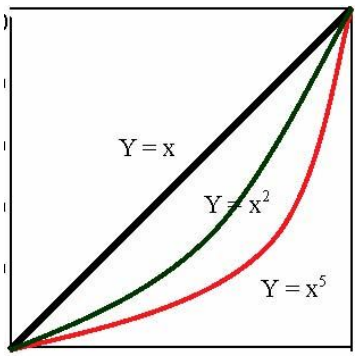
Example: Sketch the Lorenz curves $y= x^2$ and $y= x^5$ on the same graph and determine the corresponding Gini coefficients.

The graphs are shown in Figure below. The required Gini coefficients are

$$g = 2\int_0^1 [x - x^2]dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

and

$$g = 2\int_0^1 [x - x^5]dx = 2\left[\frac{x^2}{2} - \frac{x^6}{3}\right]_0^1 = \frac{2}{3}$$



2.2.4.2. Continuous Money Streams within a Period

There are many situations in business and industry where it is useful to think of money as flowing continuously into an account. For example, the ABC Company plans to buy an expensive machine which it estimates will increase the company's net income by $10,000 per year. But this income won't come at the end of each year; it will come in dribbles throughout the year. As a model for what will happen, it is convenient to think of the machine as if it will produce income continuously.

Suppose that the annual rate of income of an income stream is R (t). Suppose further that this income stream will last over the period $0 \leq t \leq T$. If interest is at rate r compounded continuously, then the present value PV of this income stream is given by:

$$PV = \int_0^T R(t)e^{-rt}\ dt$$

Example:

Suppose the ABC Company takes a more optimistic view and estimates that the new machine will produce income at the rate R (t) = 10,000 + 200t. What is the present value of this income stream, again assuming interest at 9% compounded continuously and a lifetime of 8 years?

$$PV = \int_0^8 (10{,}000 + 200t)e^{-0.09t}\,dt$$

$$= \int_0^8 10{,}000e^{-0.09t}\,dt + \int_0^8 200te^{-0.09t}\,dt$$

Let us modify the problem in the above example in another way by assuming that the machine will last indefinitely. More generally, let us ask for the present value PV of a perpetual (meaning infinitely long) income stream which produces income at an annual rate R(t), assuming that interest is at an annual rate r compounded continuously. The result in this case is

$$PV = \int_0^\infty R(t)e^{-rt}\,dt$$

Here, we face a new kind of integral, one with an infinite limit. Such integrals are called improper integrals and must be given a clear definition. We do this by defining

$$\int_0^\infty R(t)^{e-rt}\,dt = Lim_{r\to\infty} \int_0^T R(t)e^{-rt}dt$$

This point leads us to the concept known as *improper integrals* which we will handle right now!

2.2.4.3. Improper Integrals

Now we have faced somewhat new, but definite integral, problem in this function. Such type of integral is known as *improper integrals*.

An improper integral is an integral which is bounded by either, $(-\infty,\infty)$, (c,∞), or $(-\infty,c)$ where c is a constant. This limit of integration is infinite. To find out the value of this integral use the formula:

a) For

$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

This holds if f is continuous over the internal [a, ∞)

b) If f is continuous on the interval (-∞ b], then

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$$

c) If f is continuous on interval $(-\infty,\infty)$, then

$$\int_a^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx$$

$$= \underline{\underline{Lim_{a\to-\infty}\int_a^c f(x)dx + Lim_{b\to\infty}\int_c^b f(x)dx}}$$

Note: In each three cases of above, if limit exists, then the improper integral is said to converge; otherwise the improper integral diverges.

Examples:

1. Find the value of $\int_1^{\infty}\frac{1}{x}dx$ and determine its status of convergence.

$$\int_1^{\infty}\frac{1}{x}dx = Lim_{b\to\infty}\int_1^b\frac{1}{x}dx = Lim_{b\to\infty}\left[\ln|x|\right]_1^b = Lim_{b\to\infty}\left[\ln b - \ln 1\right]$$

Since limit does not exist, the integral diverges.

2. Find $\int_1^{\infty}\frac{1}{x^2}dx$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = Lim_{b\to\infty}\int_1^{b} \frac{1}{x^2}\,dx = Lim_{b\to\infty}\left[\frac{-1}{x}\right]_1^{b} = Lim_{b\to\infty}\left[\frac{-1}{b}+1\right] = \underline{1}$$

Since limit does exist, the integral converges.

With the counter-revolution of new classical macroeconomics, there has been a shift by economists to consider an infinitely lived agent. This agent is widely used in perfect foresight-model, rational expectation model, Ricardian equivalence theorem, endogenous economic growth models, among others. Hence, saving and investment decision of households is governed by the utility function agents maximize throughout, that is,

$$U(c) = \int_0^{\infty} e^{-Bt}\left(\frac{C^{1-\delta}-1}{1-\delta}\right)L\,dx$$

Where, U(c) is the utility function the social planner maximizes, B- is the discount rate agents assign to future consumption, C is per-capita consumption, L is the size of population and δ is the inverse of the intertemporal elasticity of substitution. In such applications the concept of improper integral is a par amount important.

Now it is time for you to check whether you have grasped the concept or not. To this end, why don't you handle this problem? Use your paper and pen right now!

$$\int_{-\infty}^{0} \frac{dx}{(1-2x)^{3/2}}$$

Self Test Exercise 2.2

Try the following problems.

1. Suppose that the marginal profit for a company is MP=100+50X-3X$^{2.}$. Find the total profit earned from the sale of three additional units if 5 units are currently being produced.

2. For a number of years the annual worldwide rate of oil consumption was growing exponentially with a growth constant of about 0.07. At the beginning 1970, the rate was about 16.1 million barrels of oil/year. Let R(t) denote the rate of oil consumption at time (t) where t is the number of years since the beginning of 1970. Then a reasonable model of R(t) is given by:

$$R(t) = 16.1e^{0.07t}$$

How much oil was consumed between 1975 and 1980?

Model Examination

Answer the following questions.

1) Suppose the marginal cost function of a firm is given by $MC = \frac{1}{Q^2} + Q^3$. If the fixed cost is given by $C_F = 200$, find the total cost function.

2) Given the marginal revenue function $MR = 4Q^2 + 16Q + 48$, find the total revenue function.

3) Find the consumer's surplus for price $p = 4$ if the price-demand equation for commodity x is given by $p = D(x) = 20 - 0.025x$.

4) Suppose that the demand and supply function are respectively given by $p = D(x) = 32 - 0.02x$ and $p = S(x) = 12 + 0.1x$. Assuming a perfectly competitive market, calculate the consumers' and the producers' surplus at the equilibrium price.

2.3. Basic Concepts of Matrix Algebra

Dear students! As you may recognize, most mathematical models used in economics analysis involve a system of linear equation that involve a large number of equations with a large number of unknowns. In order to deal with such enormous systems of equations, you need to have a suitable method that allows you to handle such equation systems easily.

In this section, we will revise a matrix algebra that simplifies the determination of solutions for very complicated system of linear equations since these systems can be put in very compact form AX= C, where A- is a technical coefficients matrix, X- are unknowns and C- are constants.

Consider the following linear equation system.

$$6x_1 + 3x_2 + x_3 = 10$$
$$x_1 + 4x_2 - 2x_3 = 12$$
$$4x_1 - x_2 + 5x_3 = 8$$

This is a linear equation system. It comprises of three different types of ingredients. The first ingredient is the set of coefficients, the second is the set of variables, and the last set is that of constants. Arranging the three sets as rectangular arrays and labeling them respectively as A, X and, C you have:

$$A = \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad c = \begin{bmatrix} 10 \\ 12 \\ 8 \end{bmatrix}$$

Each of these three arrays constitutes a matrix. Here is formal definition of a matrix.

Definitions:

- A matrix is defined as a rectangular array of numbers, parameters, or variables. It is usually enclosed in brackets and sometimes in parentheses or double vertical lines.
- The members of the array are referred to as the elements of the matrix.
- The number of rows and the number of columns in a matrix together define the dimension of the matrix. If a matrix has m rows and n columns, then that matrix is said to be of dimension $(m \times n)$.

2.3.1. Matrix Operations

a) Equality of Matrices

Suppose you have two matrices. The two matrices are said to be equal if and only if they have the same dimension and identical elements in the corresponding locations in the arrays. Consider the following matrices.

$$A=\begin{bmatrix}1 & 3 & 5\\ 4 & 1 & 2\\ 2 & 3 & 0\end{bmatrix} \qquad B=\begin{bmatrix}2 & 1 & 1\\ 4 & 1 & 2\\ 2 & 3 & 0\end{bmatrix} \qquad C=\begin{bmatrix}1 & 3 & 5\\ 4 & 1 & 2\\ 2 & 3 & 0\end{bmatrix}$$

It can be conclude that $A = C$, because the two matrices have the same dimension and identical elements. Though, matrices A and B have the same dimension, they are not equal since the elements in their first row are different. Thus, we conclude $A \neq B$.

b) Addition and Subtraction of Matrices

Two matrices can be added or subtracted if and only if they have the same dimension (i.e. they have the same number of rows and columns).

c) Matrix Multiplication

i) Multiplication by a Scalar

If k is a real number and A is a matrix, then multiplying the matrix A by the number k is referred to as the multiplication of matrix A by scalar. In this case, every element of the matrix is multiplied by the constant k.

ii) Multiplication of Matrices

Whereas a scalar can be used to multiply a matrix of any dimension, the multiplication of matrices is dependent on the *satisfaction of a requirement*. The product of the two matrices is defined if and only if the number of columns of matrix A is the same as the number of rows of matrix B. Matrices that fulfill this condition are said to be *conformable* for multiplication.

Symbolically, $A_{(m\times n)} \times B_{(n\times p)} = AB_{(m\times p)}$, where the subscripts denote dimension.

iii) Multiplication of Vectors

If A is an $(m\times 1)$ column vector and B is a $(1\times n)$ row vector, then their product yields a matrix of dimension $(m\times n)$.

Example

a) Given

$$A=\begin{bmatrix}1\\2\\3\end{bmatrix}_{(3\times1)} \quad \text{and } B=\begin{bmatrix}4 & 5\end{bmatrix}_{(1\times2)}, \text{ find } AB.$$

$$A\times B=\begin{bmatrix}1\\2\\3\end{bmatrix}_{(3\times1)}\times\begin{bmatrix}4 & 5\end{bmatrix}_{(1\times2)}$$

$$A\times B=\begin{bmatrix}1(4) & 1(5)\\2(4) & 2(5)\\3(4) & 3(5)\end{bmatrix}=\begin{bmatrix}4 & 5\\8 & 10\\12 & 15\end{bmatrix}$$

Therefore, the product of vectors A and B yields another matrix with dimension (3×2).

NB: However, if the row matrix/vector pre-multiplied by the column matrix/vector, the resultant matrix is a *single numeric number*.

2.3.2. Determinants and Inverse of a Matrix

In the previous section, you have been introduced to the basic concepts of matrix algebra. In this section, you will study the concept of a determinant and its use in determining the inverse of a matrix.

2.3.2.1. The Concept of Determinants, Minor and Cofactor

a) Determinants

The determinant of a square matrix is a uniquely defined scalar (number) associated with that matrix. The determinant of a matrix A is denoted by $|A|$. Determinants are defined only for square matrices. The determinant of an $(n\times n)$ matrix is known as determinant of order n. How can we find the determinant?

Suppose matrix A is a (2×2) square matrix and it is given as

$$A=\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}$$

The determinant of this matrix is defined as the difference of two terms as shown below:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example:

a) Find the determinant of the matrix

$$A = \begin{bmatrix} 10 & 4 \\ 8 & 5 \end{bmatrix}$$

Using the formula given above, the determinant of matrix *A* is

$$|A| = \begin{vmatrix} 10 & 4 \\ 8 & 5 \end{vmatrix} = 10(5) - 8(4) = 18$$

A determinant of order three is associated with a (3×3) matrix. Given

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the determinant is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

After rearranging, you get

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

This can be extended in to *Laplace Expansion* as follows:

b) Minor and Cofactor

Let us begin with the concept of minor.

Suppose matrix *A* is given as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Hence,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Deleting the 1st row and the 1st column of the determinant of matrix A gives the sub-determinant $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$. This sub-determinant is called the minor of element a_{11} and it is denoted by $|M_{11}|$.

Therefore, the minor of element a_{11} is written as

$$|M_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

In general, if matrix A is a square matrix of dimension (3×3) or more, then canceling the i^{th} row and the j^{th} column of the determinant of that matrix gives the minor of the element a_{ij} which is a sub-determinant denoted by $|M_{ij}|$.

The cofactor denoted by C_{ij} is obtained by $C_{ij} = (-1)^{i+j} M_{ij}$ where M_{ij} is the minor of the element a_{ij} obtained by canceling the i^{th} row and the j^{th} column of the determinant of a particular matrix. i and j refer respectively to the i^{th} row and the j^{th} column.

Therefore, this suggests that a cofactor is a minor with a prescribed algebraic sign attached to it.

For instance, considering matrix A given above, the cofactor C_{11} is

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

Thus, the determinant of matrix A is, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

Since, $C_{11} = (-1)^{1+1} M_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$,

Therefore,

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

After rearranging, you get

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

This exactly what you had when we defined the determinant of a (3×3) matrix.

Activity:

Compute the cofactor of the element at the 2nd row-2nd column of matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

Check your progress

First you need to determine the minor of the element at the 2nd row-2nd column. Thus, the minor in this case is obtained by canceling the 2nd row-2nd column. Therefore,

$$|M_{22}| = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (1)(1) - (2)(1) = -1$$

Hence, the cofactor will be

$$C_{22} = (-1)^{2+2}|M_{22}| = M_{22} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = (1)(1) - (2)(1) = -1$$

2.3.2.2. The Inverse of a Matrix

If matrix A is a square matrix, then the inverse of that matrix denoted by A^{-1} exists if it satisfies the property that $AA^{-1} = A^{-1}A = I$. If the inverse exists, then matrix A is said to be invertible. Matrix A is also an inverse of A^{-1} and hence both matrices are inverses of each other.

Note that it is only square matrices that can have inverses.

Suppose you have the following matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

In finding the inverse of matrix A, the first step is to obtain the determinant. That is $|A|$. Secondly, you need to find the cofactor of all elements of matrix A.

Arranging them as a matrix gives you what is known as a cofactor matrix denoted by C. That is

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$, where $C_{11},...,C_{nn}$ denote the cofactors of the elements in matrix A

Thirdly, you take the transpose of the cofactor matrix C. This transpose matrix is known as the adjoint of matrix A which is denoted by $adj(A)$. Hence, we have

$$adj\,(A) = C' = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

The last step in finding the inverse of matrix A is dividing the adjoint of matrix A by the determinant. Therefore, it follows that

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

The inverse of a matrix, say A, exits if and only if the determinant of that matrix is different from zero i.e., $|A| \neq 0$. Otherwise, with $|A| = 0$, the inverse becomes undefined and hence, no need to proceed with the subsequent steps of matrix inversion.

Example: Given $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$, find the inverse.

First, find the determinant. Thus, you have

$|A| = \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} = (3)(0) - (1)(2) = -2$ (Since $|A| \neq 0$, the inverse for matrix A is defined and hence, you can proceed to the next steps). Then, determining the cofactor matrix, you get

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix}$$

Note that since the matrix is a (2×2), the minor of each element in this case is a (1×1) determinant. Transposing the cofactor matrix, we get the adjoint matrix

$$adj\,(A) = C' = \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}$$

Lastly, dividing the adjoint matrix by the determinant gives us the inverse matrix. That is

$$A^{-1} = \frac{1}{|A|}adj\,(A) = -\frac{1}{2}\begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Activity: Try the following question and check your progress below

Given $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$, show that it has an inverse and find its inverse.

Check your progress

In this case, $|A| = -5$ and this implies that the inverse matrix is defined and you can proceed with the subsequent steps. The cofactor matrix is defined by

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Since

$$C_{11} = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10 \qquad C_{12} = -\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = -15 \qquad C_{13} = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -4 \qquad C_{22} = \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} = 4 \qquad C_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -1$$

$$C_{31} = \begin{vmatrix} 3 & 4 \\ 3 & 1 \end{vmatrix} = -9 \qquad C_{32} = -\begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = 14 \qquad C_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = -6$$

it follows that $C = \begin{bmatrix} 10 & -15 & 5 \\ -4 & 4 & -1 \\ -9 & 14 & -6 \end{bmatrix}$

Therefore, the adjoint matrix is $adj\ (A) = C^{-1} = \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$

Hence, the desired inverse matrix is

$$A^{-1} = \frac{1}{|A|} adj\ (A) = -\frac{1}{5}\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

NB: *Having studied the basics of matrix algebra and matrix inversion, it is now time to revise and you come across with the application of matrix algebra in solving systems of linear equations. You are expected to revise the Inverse method, the Cramer's Rule, and the Gauss-Jordan Elimination technique as your reading assignments from the course Linear Algebra for Economists.*

2.3.3. Input output analysis (Leontief model)

Input-output analysis attempts to establish equilibrium conditions under which industries a an economy have just output to satisfy each other demands in addition to final (outside) demand. Commonly speaking the production of one good required the input of many other goods as intermediate inputs in the production process. Hence, the, total demand for a product is the summation all the intermediate demand for the product plus the final demand for the product arising from consumers, investors, government, exporters, etc as ultimate users.

Assume in hypothetical economy there are n- numbers of sectors that produce their own homogenous products x_1 $x_{2,}$ -----, $x_{n.}$

Therefore, total demand (x_1) = demand from inter mediate users in the economy + ultimate users demand for their final consumption.

i,e. $x_1 = x_{11} + x_{12} + \ldots\ldots + x_{1n} + f_1$

$x_2 = x_{21} + x_{22} + \ldots\ldots + x_{2n} + f_2$

" " " "

" " " "

$x_n = x_{n1} \quad x_{n2} + \ldots + x_{nn} + f_{nn}$

Where, x_i - refers total output produced by firm i

x_{ij} – refers total output part produced by sector i and used by sector j or

the delivery of sector i's output to sector j as input

fi – refers output requirement of sector i for final demand.

- Let's introduce a new variable in to the model.

aij = $\frac{xij}{xj}$, where aij – refers coefficient matrix elements or

an input requirement by sulfur j from sector j to produce one unit B xj

Therefore, from the above relationship

$$\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}\, a_{12} \ldots . a_{1n} \\ a_{21}\, a_{22} \ldots a_{2n} \\ . \\ . \\ . \\ a_{n1}\ a_{n2} \ldots . a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ . \\ . \\ f_n \end{bmatrix}$$

When we condense the above equation, we get

$$x = A\,X + F$$
$$X - AX = F$$
$$(I - A)X = F$$

$$X = \frac{F}{I - A}$$

$X= (I-A)^{-1}F$

So by using the above formula we can calculate the total out put produced by each sectors that meet needs of inter industry (intermediate) and final users.

Example:

Given a technical coefficient matrix and the vector of final demand as

$A=\begin{bmatrix} 0.3 & 0.2 \\ 05 & 0.6 \end{bmatrix}$, $F=\begin{bmatrix} 15 \\ 20 \end{bmatrix}$ Find the equilibrium output levels of the sectors outputs

Solution

$$x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [I - A]^{-1} * F$$

$$(I - A) = \begin{bmatrix} 1\ 0 \\ 0\ 1 \end{bmatrix} - \begin{bmatrix} 0\ .3 & 0.2 \\ 0.5 & 0.6 \end{bmatrix}$$

$$I - A = \begin{bmatrix} 0\ .\ 7 & -0.2 \\ 0.5 & 0.4 \end{bmatrix}$$

$$(I - A)^{-1} = \frac{Adj\,(I - A)}{|I - A|}$$

⇨ So the determinant (I – A) = 0.18

Adj (I –A) = cofactors transpose (C^T)

$$C \begin{bmatrix} 0.4 & 0.5 \\ 0.2 & 0.7 \end{bmatrix}$$

$$C' = \begin{bmatrix} 0.4 & 0.2 \\ 0.5 & 0.7 \end{bmatrix}$$

$$Thus \quad , (I - A) - 1 = \frac{1}{0.18} \begin{bmatrix} 0.4 & 0.2 \\ 0.5 & 0.7 \end{bmatrix}$$

$$x = (I - A)^{-1} , F$$

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{0.18} \begin{bmatrix} 0.4 & 0.2 \\ 0.5 & 0.7 \end{bmatrix} \begin{bmatrix} 15 \\ 20 \end{bmatrix}$$

$$X_1 = \frac{0.4}{0.18} X \ 15 + \frac{0.2}{0.18} X \ 20 = 55.5$$

$$X_2 = \frac{0.5}{0.18} X + \frac{0.7}{0.18} X \ 20 = 119.4$$

Self-Test Exercise 2.3

Given hypothetical economy consists of four inter connected industries steel, Coal, iron and auto. Based on the following inter industry transaction demand table in millions of dollars below

A. Find the matrix of technical coefficient?

B. Check that the inter mediate demand x-F and compare it with AX

		Consuming sector					
		Steel	Coal	Iron	Auto	Final	Total
Producing sector	Steel	80	20	110	230	160	600
	coal	200	50	90	120	140	600
	Iron	220	110	30	40	0	400
	Auto	60	140	160	240	400	1000
Value added		40	280	10	230	-	-
Gross production		600	600	400	1000		

Model Examination

Determine the total demand for industry's 1 , 2 & 3 given the following technical coefficient matrix and find demand vector

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0.5 & 0.2 \end{bmatrix} \quad F = \begin{bmatrix} 150 \\ 200 \\ 210 \end{bmatrix}$$

2.3.4. Linear Programming

Up until now we have been dealing with the classical approach in order to solve optimization problems: concepts in calculus. Here up on we introduce non-classical approach of optimization called *mathematical programming*. *Linear programming* deals with mathematical programming problems in which both objective function and constraints are all linear.

The objective of such programming is to determine the optimal allocation of scarce resources among competing activities in the economy. It deals commonly with optimization of linear function subject to more than one linear inequality constraints.

In any linear programming problem, there are three essential ingredients: ***the objective function, a set of technical constraints, and a set of non negativity restrictions.***

In solving Linear programming problems we have ***three*** commonly used methods

1. Graphic approach
2. Simplex method
3. Duality approach

It is know that Graphic approach is applicable only for the case of two unknowns, but simplex method is an algebraic method of finding the extreme points of non-graphable constraints which are supposed to result in an optimal value of the objective function. Duality approach is frequently used method to solve minimization problems. Every maximization /minimization problem has a corresponding minimization / maximization problem. The original problem is called the ***primal*** and the corresponding problem is called the ***dual***. Every dual problem can be transformed in to its primal counterpart and vice versa. Thus, for given linear programming problem we can use either of the approaches to calculate the optimal values.

Dear Distance learners! In order to refresh your memory solve the following linear programming problem.

Example: Given a Primal problem

$$\text{Min } C = 20Z_1 + 30Z_2 + 16Z_3$$

$$\text{Subject to: } 2.5Z_1 + 3Z_2 + Z_3 >= 3$$

$$Z_1 + 3Z_2 + 2Z_3 >= 4$$

$$Z_1, Z_2, Z_3 >= 0$$

Form the dual counter part of the problem

The dual problem is = Max **C*** = $3x_1 + 4x_2$

Subject to $2.5X_1 + X_2 <= 20$

$3X_1 + 3X_2 < =30$

$X_1 + X_2 < =16$

$X_1, X_2 >= 0$

Then, you can apply one of the above approaches to calculate the values of unknowns or optimal solutions. (***Refer Linear Algebra for Economists***)

Self-Test Exercise 2.4

Form the dual counter part of the following equation

Min Z = $5X_1 - 6X_2 + 4X_3$

S.t $3X_1 + 4X_2 + 6X_3 >= 9$

$X_1 + 3X_2 + 2X_3 >= 5$

$7X_1 - 2X_2 - X_3 < =10$

$X_1 - 2X_2 + 4X_3 >= 4$

$2X_1 + 5X_2 - 3X_3 > =3$

$X_1, X_2, X_3 >= 0$

Model Examination

Express the following primal problems interims of their dual counter parts

1. Max Z = $120X_1+30X_2+10X_3$

   S.t $10X_1+2X_2+X3 \leq 100$

   $7X_1 +3X_2 +2X_3 \leq 77$

   $2X_1 + 4X_2 + X_3 \leq 80$

   $X_1, X_2, X_3 \geq 0$

2. Min Z = $6000\ X_1 + 4000X_2$

   S. t. $4X_1+X_2 \geq 12$

   $9X_1 + X_2 \geq 20$

   $7X_1 +3X_2 \geq 18$

   $10X_1 +40X_2 \geq 40$

   $X_1, X_2 \geq 0$

CHAPTER THREE

DERIVATIVE IN USE

Chapter Contents

Dear students, this chapter comprises of the following major sub-topics

- Derivatives Application in Elasticities
- Higher Order Derivatives: Concavity and Convexity of a Function
- Approximation of functions (Taylor and Maclaurin series)
- Differential function
- Estimation of Functions and Multivariate Calculus
- Homogenous Functions and Euler's Theorem

Learning Objectives

After going through this chapter, students are required to:

- ✓ be acquainted with the basic uses of derivatives in computing different forms of elasticities.
- ✓ know how to differentiate the concavity and convexity of a function applying the rules of derivatives.
- ✓ employ derivatives for approximating functions in to different orders of polynomial using the approximation methods introduced.
- ✓ estimate functions using Mc Claurin Series, Taylor Series and Newton's method.
- ✓ realize the concepts of homogenous functions, Euler's theorem

3.1 Elasticity

Dear students, do you have any idea about the term elasticity? Use your own words and define it in the space given below and then compare your answer with the following definition._____

_____

_____

Fine! Elasticity is a valuable measure of responsiveness for many functional relationships of economic interest or it measures the rate at which the dependent variable changes when independent or explanatory variable(s) changes by a unit.

Dear distance learners, *we hope you discussed different types of elasticities in your microeconomics in detail. However, just to refresh your memory, let's outline the major ones.*

Basically, there are three types of elasticities. These are elasticity of demand, elasticity of supply and elasticity of substitution.

i. **Elasticity of demand**- this further can be divided in to three: price elasticity of demand, income elasticity of demand and cross elasticity of demand.

A. Price elasticity of demand

Consider a simple linear demand function

$$Y = a - bP, \qquad b > 0$$

The value 'a' measures the quantity demanded at $P = 0$, and $'b' = {}^{-dY}/_{dP}$ measures the amount by which demand falls as a result of a unit increase in price. One might be inclined to think that the slope, -b, is a good measure of the responsiveness of demand to price changes. However, the slope of the demand function is dependent on the units in which P and Y are measured. (This becomes a great problem.)

Suppose $Y = 10 - 0.02P$

Expresses the demand for steel when Y is measured in tones and 'P' refers to the price in dollars per tone. At P = 100, Y = 8 tones, At P = 200, Y = 6 tones

These are two points in this demand function.

If we measure Y in kilograms (1000kgs = 1 tone) and 'P' in dollars per kilogram, and this demand function becomes

Y = 10,000 – 20,000P

This can be checked by nothing that the two points

At P = 0.1 $ per kg, Y = 8000

At P = 0.2 $ per kg, Y = 6000.

The slopes of these two functions are very different, yet they explain the same demand behavior. Slope is sensitive to the unit of measurement.

By defining a measure of responsiveness in terms of average percentage changes in each of the variables, P and Y, we escape the "units of measurement" problem above. First, we develop the idea of "***arc elasticity***" which is the average elasticity of demand between two points on the demand function.

Letting (Y_1, P_1) and (Y_2, P_2) be two points on the demand function, the average percentage change in the price between two points is

$$\%\Delta P = \frac{P_2 - P_1}{(P_1 + P_2)/2} \times 100$$

And the average percentage change in the quantity is

$$\%\Delta Y = \frac{Y_2 - Y_1}{(Y_1 + Y_2)/2} \times 100$$

Note that these values are the same for the demand function above regardless of which equations we use to express it. That is

$$\%\Delta P = \frac{200 - 100}{150} \times 100 = \frac{0.2 - 0.1}{0.15} \times 100 = 66.7\%$$

And

$$\%\Delta Y = \frac{6 - 8}{7} \times 100 = \frac{6000 - 8000}{7000} \times 100 = -28.6\%$$

Using the standard notation that $\Delta Y = Y_2 - Y_1$ and $\Delta P = P_2 - P_1$

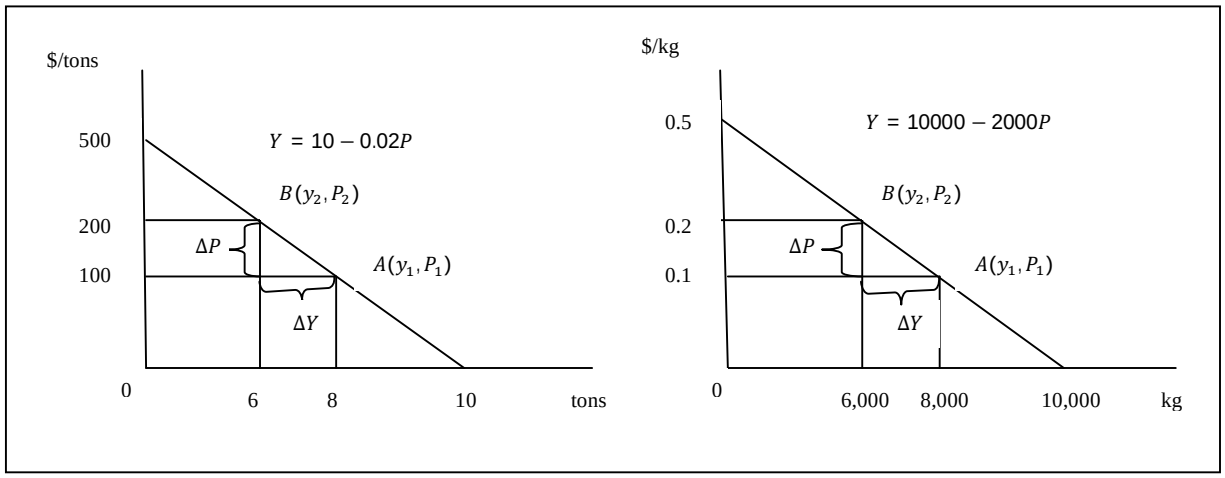
The Arc Elasticity between the two points,

$$\left[\frac{-\%\Delta Y}{\%\Delta P}\right] = \frac{-\dfrac{Y_2 - Y_1}{(Y_2 + Y_1)/2}}{\dfrac{P_2 - P_1}{(P_2 + P_1)/2}} = \frac{-\dfrac{\Delta Y}{(Y_1 + Y_2)}}{\dfrac{\Delta P}{(P_2 + P_1)}} = \left(\frac{\Delta Y}{\Delta P}\right)\left(\frac{P_2 + P_1}{Y_2 + Y_1}\right)$$

For the example above, the arc elasticity between point

$\left[\dfrac{-\%\Delta Y}{\%\Delta P}\right] = \dfrac{-(-28.6)}{66.7} = 0.43$. That is 43%. This elasticity represents, in absolute value, the percentage change in demand (Y) due to a given percentage change in price.

Fig 3.1: Single Demand Relationship Expressed in Terms of Different Units



One of the problems with arc elasticity is that the value depends on the size of the price change, ΔP. Take the same initial point as in the example above, P_1 = \$ 100 per ton and Y_1 = 8 tones, but use ΔP \$50 to get P_2 = \$150 per tone and Y_2 = 7 tones, in this case the arc elasticity is:

$$\left[\frac{-\%\,\Delta Y}{\%\,\Delta P}\right] = \frac{-\dfrac{\Delta Y}{(Y_1 + Y_2)}}{\dfrac{\Delta P}{(P_2 + P_1)}} = -\left(\frac{-1/15}{50/250}\right) = \frac{5}{15} = 0.33$$

Since there is no obvious amount by which one should change price to determine the elasticity value, the concept of arc elasticity is troublesome. The source of the difficulty is that the arc elasticity is really an average elasticity between the points (Y_1, P_1) and (Y_2, P_2). By taking the limit of the arc elasticity formula as ΔP→0, we get the two points converging and in so doing we find the elasticity at a point (Y_1, P_1), the point elasticity of demand is developed below.

In a continuous demand function, ΔY→0 as ΔP→0 and we get the following expression for (point) price elasticity of demand:

$$\varepsilon = \lim_{\Delta P \to 0} (-)\left(\frac{\Delta Y/(Y_1 + Y_2)}{\Delta P/(P_1 + P_2)}\right)$$

$$\varepsilon = \lim_{\Delta P \to 0} (-)\frac{\Delta Y}{\Delta P}\cdot\frac{P}{Y} = (-)\frac{dY}{dP}\cdot\frac{P}{Y}$$

There are three things to note about this formula

1) As ΔP→0, the two points (on the fig) (A and B) converge to as 'single' point (A), hence the term point elasticity.
2) If we are referring to the impact of the change in the product's price on the demand, we generally refer to price elasticity of demand. We could, of course, measure other elasticities such as income elasticities or cross-price elasticities, which will be discussed later.
3) The negative sign in the formula is used to convert to a positive number in the standard case for down ward slopping demand function.

Return to our example, a demand function for steel using the first form (units of measurement) of the demand function.

Y = 10 – 0.02 P, the point elasticity of demand at $(Y_1, P_1) = (8,100)$ is

$$\varepsilon = (-)\left(dY/dP\right)\frac{P}{Y} = (-)(-0.02)\left(\frac{100}{8}\right) = \frac{2}{8} = 0.25$$

Using the form: Y = 10,000 – 20,000P, the point elasticity of demand at the equivalent point (8,000, 0.1) = $\varepsilon = (-)(-20{,}000)\left(\frac{0.1}{8000}\right) = 0.25$

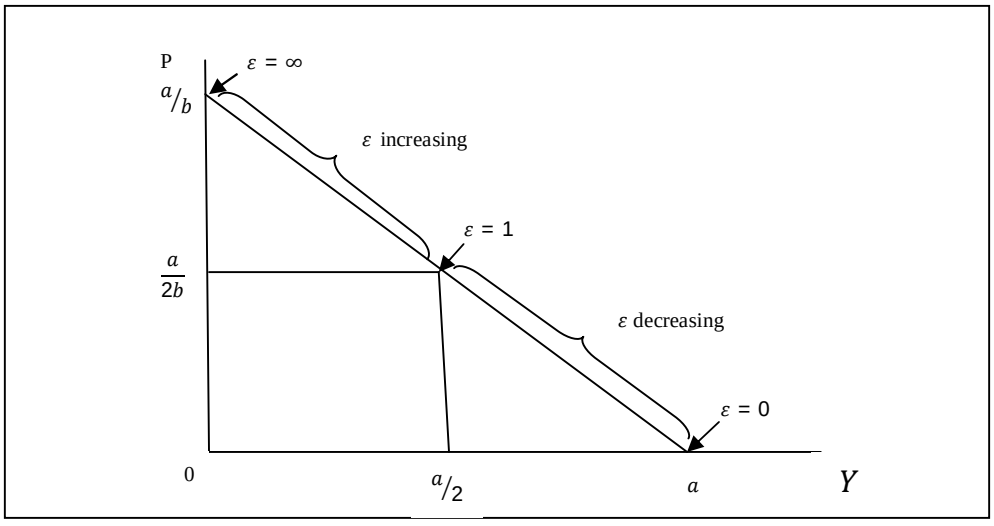
NB: The choice of units of measurement does not affect the elasticity value.

For the general demand function: Y = a-bP, /b/ > 0, we get

$$\varepsilon = \frac{-dY}{dP}\cdot\frac{P}{Y} = -(-b)\left(\frac{P}{Y}\right) = b\left(\frac{P}{Y}\right)$$

Thus, the slope of a linear demand function is constant but its elasticity is not.

Figure 3.2: Elasticity Changes along a Linear Demand Curve



We see that the elasticity of demand falls as one moves from point A to B. This is mathematically apparent that given the constant (dY/dP) (Slope), P/Y decreases implies the elasticity decreases. When we approach the P intercept, we have Y→0 and have Є→0. At the mid point i.e. at Y = $a/2$ and P = $a/2b$, the elasticity is 1.

Another functional form which is frequently used to express the relationship between price and quantity is the so called constant elasticity of demand: Y = $\alpha P^{-\beta}$, α, β > 0

Consider first the particular case with β = 1 (i.e., Y = α/P), constant unitary elasticity of demand. For this demand function it is easy to see that total sales revenue is the same at any price is always matched by the same percentage increase (reduction) in sales, which keeps total revenue constant. Thus, the elasticity of demand is a constant value equal to one at every point on the demand function.

$$\varepsilon = (-)\left(dY/dP\right)\left(P/Y\right)$$

$$= (-)\left(\frac{-\alpha}{P^2}\right)\left(\frac{P}{Y}\right) = (-)\frac{-\alpha}{P^2}\cdot\frac{P}{\alpha P^{-1}} = 1$$

For the more general form Y = $\alpha P^{-\beta}$, we get the following result

$$\varepsilon = (-)\left(dY/dP\right)\left(P/Y\right) = (-) - \beta\alpha P^{-\beta-1}\cdot\frac{P}{Y} = (-) - \beta\alpha P^{-\beta-1}\cdot\frac{P}{\alpha P^{-\beta}} = \beta$$

An alternative, easier method of computing the elasticity in the case is to transform the function in to logarithms

$Y = \alpha P^{-\beta}$

$\ln Y = \ln\left[\alpha P^{-\beta}\right]$

$\ln Y = \ln\alpha - \beta \ln P$

If we let $\hat{Y}$ represent ln Y, $\hat{P}$ represent ln P and $\hat{\alpha}$ represent ln α, then we can rewrite the equation as: $\hat{Y} = \hat{\alpha} - \beta\hat{P}$

It follows that: $\frac{d\hat{Y}}{d\hat{P}} = -\beta$ or $\frac{d \ln Y}{d \ln P} = -\beta$

In general, $-\frac{d \ln Y}{d \ln P} = \varepsilon = -\frac{dY}{dP}\cdot\frac{P}{Y}$

To see this note that

$$\frac{d \ln Y}{dY} = \frac{1}{Y} \Rightarrow d \ln Y = \frac{dY}{Y}$$

and

$$\frac{d \ln P}{dP} = \frac{1}{P} \Leftrightarrow d \ln P = \frac{dP}{P}$$

By straightforward substitution we get, $-\frac{d \ln Y}{d \ln P} \Rightarrow -\frac{dY/Y}{dP/P} = -\frac{dY}{dP} \cdot \frac{P}{Y}$

Example: Find the point elasticity of demand for the demand function Y = 50 – 2P, at P = 5. Over what range of prices is ε less than 1 and over what range of prices is it greater than 1?

Solution:

$dY/dP = -2$ and so

$$\varepsilon = (-)(-2)\left(\frac{P}{Y}\right) = 2\left(P/50 - 2P\right) = 2\left(\frac{5}{40}\right) = 0.25$$

For $\varepsilon = 1$ we have

$$1 = \frac{2P}{50 - 2P} \Rightarrow 2P = 50 - 2P \Rightarrow P = 12.5$$

$\varepsilon < 1$ when $\frac{2P}{50 - 2P} < 1$ (i.e. P<12.5)

$\varepsilon > 1$ when $\frac{2P}{50 - 2P} > 1$ (i.e. P>12.5)

Activity: Find the point elasticity of demand ε for the demand function

$Y = 100\ P^{-2}$.

Check your progress

$$\varepsilon = -\frac{dY}{dP} \cdot \frac{P}{Y} = (-) - 200 \cdot P^{-3} \cdot \frac{P}{100P^{-2}} = 2$$

Alternatively,

$$\ln Y = \ln 100 - 2 \ln P$$

$$\varepsilon = -\frac{d \ln Y}{d \ln P} = 2$$

B. Income Elasticity of Demand

Income elasticity is defined as the proportionate change in the quantity demanded resulting from a proportionate change in income. Symbolically, we may write

$$\varepsilon_y = \frac{dQ/Q}{dY/Y} = \frac{dQ}{dY}\cdot\frac{Y}{Q}$$

The income elasticity is positive for normal goods. A commodity is considered to be a luxury if its income elasticity is greater than one. A commodity is necessity if its income elasticity is small (less than unity, usually).

C. Cross-Elasticity of Demand

The cross-elasticity of demand is defined as the proportionate change in quantity demanded of X resulting from a proportionate change in the price of Y.

Symbolically, $\varepsilon_{XY} = \frac{dQ_X/Q_X}{dP_Y/P_Y} = \frac{dQ_X}{dP_Y}\cdot\frac{P_Y}{Q_X}$

Note: *The sign of the cross-elasticity is negative if X and Y are complementary goods; and positive if X and Y are substitutes.*

The higher the value of cross elasticity, the stronger will be the substitutability or complementarities of X and Y.

The main determinant of cross-elasticity is the nature of the commodities relative to their use. If two commodities can satisfy equally well the same need, the cross-elasticity is high and vice versa. It equals zero if the two commodities are unrelated.

Revenue and Elasticity

A monopolist with inverse demand function $P = f(x)$ has a revenue function $R = f(x)\cdot X = R(x)$. Then, the Marginal Revenue

$$R'(x) = f(x) + x\cdot\frac{df(x)}{dx}$$

$$R'(x) = p + x\cdot\frac{dp}{dx}$$

Price elasticity of demand is

$$\varepsilon = \frac{-dx}{dp}\cdot\frac{p}{x} \Rightarrow -\varepsilon = \frac{dx}{dp}\cdot\frac{p}{x}$$

MR may be rewritten as

$$MR = p(1 + \frac{x}{p} \cdot \frac{dp}{dx})$$

$$MR = p(1 + \frac{1}{-\varepsilon}) \Rightarrow MR = p\left(1 - \frac{1}{\varepsilon}\right)$$

$\varepsilon \prec 1 \Rightarrow R'(x) \prec 0$,

Thus, when demand is inelastic and increase in output (decrease in price) reduces revenue.

$\varepsilon \succ 1 \Rightarrow MR \succ 0$, so when demand is elastic an increase in output (decrease in price) increases revenue.

Moreover, the output X* at which revenue is maximized must satisfy

$MR = p(1 - \frac{1}{\varepsilon}) = 0$ Implying that revenue is maximized at the point on the demand curve when $\varepsilon = 1$.

ii. Elasticity of supply

In similar fashion with that of demand, we can formulate elasticity of supply.

iii. Elasticity of Substitution

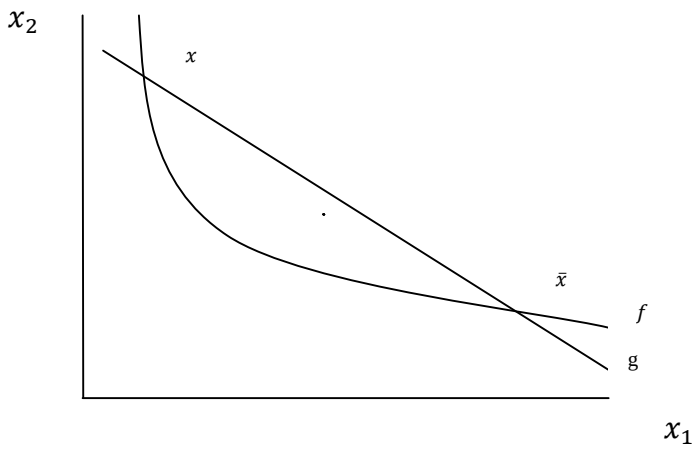
In a bit different approach, we can develop the elasticity of substitution, which has a good application in economic analysis.

Elasticity of Substitution between inputs, σ, is defined as

$\sigma =$ Proportionate rate of change of the input ratio

Proportionate rate of change of the MRT

Figure 3.3: Two Isoquant Curves of Different Production Functions



We have two isoquants for two different production functions f & g. The proportionate rate of change in the input ratio (the numerator) in moving from point X to $\overline{X}$ is the same for either isoquant f or g.

However, the proportionate rate of change of the slopes of the isoquants- the MRTS (the denominator)-is less for g than f. Thus g displays greater elasticity of substitution.

Mathematically, we have the following definitions:

The *elasticity of substitutions* between inputs for a production function y = f(x) which has a continuous marginal product function is defined as

$$\sigma = \frac{d \ln(x_2 / x_1)}{d \ln(f_1 / f_2)}$$

The Constant elasticity of substitution (CES) production function has the property that the value of σ is the same at any point on any isoquant. The cobb-Douglas production function has this property, with σ =1.

Example 1: Show that the Cobb-Douglas production function $Y = AX_1^{\alpha} X_2^{\beta}$,

$A, \alpha, \beta > 0$ defined on real numbers has σ =1 everywhere.

Solution:

First we find the partial derivatives, f_1 and f_2

$f_1 = \alpha A X_1^{\alpha-1} X_2^{\beta}$

$f_2 = \beta A X_1^{\alpha} X_2^{\beta-1}$

$$\text{MRTS} = \frac{f_1}{f_2} = \frac{\alpha A X_1^{\alpha-1} X_2^{\beta}}{\beta A X_1^{\alpha} X_2^{\beta-1}} = \frac{\alpha X_2}{\beta X_1}$$

$$\text{Or } \frac{X_2}{X_1} = \frac{\beta f_1}{\alpha f_2}$$

Taking their logs, we have

Taking their derivatives,

$$d\ln\left(X_2/X_1\right) = d\ln\left(f_1/f_2\right) \qquad \sigma = \frac{d\ln\left(X_2/X_1\right)}{d\ln\left(f_1/f_2\right)} = 1$$

Example 2: Find the elasticity of substitution for CES production function

$$Y = f(L,K) = \left[\delta L^{-r} + (1-\delta)K^{-r}\right]^{1/r}$$

Where $0 \prec \delta \prec 1$

$r \succ -1$

$L, K \succ 0$

Note that inputs L & K referring to labor and capital respectively

Solution

$$f_L = \frac{-1}{r}\left[\delta L^{-r} + (1-\delta)K^{-r}\right]^{\frac{-1-r}{r}} \cdot -r\delta L^{-r-1}$$

$$f_K = \frac{-1}{r}\left[\delta L^{-r} + (1-\delta)K^{-r}\right]^{\frac{-1-r}{r}} \cdot -r(1-\delta)K^{-r-1}$$

This implies that,

$$\text{MRTS} = \frac{f_L}{f_K} = \frac{\delta L^{-r-1}}{(1-\delta)K^{-r-1}} = \frac{\delta}{1-\delta}\left(K/L\right)^{r+1}$$

Taking the logs, we have

$$\ln\left(f_L/f_K\right) = \ln\left(\delta/1-\delta\right) + (r+1)\ln\left(K/L\right)$$

The first derivative is

$$d\ln(f_L/f_K) = (r+1)d\ln(K/L)$$

$$\sigma = \frac{d\ln(K/L)}{d\ln(f_L/f_K)} = \frac{1}{r+1}$$

NB: The smaller r, the greater is σ.

The following special cases are of interest:

Case 1:- r $\rightarrow +\infty$, which implies $\sigma \rightarrow 0$. Here, MRTS approaches zero if L > K and $+\infty$ if K > L. This means the isoquants become right angle and it is not possible to substitute between inputs.

Case 2:- r = 0, which implies that $\sigma = 1$. We cannot use the formula for the CES in this case because $-1/r$ is undefined at r = 0. However, we know that $\sigma = 1$corresponds to the Cobb-Douglas case and it can be shown that the cobb-Douglas is a limiting case of the CES as $r \rightarrow 0$.

Case 3:- $r \rightarrow -1$ (from the right since $r \succ -1$), which implies that $\sigma \rightarrow +\infty$. This is the case of perfect substitutability and the isoquants are linear.

Activity:

Show that the MRTS for the Cobb-Dougals production function $f(X_1, X_2) = AX_1^{\alpha} X_2^{\beta}, A \succ 0, \alpha, \beta \succ 0$ is constant along any ray from the origin. Also show that the marginal product change value (for different values of K or L) along such a ray unless $\alpha + \beta = 1$. (Hint: along a ray that passes through the origin $X_2 = KX_1$)

3.2. Higher Order Derivatives:

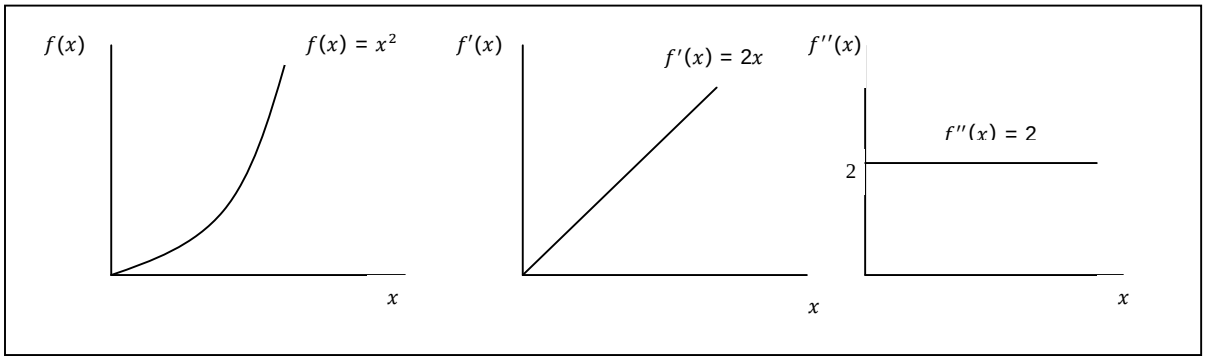
3.2.1. Concavity and Convexity of a Function

Since the derivative of a function is also a function, we can write $dy/dx = f'(x)$ and its derivative, $d[f'(x)]/dx$. We call f' the first derivative function of the function f and the derivative of the first derivative, $\frac{d\left(dy/dx\right)}{dx}$ or $\frac{d[f'(x)]}{dx}$ or $\frac{d^2y}{dx^2}$ or $f''(x)$ the second derivative function. Since the second derivative is also a function, we can also find its derivative. We call $d[f''(x)]/dx$ the third derivative function and write it as $f'''(x)$ or $f^{(3)}(x)$ to indicate that this function is found by three successive operation of differentiation, starting with the function f. Of course, this process may continue indefinitely and so we use the general notation $f^{(n)}(x)$ to indicate the n^{th} derivative of the function f.

Exercise: Find the first four derivatives of the function $f(x) = x^4$.

If the first two derivatives of a function exist, we say the function is twice differentiable. Concentrating at first and second derivative leads us to the concept of concavity and convexity. Consider the function $f(x) = x^2$ for the domain $x > 0$. On $x > 0$, this function is upward slopping and we can see that its slope increases as x increases. This means that the first derivative function is increasing in x, and so the derivative of this, the function's second derivative, must be positive valued for all $x > 0$. Up on finding the derivatives, we get $f'(x) = 2x$ and $f''(x) = 2$. Thus the second derivative is indeed positive for any value of x.

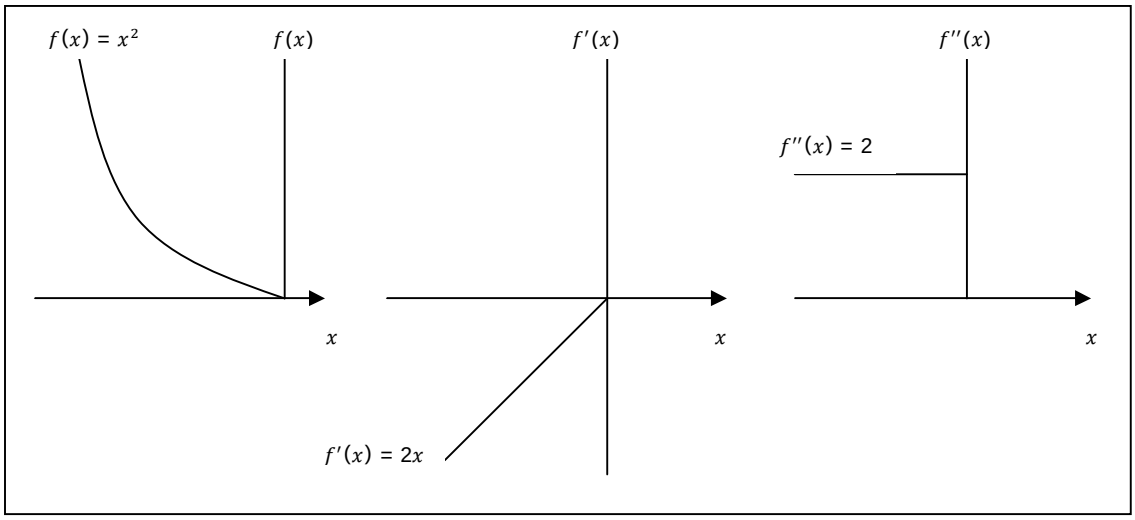
Figure 3.4: Function $f(x) = x^2$ $(x \geq 0)$ and its Derivatives



Now consider the graph of the same function defined on $x < 0$. On this set of values the function is negatively sloped.

$f'(x) = 2x < 0$ on $x < 0$. The greater the value of x, the less steep is the curve in absolute values.

Figure 3.5: Function $f(x) = x^2$ for $x < 0$ and its derivatives



Defining this function on the domain R, we see that the second derivative is positive throughout. A function with this shape, as determined by the second derivative being positive, is convex.

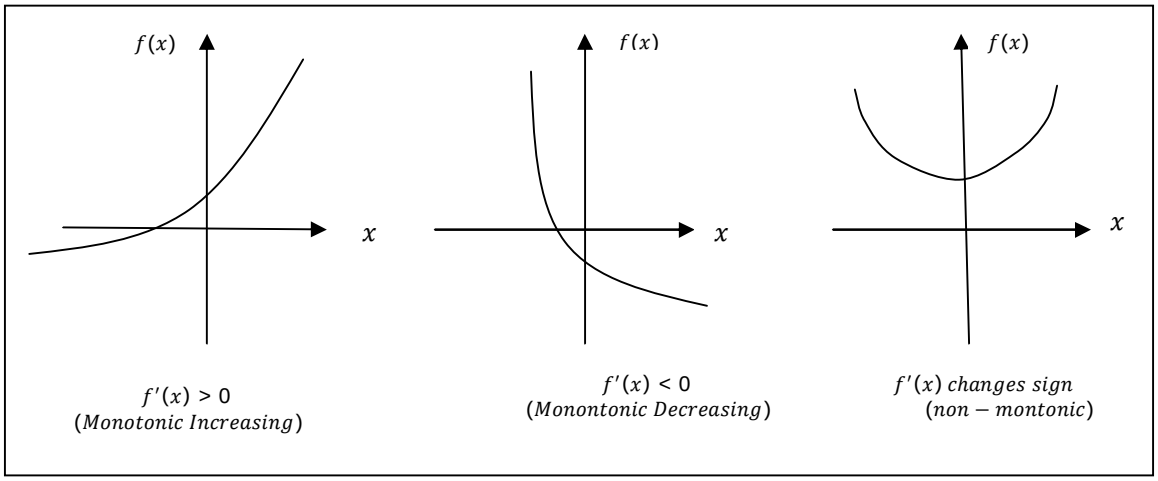
Definition: *A twice differentiable function* $f(x)$ *is convex if at all points on its domain* $f''(x) \geq 0$. According to the this definition, a linear function is convex. To exclude linear functions, we come to the concept strict convexity. This is done by replacing the week inequality $(\geq)$ with strict inequality (>).

Definition: *A twice differentiable function* $f(x)$ *is strictly convex if* $f''(x) > 0$ *except possibly at a single point.*

Notice that the function $f(x) = x^4$ has the second derivative $f''(x) = 12x^2$ which is positive for all x except $x = 0$ where the second derivative becomes zero.

This function is, however, strictly convex, and hence the qualification in the above definition requirement $f''(x) > 0$ except possibly at one point. We may have strictly convex functions with monotonic increasing, monotonic decreasing or non-monotonic functions.

Figure 3.6: Illustrations of Monotonic and Non-monotonic Functions



Definition: *A twice differentiable function* $f(x)$ *is concave if* $f''(x) \le 0$ *on all points of is domain.* As for the case of convex functions, a linear function satisfies the definition of concavity ($f''(x) = 0$ for all x). To exclude it, we should redefine it as strict concavity.

Definition: *A twice differentiable function* $f(x)$ *is strictly concave if* $f''(x) < 0$ *on all points of its domain except possibly at a single point.*

Alternatively, since multiplying a function by (-1) reverses the inequality, we could say that f(x) is (strictly) concave if –f(x) is (strictly) convex.

A function whose second derivative is sometimes positive and sometimes negative is neither convex nor concave everywhere. However, we can sometimes find intervals over which the function is convex or concave.

Example: Check curvature of the function, $f(x) = -\frac{1}{3}x^3 + 3x^2 - 5x + 10$ on $x \ge 0$

$f'(x) = -x^2 + 6x - 5$ and

$f''(x) = -2x + 6$. Thus, since $f''(x) > 0$ for $x < 3$ and $f''(x) < 0$ for $x > 3$, the function is convex on [0,3]and concave [3, $+\infty$].

Exercise: Show that the production function $f(x) = \frac{-2}{3}x^3 + 10x^2 + 5x$ has both a concave and a convex section.

Note: Production functions are concave means additional labor units yield lesser and lesser product and hence are the reflection of diminishing marginal product low. If production functions are convex, it becomes difficult to explain it in economic background.

Example: A single input, x, is used to produce output y. show that if the production function is $y = x^{1/3}, x > 0,$ then the cost function, c(y) is convex while the production function is concave.

Solution

Production function $f(x) = x^{1/3}$

First derivative $f'(x) = \frac{1}{3}x^{-2/3}$

Second derivative $f''(x) = -\frac{2}{9}x^{-\frac{5}{3}} = -\frac{2}{9x^{5/3}} < 0.$

Thus, if x is greater than zero, so the production function is concave.

Take the short-run cost function

$C(x) = c_0 + rx$ in terms of fixed cost c_0 and variable cost x needed to produce output y.

But since $y = x^{1/3} \Rightarrow x = y^3$ and we can write the cost function in terms of y.

$C(y) = c_0 + ry^3$

$C'(y) = 3ry^2$

$C''(y) = 6ry \geq 0$ for $y \geq 0$

And, so the cost function is convex.

Example: Given the following function, determine the slope of marginal cost at x=7.

$c(x) = x^3 - 9x^2 + 33x + 30$

Solution:

c'(x)=3x^2-18x, measures the slope of total cost curve and to find the slope of marginal cost curve take the derivative of MC, which is the second order derivative of total cost function.

C"(x) =6X-18

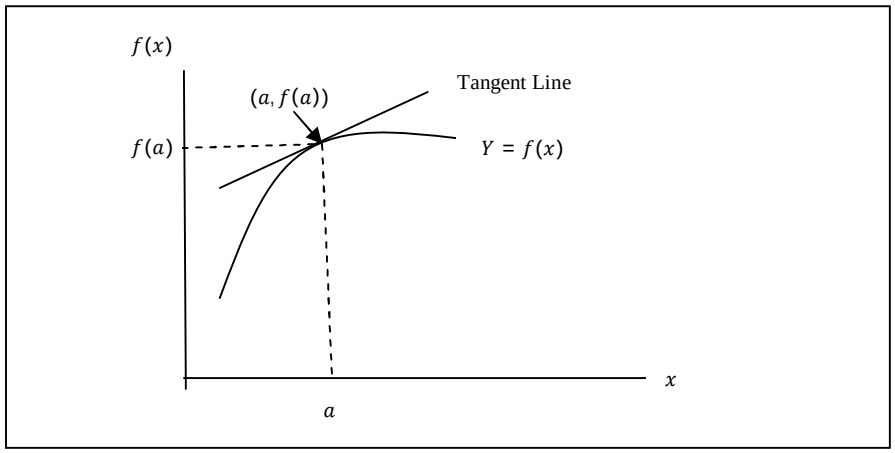
C" (7) =6[7]-18=24

3.2.2. Linear Approximation

To avoid working with a complicated function we sometime try to find a simpler function that can approximate the original one. Since linear functions are easy to work with, we first try to find a linear approximation to a function.

Suppose the function $y = f(x)$ is differentiable at $x = a$, what is the equation of the tangent to the graph of the function at the point $(a, f(a))$.

Figure 3.7: Illustrating Graph for Linear Approximation



Given $y = f(x)$ and (x_1, y_1)

$y - y_1 = b(x - x_1)$ where b is slope

$b = \frac{y - y_1}{x - x_1}$. This is the relation for the tangent straight line.

The equation of a straight line passing through (x_1, y_1) and having a slope

$b = \frac{y - y_1}{x - x_1}$ is $y - y_1 = b(x - x_1)$

Therefore, the equation of a tangent to the graph of $y = f(x)$ to the point $(a, f(a))$ is

$y - f(a) = b(x - a)$

$y - f(a) = f'(a)(x - a)$

$y = f(a) + f'(a)(x - a)$

If we approximate the graph of $f(x)$ by its tangent line at $x = a$, the resulting approximation is called Linear Approximation.

Examples

1) Find the linear approximation of a function $f(x) = \sqrt[3]{x}$ about $x = 1$.

Solution

$f'(x) = \frac{1}{3}x^{-2/3}$

$f'(1) = \frac{1}{3}(1)^{-2/3} = 1/3$

$f(1) = 1$

$y = f(1) + f'(1)(x - 1)$

$f(x) = 1 + 1/3(x - 1)$

$f(x) = \frac{1}{3}x + \frac{2}{3}$

If $x = 1.03$

$\sqrt[3]{x} \cong 1 + \frac{1}{3}(x - 1)$

$\cong 1 + \frac{1}{3}(1.03 - 1)$

$\cong 1.01$

But the actual value of the original function at $x = 1.03$ is $f(x) = \sqrt[3]{x}$

$f(1.03) = \sqrt[3]{1.03}$

$f(1.03) = 1.0099$

$f(1.03) \cong 1.01$

2) Find the linear approximation of a function

$f(x) = (1 + \frac{3}{2}x + \frac{1}{2}x^2)^{\frac{1}{2}}$ at $x = 0$.

Answer: $f(x) = 1 + \frac{3}{4}x$

3) Find the linear approximation value for $(1.001)^{50}$.

It can be written as a function of x as;

$f(x) = x^{50}$ at $x = 1$

To linearly approximate it we use the formula $f(x) = f(a) + f'(a)(x - a)$

$f(x) = x^{50} \Rightarrow f(1) = 1^{50}$

$f'(x) = 50x^{49} \Rightarrow f'(1) = 50(1^{49}) = 50$

$f(x) = 1 + 50(x - 1) = 50x - 49$

Evaluating this result around $x = 1$ such as x = 0.001, we get a linear estimation of

$$f(x) = 50x - 49$$

$$f(1.001) = 50(1.001) - 49 = 1.05.$$

But, the actual value or the correct value of $f(x) = x^{50}$ at x = 1 is 1.0512.

Activity:

1. Show that $\sqrt{1 + x}$ is approximately equal to $1 + \frac{1}{2}x$ for x close to zero.
2. Prove that $(1 + x)^m \cong 1 + mx$ for x close to zero.

3.2.3. The Differential Function

Given $Y = f(x)$, then we know that $\frac{dY}{dx} = f'(x)$. It follows that $dY = f'(x)dx$ – this is the differential of the function $f(x)$. It measures the change in the value of Y resulting from a small change in the value of x.

Let's compare dY and ΔY.

dY and dx are proportional to each other since $dY = f'(x)dx$.

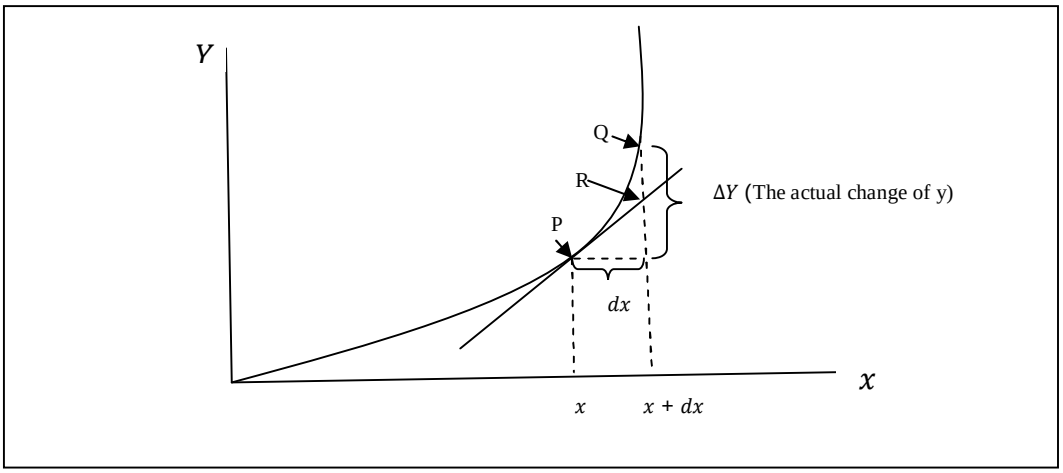
Let x changes from $x \to x + dx$, then $\Delta Y = f(x + dx) - f(x)$.

Given the linear Approximation formula $f(x) \cong f(a) + f'(a)(x - a)$ (x close to a), Substituting a by x and x by x + dx, we have

$$f(x + dx) = f(x) + f'(x)(x + dx - x)$$

$$\underbrace{f(x + dx) - f(x)}_{\Delta Y} = \underbrace{f'(x)dx}_{dY}$$

Therefore, in linear approximation, we approximate (estimate) ΔY by dx.



When x changes by dx, the actual change in Y is given by ΔY (movement along the curve). But when we consider a tangent or linear approximation or if we approximate the function by its tangent at point P, the change in the value of Y become dY (movement along the tangent line from P to R. Thus, $dY \cong \Delta Y$

The line segment $\overline{RQ}$ measures the distance that shows the difference between ΔY and dY (error term). The error term tends to be zero as $dY \to 0$.

Rules of Differentials

All the rules of differentiation can be applied in the case of differentials.

Given two differential functions (f and g) the following rules hold true:

1) $d(f \mp g) = df \mp dg$
2) $d(af + bg) = adf + bdg$
3) $d(fg) = fdg + gdf$
4) $d\left(f/g\right) = \left(\frac{gdf - fdg}{g^2}\right) \; for \; g \neq 0.$

All other rules of differentiations, discussed in your Calculus for Economists course you have taken earlier, are all applicable in this context too.

Example:

Given $Y = (2x-5)^2$, then

$$dY = f'(x)dx$$

$$f'(x) = 2(2x-5)(2) = 4(2x-5)$$

$$dY = 4(2x-5)dx$$

3.2.4. Polynomial Approximation

Approximation by a linear function may be insufficiently accurate, so it is natural to try quadratic approximation or approximation by polynomial of higher order.

i. Quadratic Approximation

We have a general quadratic formula

$P(x) = A + B(x-a) + C(x-a)^2 \qquad (for\ x\ close\ to\ a)$

In the above equations we have three unknown coefficients A, B and C. We impose three conditions on the polynomial to determine the value of the three unknown variables.

Assume at $x = a$,

$$f(a) = P(a)$$
$$f'(a) = P'(a)$$
$$f''(a) = P''(a)$$

Given $P(x) = A + B(x-a) + C(x-a)^2$

$P'(x) = B + 2C(x-a)$

$P''(x) = 2C$

When $x = a$,

$f(a) = A$

$f'(a) = B$

$f''(a) = 2C \Rightarrow C = \frac{1}{2}f''(a)$

Hence, the quadratic approximation of $f(x)$ is given by

$$f(x) \cong P(x) = A + B(x-a) + C(x-a)^2$$

$$= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

This is just similar to the linear approximation except one additional term

$$\frac{1}{2}f''(a)(x-a)^2$$

Example:

Find the quadratic approximation to $f(x) = \sqrt[3]{x}$ about$x = 1$.

Solution

$$f(x) = x^{1/3}$$
$$f(1) = 1$$
$$f'(1) = \frac{1}{3}x^{-2/3} = 1/3$$
$$f''(1) = -2/9\ x^{-5/3}$$
$$= -2/9$$

Then, the quadratic approximation will be

$$P(x) = 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2$$

If $x = 1.03$, then $\sqrt[3]{1.03}$ is quadratically approximated to be equal to $\sqrt[3]{1.03} = 1 + \frac{1}{3}(1.03-1) - \frac{1}{9}(1.03-1)^2 = 1.0099$ which is nearer to the exact value than the previous linear approximation.

ii. Higher Order Approximation

Functions in higher order derivatives can be better approximated near a point by using polynomial of higher degree.

Suppose we want to approximate a function $f(x)$ around $x = a$ of the form

$$\underbrace{P(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + A_3(x-a)^3 + \cdots + A_n(x-a)^n}_{n+1\ Coefficients}$$

It has $n + 1$ coefficients, so we impose $n + 1$ conditions on the polynomial to determine the value of $n + 1$ unknown coefficients.

$$f(a) = P(a)$$
$$f'(a) = P'(a)$$
$$f''(a) = P''(a)$$
$$\vdots \qquad \vdots$$
$$f^{(n)}(a) = P^{(n)}(a)$$

The implication is $f(x)$ and its n-derivatives are equal to $P(x)$ and its n-derivatives.

$P(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + A_3(x-a)^3 + \cdots + A_n(x-a)^n$
$P'(x) = A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + \cdots + nA_n(x-a)^{n-1}$
$P''(x) = 2A_2 + 6A_3(x-a) + \cdots + n(n-1)A_n(x-a)^{n-2}$
$P'''(x) = 6A_3 + \cdots + n(n-1)(n-2)A_n(x-a)^{n-3}$
.
.

.

$P^{(n)}(x) = n(n-1)(n-2)\ldots(2)(1)A_n$
When $x = a$,
$P(a) = A_0 = 0!\,A_0$
$P'(a) = A_1 = 1!\,A_1$
$P''(a) = 2A_2 = 2!\,A_2$
$P'''(a) = 6A_3 = 3!\,A_3$
.
.
.
$P^{(n)}(a) = n(n-1)(n-2)\ldots.(2)(1)A_n = n!\,A_n$

Therefore,

$P(a) = 0!\,A_0 = f(a) \Rightarrow A_0 = \frac{f(a)}{0!}$
$P'(a) = 1!\,A_1 = f'(a) \Rightarrow A_1 = \frac{f'(a)}{1!}$
$P''(a) = 2!\,A_2 = f''(a) \Rightarrow A_2 = \frac{f''(a)}{2!}$
$P'''(a) = 3!\,A_3 = f'''(a) \Rightarrow A_3 = \frac{f'''(a)}{3!}$
.
.
.
$P^{(n)}(a) = n!\,A_n = f^{(n)}(a) \Rightarrow A_n = \frac{f^{(n)}(a)}{n!}$

Substituting the values on the coefficients we get the following approximation to $f(x)\ at\ x = a$ by a n^{th} degree polynomial.

$$f(x) \cong P(x) = f(a)\frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

Example: Find the third degree polynomial approximation of

$f(x) = \sqrt{1+x}$ about $x = 0$.

$$f(0) = 1$$
$$f'(0) = {}^{1}/_{2}$$
$$f''(0) = {}^{-1}/_{4}$$
$$f'''(0) = {}^{3}/_{8}$$

$$P(x) = f(a)\frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$$
$$= 1 + \frac{1}{2}x + \frac{{}^{-1}/_{4}}{2}x^2 + \frac{{}^{3}/_{8}}{6}x^3$$

Therefore, $P(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

3.2.5. Estimation of Functions (Mclaurin and Taylor series)

This is expansion of a function $f(x)$by Mclaurin and Taylor series.

(a) Mclaurin Series

This is expansion of the function $f(x)$around $x = 0$. Let's consider an expansion of n^{th}degree polynomial by Mclaurin series.

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$

Successive differentiation of a function yields:

$$P'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$$
$$P''(x) = 2a_2 + 6a_3x + \cdots + n(n-1)a_nx^{n-2}$$
$$P'''(x) = 6a_3 + \cdots + n(n-1)(n-2)a_nx^{n-3}$$
$$\vdots$$
$$P^{(n)}(x) = n(n-1)(n-2)(n-3)\ldots(3)(2)(1)a_n$$

Evaluating the Maclaurin expansion around $x = 0$ yields

$$P(0) = a_0 = 0!\,a_0 \Rightarrow a_0 = \frac{P(0)}{0!}$$
$$P'(0) = a_1 = 1!\,a_1 \Rightarrow a_1 = \frac{P'(0)}{1!}$$
$$P''(0) = a_2 = 2!\,a_2 \Rightarrow a_2 = \frac{P''(0)}{2!}$$
$$P'''(0) = a_3 = 3!\,a_3 \Rightarrow a_3 = \frac{P'''(0)}{3!}$$
$$\vdots$$

$$P^{(n)}(0) = n!\,a_n \Rightarrow a_n = \frac{P^{(n)}(0)}{n!}$$

Therefore, the above polynomial function can be rewritten as

$$P(x) = \frac{P(0)}{0!} + \frac{P'(0)}{1!}x + \frac{P''(0)}{2!}x^2 + \frac{P'''(0)}{3!}x^3 + \cdots + \frac{P^{(n)}(0)}{n!}x^n$$

This power series representation is called Mclaurin Series.

Note: *Expansion using Mclaurin series yields exactly the same function as the original one.*

Example: Find the Mclaurin series for the following function

$$f(x) = 8 + 36x + 12x^2 + x^3$$

Given the above function, evaluating it around $x = 0$ based on Mclaurinseries, we expand until the third derivative, because the highest power is 3.

$$f(0) = 8$$

$$f'(0) = 36 + 24(0) + 3(0)^2 = 36$$

$$f''(0) = 24 + 6(0) = 24$$

$$f'''(0) = 6$$

In the Maclaurin series

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= \frac{8}{0!} + \frac{36}{1!}x + \frac{24}{2!}x^2 + \frac{6}{3!}x^3$$

$$\underline{f(x) = 8 + 36x + 12x^2 + x^3}$$

(b) Taylor's Series

This is expansion of a function around $x = x_0$. Suppose now we want to expand an nth degree polynomial at some arbitrary point $x = x_0$.

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$

Rewriting the function in terms of the variable $(x - x_0)$, we get

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

Let's find the successive derivatives as follows

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1}$$

$$f''(x) = 2a_2 + 6a_3(x - x_0) + \cdots + n(n-1)a_n(x - x_0)^{n-2}$$

$$f'''(x) = 6a_3 + \cdots + n(n-1)(n-2)a_n(x - x_0)^{n-3}$$

.

.

$$f^{(n)}(x) = n(n-1)(n-2)\ldots.(3)(2)(1)a_n$$

We then find the derivatives at $x = x_0$ as follows:

$$f(x_0) = a_0 = 0!\,a_0 \Rightarrow a_0 = {f(x_0)}/{0!}$$

$$f'(x_0) = a_1 = 1!\,a_1 \Rightarrow a_1 = {f'(x_0)}/{1!}$$

$$f''(x_0) = 2a_2 = 2!\,a_2 \Rightarrow a_2 = {f''(x_0)}/{2!}$$

$$.f'''(x_0) = 6a_3 = 3!\,a_3 \Rightarrow a_3 = {f'''(x_0)}/{3!}$$

.

.

$$f^{(n)}(x) = n(n-1)(n-2)\ldots.(3)(2)(1)a_n = n!\,a_n \Rightarrow a_n = \frac{f^{(n)}(x_0)}{n!}$$

Let's substitute the values of the coefficients on the original equation:

$$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{4!}(x-x_0)^n$$

This power series representation is known as Taylor's series representation.

Example: Find the Taylor's series expansion for $f(x) = e^x \; around \; the \; point \; x_0 = 0$.

Solution

Given $f(x) = e^x$ evaluating the function at $x_0 = 0$, we get

$$f(0) = e^0 = 1$$

The derivative of $e^x = e^x$ and never changes for successive derivatives so that

$$f'(0) = e^0 = 1, f''(0) = 1 \text{ and continues.}$$

The Taylor's series expansion will be

$$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{4!}(x-x_0)^n$$

$$= \frac{f(0)}{0!} + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x)^2 + \cdots + \frac{f^{(n)}(0)}{4!}(x)^n$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

(C) Expansion of Arbitrary Function (Taylor's Theorem)

According to Taylor's theorem, given an arbitrary function$\emptyset(x)$, which is continuous with a finite power n, its expansion around the point $x_0$yields.

$$\emptyset(x) = \frac{\emptyset(x_0)}{0!} + \frac{\emptyset'(x_0)}{1!}(x - x_0) + \frac{\emptyset''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{\emptyset^{(n)}(x_0)}{4!}(x - x_0)^n + R_n$$

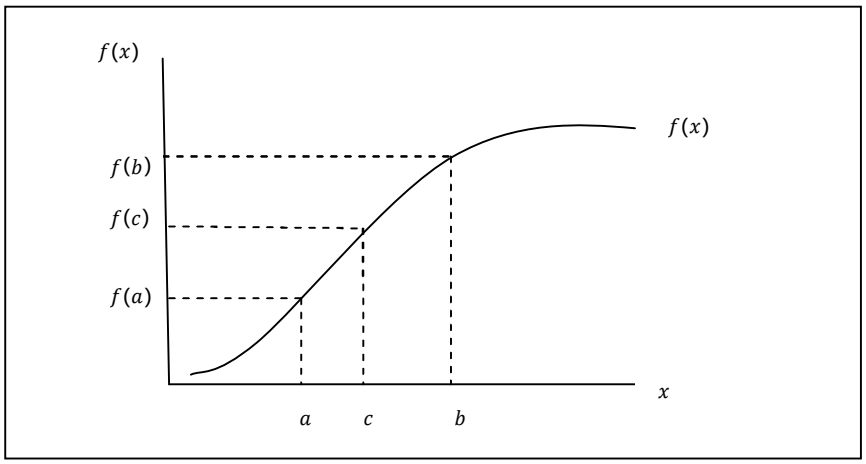
$\emptyset(x) = P_n + R_n$ *where* P_n *is polynomial approximation and* R_n *is remainder.*

If the arbitrary function $\emptyset(x)$ is an n^{th}degree polynomial and the expansion is in to another n^{th} degree polynomial is to be expanded in to a lesser degree the latter can only be considered as an approximation and the remainder is bound to appear.

Activity: Expand the function $f(x) = 2 + 3x + 4x^2 + 3x^2$ by using Taylor's series to a third degree polynomial around the point $x = 1$.

3.2.6. The Intermediate Value Theorem

Let $f(x)$ be a function that is continuous for all x in the interval $[a, b]$and assume that $f(a) \neq f(b)$. As x varies between a and b, so $f(x)$ takes on every value between $f(a)$ and $f(b)$.

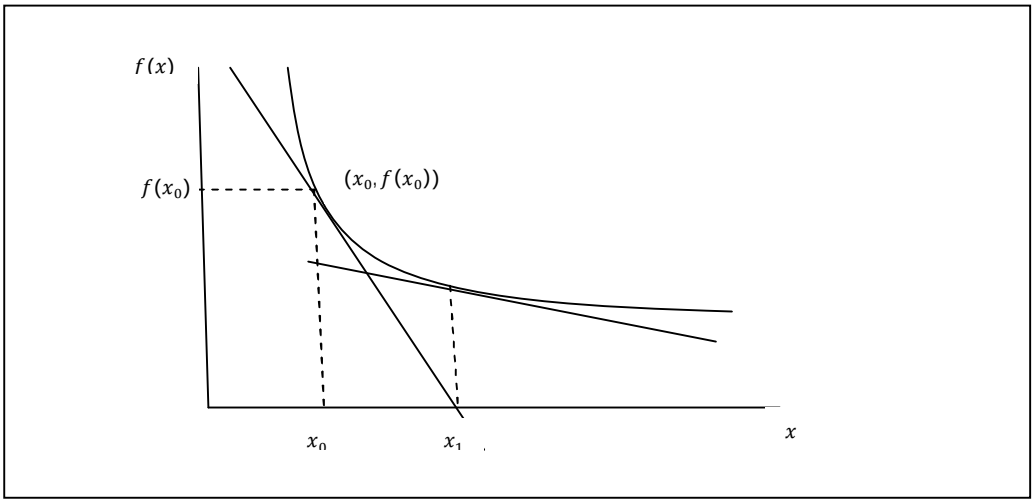


The theorem implies that the graph of the continuous function $f(x)$ must intersect the line $y = m = f(c)$ at least at one point (c, m).

An important result of the theorem is let $f(x)$ be a function continuous in (a, b) and assume that $f(a)$ and $f(b)$ have different signs, then there is at least one $c \in (a, b)$ such that $f(c) = 0$.

Newton's Method

The intermediate value theorem shows that a given equation $f(x)$ has a solution in a given interval; however, it doesn't provide additional information about the location of the solution. Newton's method leads us to a better approximate solution.



By constructing a tangent at $(x_0, f(x_0))$, we obtain x_1. Repeat the procedure by constructing another tangent line at $(x_1, f(x_1))$ and then equation x_1 derived by the Newton's Method. What is the equation of the tangent at $(x_0, f(x_0))$?

$y = f(x_0); y_0 = f(x_0) y - y_0 = m(x - x_0)$. At $(x_0, f(x_0))$, we have

$$y - f(x_0) = m(x - x_0) \text{ but } m = f'(x_0) \text{ so that we can rewrite it as}$$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

And at $x = x_1$, the tangent line intersects the x axis and hence the value of y assumes zero, so that it can be written as:

$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$
$$-f(x_0) = x_1 f'(x_0) - x_0 f'(x_0)$$
$$x_1 f'(x_0) = -f(x_0) + x_0 f'(x_0)$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If we compute for x_2, x_3 and continue in the same way, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

In general the points generated by Newton's method are given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example:

Find an approximate value of the function $x^{15} = 2$ using Newton's method twice.

Solution

First of all, for an intermediate solution we need to equate $f(x) = 0$ so that our reformulated equation is written as: $f(x) = x^{15} - 2 = 0$

Besides, we have to try manually to find the nearest integer that makes our function to be approximately equal to zero and that integer is 1 so that our $x_0 = 1$.

Finding and evaluating the function and its first derivative at the initial vale of x ($x_0 = 1$), we get:

$$f(1) = x^{15} - 2 = 1^{15} - 2 = -1$$

$$f'(x) = 15x^{14} \Longrightarrow f'(1) = 15(1)^{14} = 15$$

Hence, we find for x_1 using the Newton's method formula once:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{15} = \frac{16}{15}$$

Finding and evaluating the function and its first derivative at ($x_1 = \frac{16}{15}$), we get:

$$f\left(\frac{16}{15}\right) = x^{15} - 2 = \left(\frac{16}{15}\right)^{15} - 2 = 0.633$$

$$f'(x) = 15x^{14} \Longrightarrow f'\left(\frac{16}{15}\right) = 15(\frac{16}{15})^{14} = 37.025$$

Once we find x_1 it is easier to find x_2 using Newton's method formula twice as

:$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{16}{15} - \frac{0.633}{37.025} = 1.05$

Therefore, the approximate value of x in the equation $x^{15} = 2$ using Newton's method twice is found to be equal to 1.05 to the hundredth digit.

3.3. Multivariate Calculus: Calculus for Multivariate Functions

Though functions of singe variable have large simplicity, they are not commonly used in economics, since the functions that the real economic world consisted of are multivariate. For example, rational consumer will not consume one commodity only; rather, s/he consumes different combinations to maximize her/his utility.

The representative production function of a firm is not only a function of labor but it is also a function of capital. The overall income of nations is not a function of physical capital stocks only. It is also a function of human capital, factor productivity, institutions, and government policies. Hence, we have to extend the concepts of univariate calculus to multivariate ones. For the time being, let's begin with a function of two independent variables so that we can simply extend it to n-variables case.

Let's take the Cobb Douglas production function of a hypothetical firm.

$$y = AL^{\alpha}K^{\beta}$$

Where: Y-is the level of output, A-is the state of technology, which reflects productivity of factors, L & K -represent the units of labor and capital, respectively.

$\alpha \& \beta$ -are elasticity of y with respect to labor and capital respectively.

Example: Suppose for a firm $y = f(L,K) = 60L^{3/4}K^{1/4}$, then

a. How many units output will be produced if the firm uses 81 labor and 16 capital

b. Show that the production function is constant returns to scale

Solution

a. $Y=f(L, K) = 60 (81)^{3/4} (16)^{1/4} =3240$

b. To check for returns to scale, add the exponents of labor and capital. If the sum is greater than one it is increasing returns to scale, less than one it is decreasing returns to scale and if it is equal to one it is constant returns to scale. Since 3/4 +1/4=1, it is constant returns to scale. For the proof, you can see the following clarification.

Proof: For a general production function, when we change all the inputs by a given proportion if output changes by equal proportion, then the production function illustrates constant returns to scale. If it changes by greater/smaller proportion, the production function illustrates increasing/decreasing returns to scale respectively.

For $Y = AL^{\alpha}K^{\beta}$ and, Let $Y_o = AL^{\alpha}K^{\beta}$

Assume labor and capital have changed by λ, the new labor and capital will be λL, and λK respectively. So the new production function will be,

$$Y_N = A(\lambda L)^{\alpha}(\lambda K)^{\beta}$$

$$= A\lambda^{\alpha}L^{\alpha}\lambda^{\beta}K^{\beta}$$

$$= A(\lambda^{\alpha+\beta})L^{\alpha}K^{\beta}$$

$$= (\lambda^{\alpha+\beta})AL^{\alpha}K^{\beta}$$

$$= (\lambda^{\alpha+\beta})Y_0$$

If $\alpha+\beta > 1$, it shows increasing returns to scale, if $\alpha+\beta < 1$, it shows decreasing returns to scale, and if $\alpha+\beta = 1$, it is constant returns to scale.

3.3.1. Partial Differentiation

As you remember from your calculus for economists, partial differentiation measures the separate effect of each explanatory variable, holding others maintain constant and is usually works in multivariate calculus.

Given $Z = f(x, y)$, which is a function of two variables.

First order partial derivatives are given as: $f_x = \frac{\partial Z}{\partial x} = f_x(\text{x, y}),\ f_y = \frac{\partial Z}{\partial y} = f_y(\text{x, y})$

Second order partial derivatives

We have two types of second order partial derivatives

1. Direct Second Order Partial Derivatives

$$f_{xx} = \frac{\partial f_x}{\partial x} = f_{xx}(x, y), \qquad f_{yy} = \frac{\partial f_y}{\partial y} = f_{yy}(x, y)$$

2. Mixed (Cross) Second Order Partial Derivatives

$$f_{xy} = \frac{\partial f_x}{\partial y} = f_{xy}(x, y), \quad f_{yx} = \frac{\partial f_y}{\partial x} = f_{yx}(x, y)$$

Young's Theorem

According to this theorem, the mixed (cross) partials for a given function will always be equal if both cross-partials exist and are continuous.

Given a function, $f(x_1, x_2)$, then young's theorem asserts that $f_{12} = f_{21}$or alternatively can be written as $f_{x_1x_2} = f_{x_2x_1}$.

Example: Given a function $U = f(x_1, x_2) = x_1 lnx_2$, show that Young's theorem is true.

The first task will be to find the first and second derivatives of the functions one by one and secondly to check the equality of the second mixed partials.

$$U_1 = lnx_2 \qquad U_2 = \frac{x_1}{x_2}$$

$$U_{11} = 0 \qquad U_{22} = \frac{-x_1}{x_2^2}$$

$$U_{12} = \frac{-1}{x_2} \qquad U_{21} = \frac{1}{x_2}$$

So, as Young's theorem predicts that $U_{12} = U_{21}$ and our result also confirmed that is the case.

Exercise 1: Given the production function is given by$Y = f(L, K) = L^{\alpha}K^{\beta}$. Prove that Young's theorem is true.

Exercise 2: For the function$Y = f(x_1, x_2) = x_1 e^{x_1 + x_1^2}$, show that$Y_{12} = Y_{21}$.

Dear students, the concept of partial derivatives can be applied for elasticity, basically for cross-elasticity determination.

Two commodities are said to be complementary when a decrease in the demand for one commodity as a result of an increase in its price leads to a decrease in the demand for the other commodity.

For example, when the goods are complementary and p_2 is held constant, then both $\partial x_1/\partial p_1 < 0$ and $\partial x_2/\partial p_1 < 0$ and when p_1 is held constant, then $\partial x_1/\partial p_2$ as well as $\partial x_2/\partial p_2$.

Thus, we conclude that two commodities are complementary if and only if the cross-partials or partial elasticites of demand ($\partial x_1/\partial p_2$ and $\partial x_2/\partial p_1$) are ***negative***.

On the other hand, two commodities are said to be substitute or competitive when a decrease in the demand for one commodity as a result of an increase in its price leads to an increase in the demand for the other commodity.

Hence, we conclude that two commodities are substitute if and only if the cross-partials are ***positive***.

Example: A demand function for hypothetical commodity is given by $x_1 = 60-3p_1-6p_2$

Find the values of direct and cross price elasticities when $p_1=p_2=5$, and what types of goods are?

Solution

$e_{11} = \partial x_1/\partial p_1 * p_1/x_1$, $x_1 = 60-3(5)-6(5) = 15$

$= (-3)\ 5/15 = -1$

$e_{12} = \partial x_1/\partial p_2 * p_2/x_1$

$= (-6)\ 5/15 = -2$

Since the cross partial elasticity of demand is negative, goods x_1 and x_2 are complementary.

3.3.2. The Multivariate Chain Rule

Many economic models involve composite functions. Differentiations of composite functions require the application of chain rule.

(i) For functions of one variable $Y = f(x)$ and $x = f(t)$, then dy/dt is given by

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

(ii) Multivariate Functions (a simple case)

Suppose, $Z = f(x, y), x = f(t)$ and $y = f(t)$,
then the multivariate chain rule is given as:

$$\frac{dZ}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

Example: Suppose $Y = \ln(x_1 + x_2)$ where $x_1 = 2t$ *and* $x_2 = t^2$, find dy/dt.

Solution

$$\frac{dy}{dt} = \frac{dy}{dx_1} \cdot \frac{dx_1}{dt} + \frac{dy}{dx_2} \cdot \frac{dx_2}{dt}$$

$$= \left(\frac{1}{x_1 + x_2}\right)(2) + \left(\frac{1}{x_1 + x_2}\right)(2t)$$

$$= \frac{1}{x_1 + x_2}(2 + 2t)$$

To write the result of dy/dt in terms of only t, substitute t for x's:

$$= \frac{1}{2t + t^2}(2 + 2t)$$

Another alternative method to find this result is by direct substitution, that is:

$$Y = \ln(2t + t^2)$$

Then, we can directly find dy/dt using direct differentiation as follows:

$$\frac{dy}{dt} = \left(\frac{1}{2t + t^2}\right)(2 + 2t)$$

Activity: Suppose $Z = f(x, y) = x^2 e^y$ *and* $x = lnt, y = t^2$.

Find dy/dt using the chain rule.

3.3.3. Homogeneous Function and Euler's Theorem

Homogenous function

Definition: A function $Y = f(x_1, x_2, \dots, x_n)$ is said to be homogeneous of degree r if and only if $f(tx_1, tx_2, \dots, tx_n) = t^r f(x_1, x_2, \dots, x_n) = t^r y$.

Multiplying all the independent variable by a factor t will multiply the value of the function by the factor t^r.

Example 1: Given $f(x, y, z) = \frac{x}{y} + \frac{2z}{3x}$, check whether the function is homogeneous or not.

Solution

First, let's multiply the function by t:

$$f(tx, ty, tz) = \frac{tx}{ty} + \frac{2tz}{3tx}$$

$$= \frac{t}{t}\left(\frac{x}{y} + \frac{2z}{3x}\right) = t^0 f(x, y, z)$$

Therefore, the function is homogeneous of degree zero.

Example 2: Consider the Cobb-Douglas production function $Y = f(L, K) = L^{\alpha} K^{1-\alpha}$.

$$f(tL, tK) = (tL)^{\alpha}(tK)^{1-\alpha}$$

$$= t^{\alpha} L^{\alpha} t^{1-\alpha} K^{1-\alpha}$$

$$= t^{\alpha+1-\alpha} L^{\alpha} K^{1-\alpha}$$

$$= t^1 L^{\alpha} K^{1-\alpha}$$

$$= t^1 f(L, K)$$

Therefore, this function is homogeneous of degree 1. Such functions are also called linear homogeneous functions.

Activity: Check the homogeneity of $f(x, y) = 3x^2 y - y^3$.

In economics, the degree of homogeneity of a function has one important implication. That is, given $f(tx_1, tx_2, \dots, tx_n) = t^r f(x_1, x_2, \dots, x_n)$

If $r = 1 \Rightarrow$ Constant returns to scale

If $r > 1 \Rightarrow$ Increasing returns to scale

If $r < 1 \Rightarrow$ Decreasing returns to scale

Euler's Theorem

Euler's theorem states that if a homogeneous function has a degree r, then

$$\frac{\partial f}{\partial x_1}.x_1 + \frac{\partial f}{\partial x_2}.x_2 + \cdots + \frac{\partial f}{\partial x_n}x_n = rf(x_1, x_2, \dots, x_n)$$

Where, r measures the degree of homogeneity.

Example: Suppose, $f(x, y) = x^4 + x^2y^2$, then check the Euler's theorem.

Solution

According to Euler's theorem $f_x.x + f_y.y$ is equal to some value (r) times the original function and the question is what is this value.

$$\frac{\partial f}{\partial x} = f_x = 4x^3 + 2xy^2$$

$$\frac{\partial f}{\partial y} = f_y = 2x^2y$$

Then, $f_x.x + f_y.y = (4x^3 + 2xy^2)x + (2x^2y)y$

$$= 4x^3 + 2x^2y^2 + 2x^2y^2$$

$$= 4x^3 + 4x^2y^2$$

$$= 4(x^3 + x^2y^2)$$

$$= 4(f(x, y))$$

Hence, since the function is homogeneous of degree one, we call it linear homogeneous function.

Self-test Exercise 3

1. If $f(x) = (5x + 3)^{-2}$ about $x = 0$, find the quadratic approximation.
2. The demand curve for a good is given by the equation $Q=100-p-p^2$, find the price elasticity of demand and describe the state of the demand when p=5
3. If the Engel curve is given by the equation 30Q-10+5Y, where Q is quantity demanded and Y is income. Find the income elasticity of demand when Y=600 and comment on type of good.
4. Given the hyper parabola demand equation $Q=ap^b$, derive elasticity of demand.

5. Find the Maculaurian series of the function f(x) =6+8x-7x^2+3x^3+2x^4.

6. Consider an agricultural production function given by Q=F(K,L,T) =A$K^a L^b T^c$, where a, b& c are positive constants. Find the marginal products.

Model Examination

1. Approximate $y = \frac{1}{\sqrt{1+x}}$ by a quadratic polynomial $x = 0$.
2. Find the n^{th}derivative of e^{rx} with respect to x.
3. Show that the function $y = \frac{1}{x}$ is convex from below for all positive values of x and concave from below for all negative values of x. Can we interpret $x = 0$ as a point of inflexion? Explain.
4. Examine the concavity/convexity of the function $y = 2^x$ at $(0,2)$.
5. Find the quadratic approximation to $y = y(x)$ given by $xy^3 + 1 = y$ about $x = 0$ when y is defined implicitly as a function of x near $(x, y) = (0,1)$.
6. What is the degree of homogeneity of the function $h(x, y, w) = 2x^2 + yw - w^2$?
7. Consider two goods x_1& x_2 with prices p_1 & p_2. Suppose that a consumer with money income M has the following demand function for x_1: x_1= M. p_2/p_1^2

   A) Show the effect of neutral inflation on demand of x_1.

   B) Check whether the Euler's theorem holds for the demand or not

8. Given the utility function U = A$x^b y^c$, A, b &c are constants. Check whther the Euler's theorem holds true or not for this function.

CHAPTER FOUR

UNCONSTRAINED OPTIMIZATION

Chapter Contents

Dear students, this chapter comprises of the following major sub-topics

- Function of one variable
- Optimization of function with several independent variables
- Implicit functions and the unconstrained Envelop Theory

Learning Objectives

After successful completion of this chapter, you will be able to:

- apply the first and second order conditions for optimizing functions
- perform determinantal test for sign definiteness
- know the application of Envelop theorem for unconstrained implicit functions.

Introduction

Optimization involves maximization or minimization of a given objective function. Most of our problems in economics are optimized subject to some constraints but to grasp the basic idea, let's begin how one can handle optimization problems functions without any constraint. In the next chapter we will extend the technique to the constrained one. We will begin by briefly describing an optimum value and proceed to the next section.

4.1. Functions with one variable

Given any function Yas a function of x= f(x), then the derivative version of the function (First Order Condition or FOC) , as described earlier, is stated in

the form dy/dx , y' or f '(x) is the instantaneous rate of change of f (x) which is given by:

$$\frac{dy}{dx} = \underset{\alpha x \to 0}{Lim} \frac{f(x+\alpha x) - f(x)}{\alpha x}$$

4.1.1. The concept of optimum (extreme) value

An extreme value of a function is the maxima or minima value of that function. We can have two types of extreme values.

a. Relative extreme values (Local extreme)

b. Absolute extreme values (Global extreme)

A. Relative Extreme Value

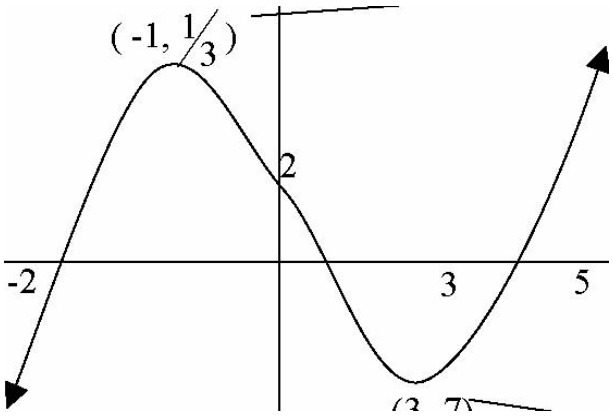
Relative extreme value: is an extreme value when compared with the values near to it. It can be relative maxima (when it is higher than any other point near to it) or relative minima (when it is lower than any value near to it)

For example, consider the function:

$$y = \frac{1}{3}x^3 - x^2 - 3x + 2$$

.

We can sketch, the graph for the function as follows:



In general, if a function f has relative maxima at $x = x_0$ if there exists an interval around x_0 on which for all x in the interval $f(x_0) \geq f(x)$

If a function f has relative minima at $x = x_0$ if there exist an interval around on which $f(x_0) \leq f(x)$ for all x in the interval. It is easy to find out relative extreme values using the derivative. Previously we have seen that a function changes from increasing to decreasing and vice versa, when its first derivative is zero or undefined.

Since a function changes from decreasing to an increasing at a relative minima, we have $f'(x) = 0 \; or \; f'(x)$ is undefined at relative minima or maxima.

Hence, we can find relative maxima and minima for a function by finding the values of x for which $f'(x) = 0 \; or \; f'(x)$ is undefined. Once we get the critical value, we can use the behavior of f '(x) near the extreme point to know whether it is a relative minima or relative maxima point.

To find out relative minima or maxima,

1st Find $f'(x)$

2nd Solve for the value of x which will make $f'(x) = 0$

3rd Use the above value to get the extreme value of the function (i.e, insert the value you got in the 2nd step in f(x) to get the extreme value).

4th Evaluate the first derivative function at some values of x to the teft and the right of each critical point

a. If to the left $f'(x) > 0$ and $f'(x) < 0$ to the right of the critical value of f(x), the critical point is relative maximum

b. If to the left $f'(x) < 0$ and $f'(x) > 0$ to the right of the critical value of f(x), the critical point is relative minima.

Example: Find the relative extrema as of the following function:

$$f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$$

Solution

$$f'(x) = x^2 - 2x - 3$$

$$f'(x) = x^2 - 2x - 3 = 0$$

$$x^2 - 2x + 1 = 3 + 1$$

$$=x - 1 = \pm 2$$

$$x = 3 \quad or \quad x = -1$$

$$f(3) = \frac{1}{3}3^3 - 3^2 - 3(3) + 2$$

$$=-7$$

$$f(-1) = -\frac{1}{3} - 1 + 3 + 2$$

$$= 4 - \frac{1}{3} = \frac{11}{3}$$

Thus, the critical values are (3, -7) and (-1, 11/3).

To find out which one relative minima or maxima, use the number line.

$f'(x)$		+	0	−
x		−2	−1	2

$f'(x)$		−	0	+
X		0	3	4

Thus, (-1, 11/3) is relative maxima while (3, -7) is relative minima.

Activity: Answer the following questions and check your progress below.

Find the relative extrema as of the following functions

a. $f(x) = 2x^3 - 6x^2 + 6x + 1$

b. $f(x) = x^3 - 3x + 6$

Check progress

a. $f(x) = 2x^3 - 6x^2 + 6x + 1$

$f'(x) = 6x^2 - 12x + 6$

$f'(x) = 6x^2 - 12x + 6 = 0$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

$x = 1$

$f(1) = 2 - 6 + 6 + 1 = 3$

.

The only critical value is (1,3).

To check whether it is relative minima or maxima, let's see the behavior of df(x)/dx near 1.

$\frac{f'(x)}{x}$	+	0	+
		1	

This shows the function has no extreme value.

b. $f(x) = x^3 - 3x + 6$

$f'(x) = 3x^2 - 3$

$= 3(x-1)(x+1) = 0$

$x = 1 \quad or \quad x = -1$

$f(1) = 4 \quad or \quad f(-1) = 8$ Thus, the critical values are (1,4) and (-1,8).

$\frac{f'(x)}{x}$	+	0	-
		-1	

Relative maxima at (-1,8)

$\frac{f'(x)}{x}$	-	0	+
		1	

Relative minima at (1,4)

Note that instead of taking the number line to check for maxima or minima, we can use the 2^{nd} derivative. If λ is the critical value which makes $f'(x)=0$, then the point $(\lambda, f(\lambda))$ is relative maxima if $f''(\lambda)<0$ and relative minima if $f''(\lambda)>0$. But when $(\lambda)=0$ we can't use the 2^{nd} derivative test and we have to use the behavior of the first derivative to check for relative optima.

Example: Use the 2^{nd} derivative test of the previous function to know whether they are relative maxima or relative minima.

a. $f(x)=\frac{1}{3}x^3-x^3-3x+2$

b. $f(x)=2x^3-6x^2+6x+1$

c. $f(x)=x^3-3x+6$

a. $f'(x)=x^2-2x-3$ critical values are (-1,11/3) and (3,-7)

$f''(x)=2x-2\ f''(-1)=-4<0$ *so* (-1, 11/3) is relative maxima

$f''(3)=6-2=4>0,\therefore(3,-7)$ is relative minima.

b. $f'(x)=6x^2-12x+6$

$f''(x)=12x-12$

critical values x=1 (1,3)

$f''(1)=12-12=0$

Use the number line in this case.

C. $f'(x)=3x^2-3$, $f''(x)=6x$

Critical values (1,4) and (-1,8)

$\begin{bmatrix} f''(1)=0 \\ f''(-1)=0 \end{bmatrix}$ Use the first derivative test in this case.

B. Absolute Extrema

Absolute extreme value is an extreme value that the function attains throughout its domain. The possible largest value that the function attains in its domain is called absolute maxima of the function. Similarly, the smallest possible value that the function attains throughout its domain is known as the absolute minima of the function. Note that a function may (or may not) have absolute extreme value like there may or may not exist relative extreme value. For example consider a function y=2x+1. As we increase the value of x, the value of y increases continuously. Again, when we decrease the value of x, the value of y also decreases continuously. However, if the domain is restricted to a closed interval [a, b] which means and the function is continuous everywhere within the interval, the function will have an absolute minima and absolute maxima within the interval. They may be located either: $a \le x \le b$

a. At the end points; i.e., a and b or

b. At the interior points

To find absolute extrema within an interval

1. Find the critical values and critical points and check whether they are within the interval
2. Calculate f(a) and f(b)
3. The largest value and smallest value obtained from the above steps represent the absolute maxima and the absolute minima of the function over the interval respectively.

Examples:

1. Find the absolute maxima and minima of $f(x) = x^2 - 6x + 2$ over the interval $0 \le x \le 5$.

Solution

f'(x)=0

$2x - 6 = 0$

$x = 3$

$f(3) = -7$

$f(0) = 2$

$f(5) = -3$

There is absolute maxima at x=0 and absolute minima at x=3 within [0, 5].

2. Find the absolute extrema of $f(x) = x^3 + 3x^2 + 4$ Over the interval [-4, 2]

Solution

$$f'(x) = 3x^2 + 6x = 0$$

$$3x(x+2) = 0$$

$$x = 0 \quad or \quad x = -2$$

$$f(0) = 4 \qquad f(-2) = 8$$

$$f(2) = 24 \qquad f(-4) = -12$$

Absolute maxima at x=2 (2, 24) and absolute minima at x=-4 (-4, -12)

Self- Test Exercise 4.1

1. Given TC$= 5x + 200$ and TR$= 10x - 0.01x^2$, what is the value of x, which will maximize profit?
2. Given TC$= 200 + 25x + 2x^2$, what is the level of x which minimizes average cost?

Model Examination

1. A farmer uses his 1000 hectar land to raise two grain crops. Grain A is sold at a market and grain B is fed to his herd of 1000 cattle. Using the data below determine how many hectars of each crop should be planted in order to realize greatest profit.
   a. A hectar planted with grain A yields 150 bushels which is sold at $5 per bushel.
   b. Feeding the herd, the yield of x hectares of grain B will result in each steer weighing $350+x-0.001x^2$ pound. Steer sells for $1 per pound.
2. A book publisher sells 100,000 copies of a certain book each year. Setting up each run costs $2000, each book costs $5 in material and labor, and storage fees are $1 per book per year. Assuming, a uniform rate of sales throughout the year, find the number of runs that minimize the publishers cost.
3. A travel agency will plan a group four for groups of size 25 or larger. If the group contains exactly 25 people, the cost is $300 per person. However, each person's cost is reduced by $10 for each additional person above 25. What size group will produce the largest revenue for the agency?
4. Given the total cost of producing x items is given by : $C(x) = 2x^3 - 12x^{-2} + 26x + 26x$, and the average cost is given by $AVC(x) = \frac{WC(x)}{x} = 2x^2 - 12x + 6$

   Find the value of x for which AVC (x) is minimum show that for this value of x, the marginal cost ($C'(x)$) equals AVC(x).

4.2. Optimization of a Function with More Than One Variable

Dear students! You have studied optimization of a function in section 4.1 in which case the discussion was restricted to a function with a single explanatory variable. But, economic phenomena in reality involve the study and analysis of large number of variables simultaneously. Therefore, in this section we extend the discussion of optimization to a function with more than a single explanatory variable.

4.2.1. First and Second Order Conditions for a Function with Two Explanatory Variables

This section is a simple extension to your study on optimization of a function with a single variable. You will begin your study with the first order necessary condition for optimization of a function of two explanatory variables and then discuss the second order sufficient condition. Before going into the discussion, you should note that a maximum and a minimum in this section refer to a relative maximum and a relative minimum.

a) First Order Condition for a Function with Two Explanatory Variables

What do you think about the first and second order conditions for a function with two explanatory variables? You can use the space below to write your answer.

_____

_____

_____

Consider the function $z = f(x, y)$. In this case, the first order necessary condition for an extremum (either a maximum or a minimum) requires the first partial derivatives of the function $z = f(x, y)$ equal to zero simultaneously.

This involves finding the values of x and y so that the first partial derivatives are zero at same time. That is $f_x = f_y = 0$ or $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$.

Can you make a conclusion based upon this first order condition for an extremum? Or, is it possible for you to conclude whether a particular point is a maximum or a minimum on the basis of the first order condition?

The answer is no because of the presence of points which satisfy the first order condition but neither a maximum nor a minimum. One good example is the case of an *inflection point*. Therefore, similar to what you have studied in the case of a single variable optimization, the first order condition is only a necessary but not a sufficient condition. As a result, you need to make use of additional criteria.

b) Second Order Condition

The second order condition for the function $z = f(x, y)$ involves restrictions on the signs of the second partial derivatives: f_{xx}, f_{yy} and f_{xy}. The second order condition which is also known as the sufficient condition follows below.

Let (x_0, y_0) be the critical values of the function (i.e., where $f_x = f_y = 0$) and the second partial derivatives are evaluated at these points. Thus,

a) $f(x_0, y_0)$ is a relative maximum if and only if $f_{xx}, f_{yy} < 0$ and $f_{xx} f_{yy} > f_{xy}^2$.

b) $f(x_0, y_0)$ is a relative minimum if and only if $f_{xx}, f_{yy} > 0$ and $f_{xx} f_{yy} > f_{xy}^2$.

c) $f(x_0, y_0)$ is a saddle point if $f_{xx} f_{yy} < f_{xy}^2$, and f_{xx} and f_{yy} have different signs.

d) $f(x_0, y_0)$ is an inflection point if $f_{xx} f_{yy} < f_{xy}^2$, and f_{xx} and f_{yy} have the same signs.

e) $f(x_0, y_0)$ could be a relative maximum, a relative minimum, or a saddle point if $f_{xx} f_{yy} = f_{xy}^2$. That is the second order test fails to tell the behavior of the function.

The table below summarizes both the first and the second order conditions for relative extremum of the function $z = f(x, y)$.

Table 4.1 Conditions for Relative Extremum

Condition	**Maximum**	**Minimum**
First order necessary condition	$f_x = f_y = 0$	$f_x = f_y = 0$
Second order sufficient condition	$f_{xx}, f_{yy} < 0$ and $f_{xx} f_{yy} > f_{xy}^2$	$f_{xx}, f_{yy} > 0$ and $f_{xx} f_{yy} > f_{xy}^2$

Generally, in finding the relative maximum/minimum of a function with two explanatory variables, you need to find first the critical values of that function using the first order necessary condition and then carry on the second order test to determine if the critical values are either maximum, minimum, saddle or inflection points.

Examples:

a) Find the critical values of the function $z = x^2 + 2y^2$ and determine whether they are maximum or minimum.

Solution

Using the first order condition, you can find the critical values. Thus, you need to find the first partial derivatives, f_x and f_y.

$$f_x = \frac{\partial z}{\partial x} = 2x, \quad f_y = \frac{\partial z}{\partial y} = 4y$$

Then, find the values of x and y (the critical or optimal values) which satisfy the condition $f_x = f_y = 0$. Hence, $2x = 4y = 0$ for $x = 0$ and $y = 0$, i.e., the critical point in this case is $(0,0)$

Now, you go to the second order test. Here, you need to find the second order partial derivative: $f_{xx} = \frac{\partial^2 z}{\partial x^2} = 2$, $f_{yy} = \frac{\partial^2 z}{\partial x^2} = 4$, $f_{xy} = \frac{\partial^2 z}{\partial x^2} = 0$

Evaluating the second partial derivatives at the critical value, you have

$$f_{xx}(0,0) = 2, \quad f_{yy}(0,0) = 4, \quad f_{xy}(0,0) = 0$$

Therefore, $f_{xx} > 0$ and $f_{yy} > 0$, and $f_{xx} f_{yy} > f_{xy}^{\ 2}$. Hence, you conclude that the critical value $(0,0)$ is a relative minimum.

b) Given the function $z = 2x^2 + xy + 3y^2$, find the stationary values and examine whether they are maximum or minimum.

First order condition

The first partial derivatives are

$$f_x = 4x + y$$

$$f_y = 6y + x$$

The first order condition requires $f_x = f_y = 0$. Thus, you have

$$4x + y = 0$$

$$6y + x = 0$$

Solving the above equations simultaneously, you get $x = 0$ and $y = 0$ and hence, the critical value is $(0,0)$.

Second order condition

The second partial derivatives are: $f_{xx} = 4$, $f_{yy} = 6$, $f_{xy} = 1$

Since $f_{xx} > 0$ and $f_{yy} > 0$, and $f_{xx} f_{yy} > f_{xy}^2$ (because $f_{xx} f_{yy} = 24$ and $f^2{}_{xy} = 1$), the critical value $(0,0)$ is a relative minimum.

Activity: Answer the following questions and then check your progress below.

Find the critical values of the following functions and examine if they are maximum or minimum.

a) $z = -x^2 + xy - y^2 + 2x + y$

b) $z = -x^3 + xy + y^2 + x$

Check your progress

a) First order condition partial derivatives

$$f_x = -2x + y + 2$$

$$f_y = -2y + x + 1$$

The first order condition requires $f_x = f_y = 0$. Thus, you have

$$-2x + y + 2 = 0$$

$$-2y + x + 1 = 0$$

Solving the above equations simultaneously, you have $x = \frac{5}{3}$ and $y = \frac{4}{3}$. That is the critical value is $(\frac{5}{3}, \frac{4}{3})$.

The second partial derivatives: $f_{xx} = -2$ $f_{yy} = -2$ $f_{xy} = 1$

Since $f_{xx} < 0$ and $f_{yy} < 0$, and $f_{xx} f_{yy} > f_{xy}^2$ (because $f_{xx} f_{yy} = 4$ and $f^2{}_{xy} = 1$), you have a relative maximum at $(\frac{5}{3}, \frac{4}{3})$.

b) The first order derivatives are

$f_x = -3x^2 + y + 1$

$f_y = 2y + x$

The first order condition requires $f_x = f_y = 0$. Hence, you have

$-3x^2 + y + 1 = 0$

$2y + x = 0$

After solving the above equations simultaneously, you get $x = \frac{1}{2}$ and $x = -\frac{2}{3}$, and the corresponding values of y are $y = -\frac{1}{4}$ and $y = \frac{1}{3}$. That is the function has two critical values: $(\frac{1}{2}, -\frac{1}{4})$ and $(-\frac{2}{3}, \frac{1}{3})$.

The second order derivatives are: $f_{xx} = -6x$, $f_{yy} = 2$, $f_{xy} = 1$.

At the critical value $(\frac{1}{2}, -\frac{1}{4})$, you have $f_{xx} = -3 < 0$, $f_{yy} = 2 > 0$ and $f_{xy} = 1 > 0$.

Since f_{xx} and f_{yy} have different signs and $f_{xx} f_{yy} < f_{xy}^2$, the point $(\frac{1}{2}, -\frac{1}{4})$ is a saddle point.

At the critical value $(-\frac{2}{3}, \frac{1}{3})$, you have $f_{xx} = 4 > 0$, $f_{yy} = 2 > 0$ and $f_{xy} = 1 > 0$.

Therefore, since $f_{xx} > 0$ and $f_{yy} > 0$, and $f_{xx} f_{yy} > f_{xy}^2$, the point $(-\frac{2}{3}, \frac{1}{3})$ is a relative minimum.

4.2.2. Deterimental Test for Sign Definiteness: Second Order Condition

A form of a polynomial expression in which each component term has a unique degree, i.e the sum of components in each term is uniform

Example: W = f(x, y, z) = ax –by +z- is linear function

W = f(x, y, z) = $ax^2 + bxy + cz^2$ – quadratique differential

$d^2z = f_{xx}dx^2 + 2f_{xy}d_xd_y + f_{yy}dy^2$- is total defferential

Let $g = d^2 z$, $U = d_x$, $V = d_y$, $a = f_{xx}$, $b = f_{yy}$, $h = f_{xy} = f_{yx}$

$g = au^2 + 2h\,uv + bv^2$

Different cases

1. If g>o – If positive definite, Sufficient condition for minimum
2. If g>o - Positive semi definite , Necessary condition for minimum
3. If g<o - If Negative definite, Sufficient condition for maximum
4. If g<o - If negative semi definite, necessary condition for maximum

If g changes sings when the variables assume different values, g is said to be indefinite

We can check whether the function (g) is definite or indefinite using determinant test – determinant test for sign definiteness.

$g = a\,u^2 + 2hu + bv^2$

$= au^2 + huv + bv^2$

$= u(au+hv) + v(hu+bv)$

Note for two matrices to be multiplied, conformability condition must hold true Thus,

$$= \begin{bmatrix} u & v \end{bmatrix}_{1x2} \begin{bmatrix} a & h \\ h & b \end{bmatrix}_{2x2} \begin{bmatrix} u \\ v \end{bmatrix}_{2x1}$$

$$= \begin{pmatrix} ua + vh & uh + vb \end{pmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$= u^2 + huv + uhv + bv^2$$

$$= \begin{vmatrix} a & h \\ h & b \end{vmatrix}$$

Therefore, the above determinant is called ***Hessian Determinant***.

The first principal minor of Hessian (H_1) is the matrix which contains the first one principal diagonal element; the second principal minor of Hessian (H_2) is a matrix which contains the first two principal diagonal elements, etc.

$$\begin{vmatrix} a & h \\ h & b \end{vmatrix} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$|H_1| = |f_{xx}| = f_{xx}$$

$$|H_2| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}\ f_{yy} - (f_{xy}, f_{yx})$$

Notes:

1. The second order differential $d^2Z>0$ if and only if $/H_1/ > 0$, $/H_2/ > 0$, it is positive definite and shows the minimum point
2. $d^2Z < 0$, if and only if $/H_1/ < 0$, but $/H_2/ > 0$, - negative definite and it refers maximum point
3. d^2Z is indefinite if and only if $/H_2/$

Example: Optimize the following function by using the Hessian determinant and state sign definiteness. Find the maximum or minimum value.

$Z = f(x, y) = 6x^2 - 9x-3xy-7y + 5y^2$

Solution

F.O.C : $Zx = 12x-9-3y = 0$

$Zy = -3x -7 +10y = 0$

By solving simultaneously, the critical values will be: $x = 1$, $y = 1$

S.O.C : $Zxx = 12$

$Zxy = Zyx = -3$

$Zyy = 10$

$$H = \begin{vmatrix} Zxx & Zxy \\ Zyy & Zyy \end{vmatrix} = \begin{vmatrix} 12 & -3 \\ -3 & 10 \end{vmatrix}$$

$$|H_1| = |Zxx| = 12 > o$$

$$|H_2| = \begin{vmatrix} Zxx & Zxy \\ Zyx & Zyy \end{vmatrix} = Zxx\ Z - (Zxy, Zyx)$$

$$= 12x10 - (-3x - 3)$$

$$= 120 - 9$$

$$= 111 > 0$$

Since both $/H_1/$and $/H_2/$are greater than Zero/ the function is at minimum point.

The sign definiteness is Positive, because $/H_1/ > 0$, $/H_2/ > 0$.

The minimum value = $6(1)^2 - 9(1) - 3(1x1) - 7(1) + 5(1)^2 = -8$

The case of three variables Quadratic form

This approach can also be applied to more than three choice variables.

Given $Z = f(X_1, X_2, X_3)$, then the form of Hessian Determinant will be:

$$|H| = \begin{vmatrix} f_{ll} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

$$|H_1| = |f_{11}| = f_{11}$$

$$|H_2| = f_{11}\ f_{22} - |f_{21}\ f_{12}|$$

$$|H| = |H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

d^2z -is positive definite if and only if $/H_1/> 0$, $/H_2/ > 0$,$/H_3/> 0$

Shows minimum point

d^2z -is negative definite if and only if $/H_1/< 0$, $/H_2/ > 0$, $/H_3/< 0$

Shows maximum point

For the Case of n - variables, $Z = f(x_1, x_2, x_3 ...Xn)$

$$|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} & \cdots\cdots\cdots & f_{1n} \\ f_{21} & f_{22} & f_{23} & \cdots\cdots & f_{2n} \\ f_{31} & f_{32} & f_{33} & \cdots\cdots & f_{3n} \\ f_{n1} & f_{n2} & f_{n3} & \cdots\cdots & f_{nn} \end{vmatrix}$$

d^2z -is positive definite if and only if, $/H_1/> 0$, $/H_2/ > 0$,$/H_3/ >0$ …. /Hn/= /H/

The function is at its minimum point

d^2z -is negative definite if and only if iff, /H1/< 0, /H2/ > 0,$/H_3/ < 0$ …./Hn/ = /H/

Where the function is at maximum

Example

A firm produces three types of products Q_1, Q_2, and Q_3. And a profit function is given by:

$\Pi = F(Q_1, Q_2, Q_3) = 180Q_1 + 200 Q_2 + 150Q_3 - 3Q_1Q_2 - 2Q_2Q_3 - 2Q_1Q_3 - 4Q_1^2 - 5Q_2^2 - 4Q_3^2$.

Try to optimize the function and find the critical values which make the function at maximum or minimum point and also the maximum or minimum value. Identify the type of definiteness.

Solution

1st – find FOC

$\Pi_1 = 180 - 3Q_2 - 2Q_3 - 8Q_1 = 0 \quad \Rightarrow 8Q_1 - 3Q_2 - 2Q_3 = -180$

$\Pi_2 = 200 - 3Q_1 - 2Q_3 - 10Q_2 = 0 \quad \Rightarrow 3Q_1 - 10Q_2 - 2Q_3 = -200$

$\Pi_3 = 150 - 2Q_2 - 2Q_1 - 8Q_3 = 0 \quad \Rightarrow 2Q_1 - 2Q_2 - 8Q_3 = -150$

2nd– Form a coefficient matrix, A

$$A = \begin{bmatrix} -8 & -3 & -2 \\ -3 & -10 & -2 \\ -2 & -2 & -8 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} -180 \\ -200 \\ -150 \end{bmatrix}$$

Here if is difficult to use simultaneous method to solve the critical values. Thus, instead we use Cramer's rule.

3rd _ find the determinant of coefficient matrix, A

$$A = \begin{bmatrix} -8 & -3 & -2 \\ -3 & -10 & -2 \\ -2 & -2 & -8 \end{bmatrix} \begin{bmatrix} Q1 \\ Q2 \\ Q3 \end{bmatrix} = \begin{bmatrix} -180 \\ -200 \\ -150 \end{bmatrix}$$

$$|A| = -8(80-4) + 3(24-4) - 2(6-20)$$
$$= -8(76) + 60 + 28$$
$$= -608 + 88$$
$$= -520 \neq 0$$

$$Q_1 = \frac{|A_1|}{|A|}, \quad A_1 = \begin{bmatrix} -180 & -3 & -2 \\ -200 & -10 & -2 \\ -150 & -2 & -8 \end{bmatrix}$$

$$|A_1| = -180\ (80-4) + 3(1600 - 300) - 2\ (400 - 1500)$$
$$= -180\ (76) + 3900 + 2200$$
$$= -13,680 + 6100$$
$$= -7580$$

$$So\ \ Q_1\ \frac{|A_1|}{|A|} = -\frac{7580}{-520}\ \ 14.58$$

$$Similarily\ \ ,\ \ Q_2 = \frac{|A_1|}{A},\ \ A_2 \begin{bmatrix} -8 & -180 & -2 \\ -3 & -200 & -2 \\ -2 & -150 & -8 \end{bmatrix}$$

$$|A_2| = -6900$$

$$Q_2 = \frac{A_2}{A} = -\frac{6900}{-520} = 13.27$$

$$Q_3\ \ \frac{A}{A}\ \ ,\ A_3 = \begin{bmatrix} 8 & 3 & 180 \\ 3 & 10 & 200 \\ 2 & 2 & 150 \end{bmatrix}$$

$$|A_3| = -6130$$

$$Q_2\ \ \frac{|A_2|}{|A|} = -\frac{6900}{-520} = 13.27$$

$$Q_3 = \frac{/\ A_3\ /}{|A|}\ ,\ A_3 = \begin{bmatrix} 8 & 3 & 180 \\ 3 & 10 & 200 \\ 2 & 2 & 150 \end{bmatrix}$$

$$|A_3| = -6130$$

$$Q_2 = \frac{A_3}{A} = -\frac{-6130}{-520} = 11.79$$

Therefore the optimal values that optimize the profit function are
$Q_1 = 14.58$, $Q_2 = 13, 27$ and $Q_3 = 11.79$.

4^{th} – S, O, C – to check the profit is really maximized or not

$\pi_{11} -8 \quad \pi_{12} -3$, $\pi_{13} = -2$

$\pi_{21} -3 \quad \pi_{22} = -10, \quad \pi_{23} = -2$

$\pi_{31} = -2, \ \pi_{32} = -2$, $\pi_{33} = -8$

5^{th} -Form Hessian matrix

$$|H_1| = \begin{vmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{vmatrix}$$

$$= \begin{vmatrix} -8 & -3 & -2 \\ -3 & -10 & -2 \\ -2 & -2 & -8 \end{vmatrix}$$

$$|H_1| = |-8| = -8 < 0$$

$$|H_2| = \begin{vmatrix} -8 & -3 \\ -3 & -10 \end{vmatrix} = 71 > 0$$

$$|H_3| = |H| = \begin{vmatrix} -8 & -3 & -2 \\ -3 & 10 & -2 \\ -2 & -2 & -8 \end{vmatrix}$$

$$|H_3| = |H| = |A| = -520 < 0$$

Since $/H_1/ < 0$, $/H_2/ > 0$ and $/H_3/ < 0$, we can conclude that profit is at maximum when evaluated at optimal values & $Q_1 = 14.58$ $Q_2 = 13.27$ and $Q_3 = 11.79$ and implies negative definiteness.

The maximum profit is obtained by substituting the optimal values in the original profit function (π)

Π =180(14.58)+200(13.27)+150(11.79)-3(14.58x13.27)-2(13.27x11.79) 2(14.58x11.79)-4(14.58)^2 -5 (13.27)^2 -4(11.79)^2

=2624.4+2654+1768.5-580.43-312.91-343.80-850.31-880.46-556.02

=7046.90-3523.93

=3522.97

Self-Test Exercise 4-2

1. For each of the following functions find the stationary pint (critical values) and determine whether the function is at a relatively maximum, maximum, inflection or saddle point
   a. Z = f (x, y) = $3x^2 + 2y^2-xy-4x-7y +12$
   b. Z = f(x, y) =$60x+34y-4xy-6x^2-3y^2+5$
2. A firm produces two products x and y. The cost of producing products x and y is C (x, y) = $0.04x^2+0.01xy+0.01y^2+4x+2y+500$.Suppose that the firm sells all its out puts at a price unit of birr 15 for x and 9 for y.

   Optimize the profit function.
3. Suppose a monopolist is practicing price discrimination in the selling of product Q by charging different prices in the two separate markets. The demand functions in the two markets are given by:

   $P_1=100-Q_1$ & $P_2= 80-Q_2$. Suppose the cost function is C=6Q, $Q=Q_1+Q_2$

   a) i) How much should be sold in the two markets to maximize profit?
      ii) What are the prices charged?
      iii) How much is the maximize profit
   b) If price discrimination is made illegal and if the firm is not practicing it.
      i. Calculate optimal output levels & corresponding price.
      ii. How much is the difference between the discrimination and non – discrimination profit? (Hint: if illegal, there is only one price (P) charged in each markets; hence, $P_1=P_2=P$, so TR=P(Q1+Q2)) .

Model Examination

1. For each of the following functions find the stationary pint (critical values) and determine whether the function is at a relatively maximum, maximum, infliction or saddle point

a) $Z = f(x, y) = 8x - 2x^2 + 12xy - 12y^2 + 6y$

b) $Z = f(x, y) = 5x^2 - 30 + 4xy - 3y^2 + 7y$

2. Consider a monopolist firm that produces two products Q_1 and Q_2. Suppose that the demands facing the firm are as follows.

$P_1 = 55 - Q_1 - Q_2$

$P_2 = 70 - Q_1 - 2Q_2$

More over the total cost function is given as $c = Q_1^2 + Q_1Q_2 + Q_2$

A. Determine the optimal levels of outputs, prices and profit by optimizing the profit function (Hint $\pi = TR_1 + TR_2 - TC$)

B. Check whether the point is relative maximum or minimum using Hessian approach Determine whether π function is positive or negative definite.

3. Suppose a monopolist firm sells a certain product in three separate markets. The demand function facing the firm is given as follows.

$P_1 = 63 - 4Q_1$, $P_2 = 150 - 5Q_2$, $P_3 = 75 - 6Q3$

The total cost function is $C = 20 + 15Q$, $Q = Q_1 + Q_2 + Q_3$

Determine

i) optimal quantities

ii) prices charged is each markets

iii) optimum profit

iv) check the point by using Hessian approach (Hint: $\pi = TR1 + TR2 + TR3 - TC$)

4.3. Implicit functions and unconstrained Envelope theory

4.3.1. Implicit functions

Consider two choice variables optimization model: $Y = f(x_1,x_2)$ – explicitly shows that the optimization of Y depends only on x_1 and x_2. However, in reality optimization of Y is affected by some implicit parameters and it is possible to optimize Y by assuming that these parameters are held constant. Optimization problems in economics usually involve functions that depend on a number of parameters like prices, tax rates, etc. Although these parameters are held constant during optimization they do vary according to the economic situations.

Let's consider the effect of change in one parameter on the optimal solution of a given equation $Y = f(x_1\ x_2, \alpha)$ where $x_1\ x_2$ are choice variables and α is the parameter.

The FOC ($f_1 = 0$, $f_2 = 0$) yields critical values for x_1 and x_2 which result in the stationary value of y. But this is possible if and only if x is fixed. Thus, the critical values of the choice variables are functions of α are $x_1^* = f(\alpha)$ and $x_2^* = f(\alpha)$. i.e. if we change the value of α we will find different optimal values for x_1 and x_2 .So the objective function is also a function of parameter.

Thus, $л(x) = f(x_1^*(\alpha), x_2^*(\alpha), \alpha)$ be the indirect objective function. This function shows the effect of α on the objective function Y. For instance, if α changes to αo, we may have different optimal values $x_1 = x_1^0$ and $x_2 = x_2^0$ if x changes. And we maximize f(direct objective function) using x_1^0 , x_2^0 and α_0.

Similarly, if α changes from α_0 to α_1, we will have another new equilibrium(Optimal) values for x_1 and x_2 , $x_1 = x_1^1$ and $x_2 = x_2^1$. The $л(\alpha)$ is a function which will be tangent to $f(x_1,x_2, \alpha)$ when x_1 and x_2 are used optimally for a given α. For a given α if f is not optimized, the tangency will not occur. Thus, we have a single point where f is equal to л and; hence, the inequality $f(x) <= л(\alpha)$ .This implies $л(\alpha)$ is an envelope of the function f(x) at different values of α.

4.3.2. Envelop Theorem

This theorem is used to determine the change in the optimum value of an optimum function due to change in the value of a parameter.

- Let $Z = f(x, y, \alpha)$ be an objective function with a parameter d. The Foc for maxima/ minima are $f_x = f_y = 0$
- Then, the stationary values are : $X^* = x(a)$ *and* $y^* = y(a)$ and the optimum value of the objective function: $Z^* = f(x(a), y(a), a)$
- Taking total derivative of Z* with respect a, we get

$$\frac{dz^*}{da} = f_x \cdot \frac{d_x}{d_a} + f_y \cdot \frac{d_y}{d_a} + f_a \quad \text{, but} \quad f_x = f_y = 0 \text{ at optimum}$$

$$\frac{dz^*}{d_a} = f_a$$

- Therefore: - The rate of change of the optimum value of Z* as d changes is equal to the partial derivative of the objective function with respect d.
  - The above result is known as the envelop theorem.

Example1: Determine the effect of an increase in the value of α on the optimum value of the function $Y = 2x^2 - d_x + a^2$

Foc:

$$\frac{d_y}{d_x} = 4x - \alpha = 0 \quad , \quad x^* = a/4$$

$$\frac{d_y}{d_x^2} = 4 > 0 \text{ ....} \quad \text{y is minimum when } x = a/4$$

$$y^* = 2(a/4)^2 - a(a/4) + a^2 = 2.\frac{a^2}{16} - \frac{a^2}{4} + a^2 = \frac{7a^2}{8}$$

$$\frac{d_y^*}{d_a} = \frac{d_y^*}{d_a} = \underline{\underline{\frac{7_a}{4}}}$$

Or $\frac{d_y^*}{d_a} = \frac{d_y}{d_a} / x = x(a) = -x + 2a = \frac{-a}{4} + 2a = \underline{\underline{\frac{7a}{4}}}$

$\therefore$ Y * increase by $7a/4$ units as α increases.

Example 2: A firm producing a certain product Q wants to maximize its profit. Suppose a tax, t birr per unit is imposed on the function of α.

A. optimize the profit function

B. Find the effect of the change in t on л

Solution

A. We know that Π = TR –TC

Π = TR –TC, TR = R(Q), C = C(Q) +tQ

= R(Q) - C(Q) – tQ = O

FOC

Π' (Q)=R'(Q)-C(Q)-t=0

= MR – MC – t = 0

t= MR – MC

SOC

Π" (Q) = R"(Q) –C" (Q)< 0

= R" (Q) < C" (Q)

= slope of MR < Sope of MC

B. Π *= R(Q*) – C(Q*) – tQ*

$$\frac{d\pi^*}{dt} R'(Q^*)\frac{dQ^*}{dt} - C'(Q^*)\frac{dQ^*}{dt} - Q^* - t\frac{dQ^*}{dt}$$

$$(R'(Q^*) - C'(Q^*))dQ^*/dt - Q^* - tdQ^*/dt$$

$$= t\frac{\delta Q^*}{\delta t} - t\frac{\delta Q^*}{\delta t}$$

$$= -Q^*$$

This shows that a unit change in tax will change profit by negative Q*

Model Examination

A company manufactures and sells X television sets per month the monthly cost and demand equations are: C(x) = 72, 000 + 60x , P = 200 – x/30 $0 \le x \le 6{,}000$

a. Find the maximum revenue

b. Find the maximum profit, the production level that will realize the Maximum profit .And the price the company should change for Each television set

c. If the government decides to tax company $5 for each set if produces, how many sets should the company manufacture each month in order to maximize its profit?

d. What should the company change for each set?

CHAPTER FIVE

CONSTRAINED OPTIMIZATION

Chapter contents

Dear students, this chapter consists of the following major sub-topics;-

- Constrained optimization with equality constraint
- Functions of Two Variables and Equality Constraints
- Techniques of Constrained Optimization
- Sufficient Conditions for optimization
- The concepts of Border Hessian and Jacobian Determinants
- Constrained Optimization and Envelope theorem
- Constrained Optimization with Inequality Constraints
- The Kuhn – Tucker Condition

Learning Objectives:

After successful completion of this section, you will be able:

- describe constrained optimization of a function and understand its difference from unconstrained optimization.
- examine optimization of a function with equality constraint
- carry out optimization of a function with equality constraint by using the various techniques of constrained optimization.
- apply sufficient conditions for optimization using Border Hessian and Jacobian determinants.
- realize the basic concepts of Envelope theorem and its application in constrained optimization.
- differentiate the application of Inequality Constraints from equality Constraints.
- know the basic concepts of the Kuhn – Tucker Condition and its application in constrained optimization.

Introduction

Now we shall consider another category of the optimization problem where the function to be maximized or minimized is not strictly independent in the manner cited in previous chapter, rather it is subject to some technical constraints. To be precise, we shall discuss the problem of extrema of a function under certain constraint(s) on its independent variable(s). The function to be maximized or minimized will be termed as the *objective function.* Such types of problems are, very common in economic analysis, for example, maximization of utility subject to budget constraint, minimization of cost subject to output constraint etc.

This section further extends your previous study on optimization of a function with two or more variables. You will study how to find the maximum and minimum values of a function with a restriction using different methods.

5.1. Constrained optimization with equality constraint

Functions of Two Variables and Equality Constraints

Do you have any idea what is meant by optimization of a function with equality constraint? You can use the space below to write your answer._____

_____

_____

In the previous discussions on optimization, all the explanatory variables were considered to be independent of each other. For instance, consider a farmer producing wheat and sorghum. The farmer can choose and produce any quantity of wheat and sorghum without the two choices limiting each other. That is, the two choices are independent of each other. However, the farmer may face a production quota such that his total production of both wheat and sorghum cannot exceed a specified quantity.

In case, the choice of quantities for both outputs is no longer independent. Therefore, finding the optimum production quantities is a constrained optimization.

How can you find the extremum for a function involving an equality constraint? The discussion that follows illustrates the techniques that allow you to find a constrained extremum.

5.1.1 Techniques of Constrained Optimization

The Techniques of constrained optimization presented in this chapter are built on the method for identifying the stationary point in unconstrained optimization.

a) Substitution Method

One method in constrained optimization is to substitute equality constraints in to an objective function. This is to convert a constrained optimization in to unconstrained optimization problem by internalizing the constraint / constraints directly in to the objective function. The constraint is internalized when we express it as a function of one of the arguments of the objective function and then substituting that argument by using the constraint. We can then solve the internalized objective function by using the unconstrained optimization technique. Such a technique of solving constrained optimization problems is called the ***substitution method.***

In order to develop a procedure for the determination of constrained extrema of a function, we consider the maximization of the utility function, $U = xy$, of a consumer subject to the budget constraint, $400 = 20x + 25y$.

Given the budget constraint $400 = 20x + 25y$, we note that now x and y cannot take values independently of each other. For example, if $x = 5$, then the value of y must be 12 etc.

One way of tackling this problem is to eliminate one of the independent variables, x or y, from the utility function, by the use of given constraint. Then, U can be maximized or minimized in the usual unconstrained manner.

To eliminate y (say) from the utility function, we first solve the constraint for y. on adjusting the terms the constraint can be written as $y = 16 - \frac{4}{5}x$. substituting this value of y in the utility function, $U = xy$ we have $U = x\left(16 - \frac{4}{5}x\right) = 16x - \frac{4}{5}x^2$

Thus, $\frac{dU}{dx} = 16 - \frac{8}{5}x = 0$ (for maxima) $\Rightarrow x = 10$. Substituting $x = 10$ in the constraint, the value of $y = 8$. Thus $U = xy$ is maximum when consumer purchases 10 units of x and 8 units of y, under the condition that his income is \$400 and price of x = \$20 and price of y = \$25. Since $\frac{d^2U}{dx^2} < 0$, the second order condition is also satisfied. This technique of constrained optimization is dubbed as substitution method.

Below an attempt will be made to introduce you with the various techniques of constrained optimization, including the substitution method.

Example 1:

Let us consider a consumer with a simple utility function: $U = x_1\, x_2 + 2x_1$

Subject to budget constraint: $4x_1 + 2x_2 = 60$

The constraint can be rewritten as

$$x_2 = 30 - 2x_1$$

Substituting (1) in to an objective function

$U(x_1, x_2) = x_1(30 - 2x_1) + 2x_1 \quad = 32\, x_1 - 2x_1^2 + 2x_1$

Internalize objective function

Optimize the internalized objective function.

First Order Conditions [FOC]:

$\frac{\partial U}{\partial x_1} = 0$ *and* $\frac{\partial U}{\partial x_2} = 0 \Rightarrow 32 - 4x_1 = 0,\ \underline{\underline{x_1^* = 8}}$

Substituting $x_1^* = 8$ in the constraint function we will have,

$$4x_1 + 2x_2 = 60 \Rightarrow 4(8) + 2x_2 = 60$$

$$x_2^* = 14$$

Second Order Condition: $\frac{\partial^2 U}{\partial x_1^2} = -4 < 0$

Thus, $U(8, 14) = 128$ is maximum when a consumer purchases 8 units of x_1 and 14 units of x_2 under the condition that his income is \$60 and price of x_1 and x_2 are \$4 and \$2, respectively.

Example 2: Maximize the function, $f(x_1, x_2) = 2\sqrt{x_1} + \frac{1}{2}\sqrt{x_2}$

subject to the linear constraint $4x_1 + x_2 = 20$

Solution

The constraint can be rewritten as $x_2 = 20 - 4x$

Internalizing the above function,

$$f(x_1) = 2\sqrt{x_1} + \frac{1}{2}\sqrt{20 - 4x_1} \quad \text{…. Internalized objective function}$$

Solve for the internalized objective function. Proceed through the first and second order conditions.

First order condition:

$$f'(x_1) = \frac{1}{\sqrt{x_1}} - \frac{1}{\sqrt{20 - 4x_1}} = 0$$ Solving for x_1 yields $x_1^* = 4$

Substituting $x_1^* = 4$ in to the constraint function: $4(4) + x_1^* = 20$

$$\underline{\underline{x_2^* = 4}}$$

Second order condition:

$$f''(x_1) = \frac{-1}{2} x_1^{-3/2} - 2(20 - 4x_1)^{-3/2}$$

Evaluating $f''(x_1)$ at $x_1 = 4$ we will have the following.

$f''(4) = -5/16 < 0$. Thus, the maximum value for the function $f(x, y)$ subject to the constraint $4x_1 + x_2 = 20$ *is* $2\sqrt{4} + \frac{1}{2}\sqrt{4} = \underline{\underline{5}}$

Activity: Answer the following questions

Find the extreme values of the following functions using substitution method

a) $z = xy + 4x$ subject to the constraint $x + y = 8$.

b) $z = xy$ subject to the constraint $x + 2y = 2$

Check Your Progress

a) From the constraint equation $y = 8 - x$

Substituting this expression into the objective function: $z = x(8 - x) + 4x$

Taking the first derivative and setting it to zero, you get $\frac{dz}{dx} = -2x + 12 = 0$. Thus, $x = 6$

Substituting $x = 6$ into the expression $y = 8 - x$ gives $y = 2$. The value of the objective function at the extreme value (that is $x = 6$ and $y = 8 - x$) is $z = 36$. Since the second derivative is $\frac{d^2z}{dx^2} = -2$, the extreme value is a maximum.

b) Solving the constraint equation for x, you have, $x = 2 - 2y$

Substitute the above expression into the objective function and you get, $z = (2 - 2y)y$

Obtain the first derivative of this function and set it to zero to get the stationary value

$$\frac{dz}{dy} = -4y + 2 = 0$$

Thus, $y = \frac{1}{2}$. Substituting $y = \frac{1}{2}$ into $x = 2 - 2y$ and solving it, you get $x = 1$.

Substituting the values $x = 1$ and $y = \frac{1}{2}$ (i.e., the extreme value) into the objective function, you obtain the value of the function at the extreme values; that is $z = \frac{1}{2}$. Taking the second derivative of the objective function, you get $\frac{d^2z}{dy^2} = -4$ and hence, the stationary value is a maximum.

B. The Lagrange Multiplier Method

When the constraint is a complicated function or when there are several constraints under consideration, the substitution method could become very cumbersome. This has led to the development of another simpler method of finding extrema of a function, the lagrangean method. This method involves forming a lagrangean function that includes the objective function, the constraint function and the variable, λ, called the Lagrange multiplier. The essence of this method is to convert a constrained extremum problem in to a form such that the first order conditions of the unconstrained optimization problem can still be applied.

We may note here that the necessary condition, obtained above under the substitution method, can also be obtained from an auxiliary function, to be termed as Lagrange Function. This function is formed by the used of objective function and the constraints. The Lagrange function corresponding to the objective function $Z = f(x, y)$ and the constraint $g(x, y) = 0$, can be written as: $L = f(x, y) + \lambda g(x, y)$, *where* λ is an undetermined multiplier known as *Lagrange Multiplier*. We note that, since $g(x, y) = 0$, the value of L at a point is same as the value of $Z = f(x, y)$ at that point. Thus, the exterema of $Z = f(x, y)$ and V occurs at the same point. The necessary conditions [first order conditions] for exterma of L are:

$$\frac{\partial L}{\partial x} = f_x + \lambda g_x = 0 \quad \ldots\ldots\ldots\

The simultaneous solutions of (1), (2) and (3) gives the stationary (or critical or equilibrium) point. Note that the stationary point obtained above will satisfy equation (3), i.e. the constraint. Thus, the unconstrained extrema of L is equivalent to the extrema of $z = f(x, y)$ subject to the constraint $g(x, y) = 0$.

On eliminating λ from (1) and (2), we get

$$\frac{f_x}{g_x} = \frac{f_y}{g_y} \quad \ldots\ldots\ldots\

Note: This total differential approach yields the same result (first order condition) as of the lagrangian method.

$$L(x, y, \lambda) = f(x, y) + \lambda (c - g(x, y))$$

$$\frac{\partial L}{\partial x} = f_x - g_x \lambda = 0 \quad , \quad f_x = g_x \lambda \quad , \quad \lambda = \frac{f_x}{g_x}$$

$$\frac{\partial L}{\partial y} = f_y - g_y \lambda = 0 \quad , \quad \lambda = \frac{f_y}{g_y}$$

$$\therefore \lambda = \boxed{\frac{f_x}{g_x} = \frac{f_y}{g_y}}$$

What is the difference?

The total differential approach yields only the value of $\overline{X}$ *and* $\overline{Y}$, the Lagrange method also gives $\overline{\lambda}$. *as* $\overline{\lambda}$ provides a measure of the sensitivity of Z to the shift of the constraint, the Lagrange method other an advantage of certain built – in comparative static information of the solution.

Interpretation of the Lagrange Multiplier

It is a measure of the effect of a change in the constraint via the parameter "C" on the optimal value the objective function.

We can demonstrate it by using the chain rule.

Consider the optimal values of the arguments of function f (x , y) are $X^*(C)$ *and* $Y^*(C)$

The "C" in the bracket reflects the fact that the optimal values are functions of the constraint.

$$g(X^*(C), Y^*(C)) = C$$

Taking differential of each side with respect to "C".

$$\frac{\partial g(x^*(c) . Y^*(c))}{\partial x} . dx^*(c) + \frac{\partial g(x^*(c) . Y^*(c))}{\partial y} d y^*(c) = dc$$

$$= \frac{\partial g\left(x^*(c).Y^*(c)\right)}{\partial x} \cdot \frac{dx^*(c)}{dc} + \frac{\partial g\left(x^*(c).Y^*(c)\right)}{\partial y} \cdot \frac{dy^*(c)}{dc} = \frac{dc}{dc}$$

Also, totally differentiate the objective function

$$d\ f\left(x^*(c),\ y^*(c)\right) = \frac{\partial f^*(x^*(c), y^*(c))}{\partial x} .d\,x^*(c) + \frac{\partial f^*(x^*(c).y^*(c))}{\partial y} .d\,y^*(c)$$

Equation (1) $= \frac{d\ f\left(x^*(c).y^*(c)\right)}{d\,c} = \frac{\partial f\left(x^*(c).y^*(c)\right)}{\partial x} . \frac{d\,x^*(c)}{d\,c} + \frac{\partial f\left(x^*(c).y^*(c)\right)}{\partial y} . \frac{d\,y^*(c)}{d\,c}$

The Foc require that

$$\frac{\partial f\left(x^*(c).y^*(c)\right)}{\partial x} = \lambda \frac{\partial g\left(x^*(c).y^*(c)\right)}{\partial x}$$

$$\frac{\partial f\left(x^*(c).y^*(c)\right)}{\partial y} = \lambda \partial \frac{g\left(x^*(c).y^*(c)\right)}{\partial y}$$

$\Rightarrow$ Substituting the Foc in to equation (1)

$$\frac{d\ f\left(x^*(c).y^*(c)\right)}{d\,c} = \lambda \left[\frac{\partial g\left(x^*(c).y^*(c)\right)}{\partial x} . \frac{d\,x^*(c)}{d\,c} + \partial \frac{g\left(y^*(c).x^*(c)\right)}{\partial y} . \frac{d\,y^*(c)}{d\,c} \right]$$

But, the term in the bracket is equal to 1. Thus.

$$\boxed{d\ \frac{f\left(x^*(c).\ y^*(c)\right)}{d\,c} = \lambda}$$

i.e the Lagrange multiplier represents the change in the optimum value of the objective function with a small change in the constraint.

Generalization

Langrange Function with Multiple Constraints

We can solve the solution to a constrained optimization problem with more than one constraint by using the lagrangian function with a Lagrange multipliers corresponding to each constraint.

Let an "n" variable function be subject, simultaneously, to the two constraints $g\left(x_1\ ,\ x_2\ ,\ .....x_n\right) = C$ *and* $h\left(x_1\ ,x_2\ ,.....x_n\right) = d$

Then adopting λ and μ as the two un constrained multipliers, we may conduct the

Lagrange function as:

$$L(x_1, x_2, ...x_n, \lambda, \mu) = f(x_1, ...x_n) + \lambda(c - g(x_1, ...x_n) + \mu(d - h)(x_1, ...x_n))$$

Foc : $\frac{\partial L}{\partial \lambda} = C - g(x_1, ... x_n) = 0$

$$\frac{\partial L}{\partial x_i} = f_i - \lambda g_i - \mu h_i = 0 \qquad (i = 1, 2, ...h)$$

$$\frac{\partial L}{\partial \mu} = d - h(x_1, x_2 .....) = 0$$

$\Rightarrow$ The extension of this model to the case of more than two arguments in the objective function is straight for ward. In this the lagrangean function takes the form

$$L(x_1, x_2, ... x_n, \lambda) = f(x_1, x_2, ...x_n) + \lambda(C - g(x_1, x_2 ....x_n))$$

$$\frac{\partial L}{\partial x_i} = 0 \qquad for \ i = 1, 2, ... n$$

$$\frac{\partial L}{\partial \lambda} = C - g(x_1, x_2, ... x_n) = 0 \ , g(x_1, ...x_n) = C$$

The Lagrangian function for a constrained optimization problem with an objective function of "n" variable subject to m equality constraints is written as:-

$$L(x_1, ...x_n) = f(x_1, x_2, ...x_n) + \phi \sum_{i=1}^{m} \lambda_i (c_i - g^i(x_1, ...x_n))$$

Where $g^i(x_1, ...x_n) = c_i$ represent the i[th] constraint.

5.1.2. Sufficient Conditions for optimization

a) Substitution method

Example: Consumption Problem

Suppose you have $ 6.00 to spend on a lunch of soup and salad. A restaurant at which you dine sells both soup and salad by weight. An ounce of soap costs $ 0.25 and an ounce of salad costs $ 0.50. How many ounce of each will you purchase to maximize your satisfaction if the utility function is: $U(x_1, x_2) = \frac{1}{4} \ln x_1 + \frac{1}{2} \ln x_2$

Solution:

The constraint function is given by,

$$\frac{1}{4}x_1 + \frac{1}{2}x_2 = 6 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{ budget constraint.}$$

It shows the different combination of salad and soup (by weight) that costs you just $6. In order to solve the optimization problem through the substitution method, let us rewrite the constraint function as: $x_1 = 24 - 2x_2$

Internalizing this in to the objective function yields:

$$U(x_2) = \frac{1}{4} In(24 - 2x_2) + \frac{1}{2} In\ x_2$$

Solving the internalized objective function, we have

$$U'(x_2) = \frac{-2}{4(24 - 2x_2)} + \frac{1}{2x_2} = 0 \qquad \Rightarrow \underline{\underline{x_2 * = 8}}$$

Substituting $x_2 = 6$ and solving the constraint function for x_2, we have

$$\frac{1}{4}x_1 + \frac{1}{2}x_2 = 6 \quad , \quad \frac{1}{4}x_1 + \frac{1}{2}(8) = 6, \quad \underline{\underline{x_1 * = 8}}$$

Second Order Condition: $U''(x_2) = \frac{-2}{(24 - 2x_2)^2} - \frac{2}{x_2^{\,2}}$

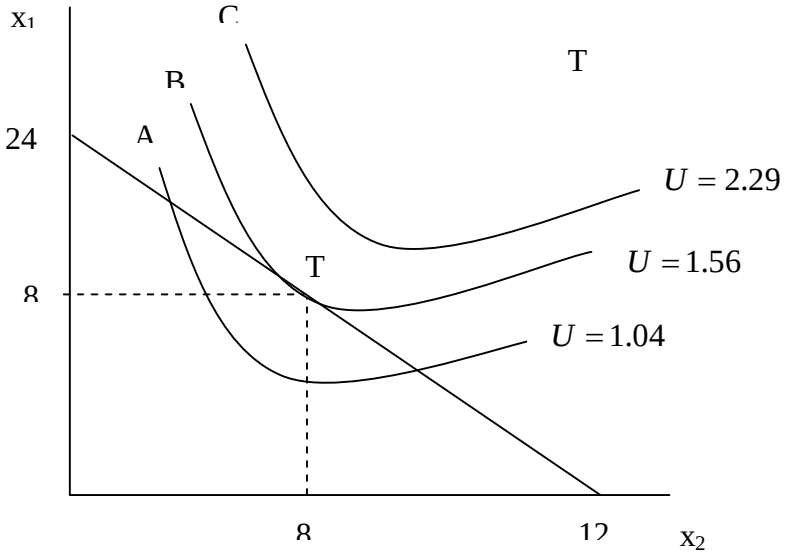
Substituting $x_2 * = 8$ for x_2 in the second order derivative we will get the following.

$$= \frac{-2}{(24 - 2(8))^2} - \frac{2}{64} < 0$$

Therefore, the optimal amount of salad (x_1) and soup (x_2) to be purchased is 8 and 8 ounces, and the maximum satisfaction from consuming these is:

$$U(8, 8) = \frac{1}{4} In\ 8 + \frac{1}{2} In\ 8 = \underline{\underline{1.56}}$$

Graphically:-



There are 3 indifference curves consistent with the utility function:

i. A: Shows all combination of x_1 and x_2 that provide a level of utility equal o 1.04.
   - All combinations are attainable but sub-optimal at which the consumer will not maximize his utility.

ii. C: Shows all combination of x_1 and x_2 that provide a level of utility equal to 2.29.
   - Given the budget constraint, all combinations are not affordable.

iii. B: All combinations except "T" is not affordable under the budget constraint.
   - Point "T" is not only attainable but also an optimal combination.

Proof

We know that, the consumer will maximum his utility at a point where the budget line is tangent to the highest possible indifference curve, where

$$\frac{-P_1}{P_2} = \frac{\Delta x_1}{\Delta x_2} \Rightarrow \frac{-P_1}{P_2} = \frac{MU x_2}{MU x_1}$$

$$\frac{-P_1}{P_2} = \frac{-1}{2}, \quad MUx_1 = \frac{1}{4x_1} \quad and \quad MU x_2 = \frac{1}{2x_2}$$

$$\frac{MU x_2}{MU x_1} = \frac{1/2x_1}{1/4x_2} \frac{2x_2}{4x_1} \Rightarrow \frac{x_2}{2x_1}$$

Then $\frac{-1}{2} = \frac{x_2}{2x_1}$ at $Po\ \text{int}$ T $(x_1 = 12\ , x_2 = 6)$

$\left|-\frac{1}{2}\right| = \frac{1}{2}$

Or at the optimal point, any decrease in utility due to a small reduction in the consumption of one good would be just matched by the increase in utility due to an increase in constraint of the other good. If salad constraint increase by d x_2, then the utility decrease by

$MU\ x_2 . \Delta x_2 = d x_2 . \frac{1}{2x_2} = \frac{1}{2}\ d x_2\ at\ x_2 = 6$

When evaluated at optimal point where MRS [the marginal rate of substitution] is -2, the reduction in salad allows the constraint of soup to increase by $\frac{1}{2}\ d\ x_1$ because

$\frac{d x_1}{d x_2} = -2$. And, utility rise by $MU\ x_1\ .\ \frac{d x_1}{2} = MU\ x_1\ .\ 2 d x_2$

$= \frac{1}{2x_1}\ .\ 2 d\ x_2 = \frac{1}{12}\ d\ x_2\ \ at\ \ x_1^* = 12$

From the first order conditions, after some rearrangement we can have,

$$\therefore\ MU\ x_2\ .\ \Delta x_2 = MU\ x_1\ .\ \Delta x_1\ \ i.e., du = 0$$

We may note here that when the constraint is a complicated function, or when there are several equality constraints to be considered, the substitution method might be difficult or simply impossible to carry out in practice. In that case, you need to use other techniques. This has led to the development of alternative methods of finding extrema of a function under constraints. One such method is the method of the ***Lagrange Multiplier*** which will be discussed in the following sub section.

b) Second Order Total Differential

As we have been discussed, Lagrangian method lends the concept of differentials:

Taking differential of $Z = f(x, y)$, we have

$$dZ = f_x dx + f_y dy$$

Further, $d^2 Z = \frac{\partial (dZ)}{\partial x}\ dx + \frac{\partial (dZ)}{\partial y}\ dy$

$$= \left[f_{xx}\ dx + f_{xy}\ dy + f_y \frac{\partial (dy)}{\partial x} \right] dx + \left[f_{xy}\ dx + f_{yy}\ dy + f_y \frac{\partial (dy)}{\partial y} \right] dy$$

Here, the constraint g (x , y) = c

$$dg = g_x\,dx + g_y\,dy = 0 \quad / \qquad dy = -\left(g_x / g_y\right)\,dx \;\ldots\ldots\ldots\ldots (1)$$

In this case, both dx and dy are no longer arbitrary.

We may take dx as an arbitrary change but dy must be regarded as dependent on dx (equation (1)).

Once dx is specified, dy will depend on gx and gy but the latter derivatives in turn depend on x and y. hence, dy will also depend on x & y.

To find an appropriate new expression for d^2 z :

$$Z = f(x, y) \;/\; dz = f_x\,dx + f_y\,dy$$

$$d^2 z = \frac{\partial}{\partial x}(dz)\,dx + \frac{\partial}{\partial y}(dz)\,dy$$

$$= \frac{\partial}{\partial x}\left(f_x\,dx + f_y\,dy\right)dx + \frac{\partial}{\partial y}\left(f_x\,dx + f_y\,dy\right)dy$$

$$= \left(f_{xx}\,dx + \left[dy\,\frac{\partial}{\partial x}\left(f_y\right) + \left(f_y \,.\frac{\partial}{\partial x}\,dy\right)\right]dx + \left(f_{xy}\,dx + dy\,.\frac{\partial}{\partial y}\right)f_y + f_y\frac{\partial}{\partial y}\,dy\right)$$

$$d^2 z = \left[f_{xx}\,f_x + \left(f_{xy}\,dy + f_y.\frac{\partial\,dy}{\partial x}\right)\right]dx + \left[f_{yx}\,dx + \left(f_{y_y}\,dy + f_y.\frac{\partial\,dy}{\partial y}\right)\right]dy$$

$$= f_{xx}\,dx^2 + 2\,f_{xy}\,dx\,dy + f_{y_y}\,dy^2 + f_y\left[\frac{\partial(dy)}{\partial x}\,dx + \frac{\partial(dy)}{\partial y}\,dy\right]$$

But, $\dfrac{\partial(dy)}{\partial x}\,dx + \dfrac{\partial(dy)}{\partial y}\,dy = d\,(dy)$

Then, $d^2 Z = f_{xx}\,dx^2 + 2\,f_{xy}\,dx\,dy + f_{yy}\,dy^2 + f_y\,d\,(dy)$

$= f_{xx}\,dx^2 + 2\,f_{xy}\,dx\,dy + f_{yy}\,dy^2 + f_y\,d^2 y \;\ldots\ldots\ldots\ldots (2)$

Here: $d^2 y \neq dy^2$, thus the presence of d^2y in the above equation d is qualifies d^2 z as a quadratic form. However, d^2 z can be transformed in to a quadratic form by virtue of the constraint g (x , y) = c.

$$d^2 g = g_{xx}\,dx^2 + 2\,g_{xy}\,dx\,dy + g_{yy}\,dy^2 + g_y\,dy^2 = 0$$

$$dy^2 = \left[-\frac{g_{xx}\,dx^2 + 2g_{xy}\,dx\,dy + g_{y_y}\,dy^2}{g_y}\right] \;\ldots\ldots\ldots\ldots\ldots (3)$$

Substituting equation (2) in to equation (3):-

$$d^2z = f_{xx}\,dx^2 + 2\,f_{xy}\,dx\,dy + f_{yy}\,dy^2 - \frac{g_{xx}}{g_y}\,dx^2 - 2\frac{g_{xy}}{g_y}\,.f_y - dxdy - .f_y\frac{g_{yy}}{g_y}\,dy^2$$

$$d^2z = \left(f_{xx} - \frac{f_y}{g_y}\,f_{xx}\right)dx^2 + 2\left(f_{xy} - \frac{f_y}{g_y}\,g_{xy}\right)dx\,dy + \left(f_{yy} - \frac{f_y}{g_y}\,g_{yy}\right)dy^2$$

But, $\lambda = f_y / g_y$

And $\dfrac{d^2z}{dx^2} = Z_{xx} = f_{xx} - \lambda g_{xx}$ / $\dfrac{d^2z}{dxdy} = Z_{xy} = f_{xy} - \lambda\, g_{xy}$

$$\frac{d^2z}{dy^2} = Z_{yy} = f_{yy} - \lambda\, g_{yy}$$

$$\boxed{d^2z = Z_{xx}\,dx^2 + Z_{xy}\,dx\,dy + Z_{yx}\,dy\,dx + Z_{yy}\,dy^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

(Here we have taken dx as an arbitrary increment and y as a function of x because of the constraint, $g(x,y) = 0$.

$$d^2Z = f_{xx}\,dx^2 + 2\,f_{xy}\,dxdy + f_{yy}\,dy^2 + f_y\left[\frac{\partial(dy)}{\partial x}\,dx + \frac{\partial(dy)}{\partial y}\,dy\right]$$

$$= f_{xx}\,dx^2 + 2\,f_{xy}\,dx\,dy + f_{yy}\,dy^2 + f_y\,d(dy)$$

$$= f_{xx}\,dx^2 + 2\,f_{xy}\,dx\,dy + f_{yy}\,dy^2 + f_y\,d^2y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

We note that d^2y is different from dy^2. dy^2 can be eliminated from the above equation with the help of second order differential of the constraint as shown below.

$$dg = g_x dx + g_y dy$$

Following the procedure used to obtain equation (5), we can write

$$d^2g = g_{xx}\,dx^2 + 2g_{xy}\,dx\,dy + g_{yy}\,dy^2 + g_y\,d^2y = 0$$

$$\therefore \quad d^2y = -\left[\frac{g_{xx}\,dx^2 + 2g_{xy}\,dx\,dy + g_{yy}\,dy^2}{g_y}\right]$$

Substituting this value in (5), we have

$$d^2z = \left(f_{xx} - \frac{f_y}{g_y}\,g_{xx}\right)dx^2 + 2\left(f_{xy} - \frac{f_y}{g_y}\,g_{xy}\right)dx\,dy + \left(f_{yy} - \frac{f_y}{g_y}\,g_{yy}\right)dy^2 \ldots\ldots(6)$$

From equation (2) we have, $\lambda = -\dfrac{f_y}{g_y}$

$$\therefore\ d^2z = \left(f_{xx} + \lambda g_{xx}\right)dx^2 + 2\left(f_{xy} + \lambda g_{xy}\right)dx\,dy + \left(f_{yy} + \lambda g_{yy}\right)dy^2$$

C) The Second-Order Condition for Lagrangian function

The discussion of the Lagrange multiplier method has focused on necessary conditions for identifying an extreme point. But, the necessary conditions do not distinguish between maxima and minima.

In this section, we present sufficient conditions for determining whether an optimum set of values represent a maximum / minimum of the objective function in the context of the *Lagrange multiplier method*.

The sufficient conditions for casing the stationary point of a multivariate function in the free-extremum case is based on whether the Hessian matrix of the function is positive definite or negative definite when evaluated at the stationary point.

Similarly, the sufficient condition for a constrained optimization problem depend up on whether the lagrangian function is positive definite or negative definite when evaluated at the stationary point.

But, we must be careful not to apply the soc developed for the unconstrained problem. As we shall be, the new conditions must be stated in terms of the second order total differential, $d^2 z$.

For a constrained extremum of Z = f (x , y) , subject to g (x , y) = c, the second order necessary and sufficient conditions still revolve around the sing of $d^2 z$, evaluated at the stationary point. But, in the present context, we are concerned with the sign definiteness/ semi definiteness of $d^2 z$, not for all possible values of dx and dy but only for those dx and dy values (not both zero) satisfying the linear constraint: $g_x \ dx \ + \ g_y \ d_y \ = 0$

The second-order necessary conditions are:

For maximum of z: d^2z is negative semi definite $\left(d^2 \ z \le 0\right), \ s \ to \ dg = 0$

For minimum of z: d^2z is positive semi definite $\left(d^2 \ z \ge 0\right) \ s \ to \ dg \ = 0$

The second order sufficient condition:

For maximum: - d^2z negative definite s to $dg \ = 0$

For minimum:- $d^2 z$ positive definite $\left(d^2 \ z > 0\right) \ s \ to \ d \ g \ = 0$

5.1.3 The concepts of Border Hessian and Jacobian Determinants

a) The Border Hessian(H)

To determine the conditions under which $d^2 z \geq 0$, we consider the quadratic form:

$$d^2 Z = f_{xx}\ dx^2 + f_{yy}\ dy^2 + 2\ f_{xy}\ dx\ dy \text{ subject to } g_x d_x + g_y d_y = 0$$

Let $a = f_{xx}$ $b = f_{yy}$, $h = f_{xy}$, $U = d_x$ $V = d_y$ $d = g_x$ $\beta = g_y$

$$aU^2 + 2huV + bv^2 \ s\ to\ dU + \beta V = 0 \ / \ V = -\left(d/\beta\right) U$$

$$q = aU^2 + 2h\left(-\alpha/\beta\right) u^2 + b\frac{d^2}{\beta^2}\ u^2 = \frac{U^2}{\beta^2}\left(a\beta^2 - 2hd\beta + bd^2\right)$$

To since $U^2/\beta^2 > 0$ the sign of θ is determined by the sing of $\alpha\beta^2 - 2hd\beta + bd^2$ which can be written in determinant:

$$\begin{vmatrix} 0 & d & \beta \\ \alpha & a & h \\ \beta & h & b \end{vmatrix} = 2h\ d\beta - d\beta^2 - bd^2$$

NB: $-\left(a\beta^2 - 2hd\beta + bd^2\right) = 2hd\beta - d\beta^2 - db^2$

Thus,

$$q \text{ is } \left\{\begin{matrix} - \text{ ve definite} \\ + \text{ ve definite} \end{matrix}\right\} \quad iff \quad \begin{vmatrix} 0 & d & \beta \\ d & a & h \\ \beta & h & b \end{vmatrix} \begin{matrix} < 0 \\ > 0 \end{matrix}$$

$$q = \begin{vmatrix} 0 & g_x & g_y \\ g_x & f_{xx} & f_{xy} \\ g_y & f_{xy} & f_{yy} \end{vmatrix}$$

Since the determinant can be obtained by bordering the determinant of the SOCs, for (the Hessian determinants unconstrained extrema, by a row and a column, it is called ***Border Hessian*** and denoted by $\left|\overline{H}_2\right|$.

Given the Lagrange function $L = f(x,y) + \lambda g(x,y)$ we can write

$$L_x = f_x + \lambda g_x,\ L_{xx} = f_{xx} + \lambda g_{xx},\ L_{xy} = f_{xy} + \lambda g_{xy}$$

$$L_y = f_y + \lambda g_y,\ L_{yy} = f_{yy} + \lambda g_{yy}$$

Thus, $d^2 z = L_{xx}\ dx^2 + 2L_{xy}\ dx\ dy + L_{yy}\ dy^2$

Note that: Lagrangian can be symbolized by L , V or Z

We recall that if $d^2Z < 0$, the stationary point will be a maxima. But, if $d^2Z > 0$, the stationary point will be a minima. To determine the conditions under which $d^2Z > 0$ or $d^2Z < 0$, we consider the quadratic form $Q = au^2 + 2huV + bv^2$ subject to $\alpha u + \beta v = 0.$ substituting the value of $V \left[= -\left(\alpha / \beta\right) u \right]$ from the constraint in Q, we get

$$Q = au^2 - 2h\frac{\alpha}{\beta}u^2 + b\frac{\alpha^2}{\beta^2}u^2 = \left(a\beta^2 - 2h\alpha\beta + b\alpha^2\right)\left(\frac{u^2}{\beta^2}\right).$$

Thus the sign of Q will be same as the sign of $a\beta^2 - 2h\alpha\beta + b\alpha^2$.

We note that $\begin{vmatrix} 0 & \alpha & \beta \\ \alpha & a & h \\ \beta & h & b \end{vmatrix} = 2h\alpha\beta - a\beta^2 - b\alpha^2$

Thus Q is positive if $\begin{vmatrix} 0 & \alpha & \beta \\ \alpha & a & h \\ \beta & h & b \end{vmatrix} < 0$ *and* Q *is negative if* $\begin{vmatrix} 0 & \alpha & \beta \\ \alpha & a & h \\ \beta & h & b \end{vmatrix} > 0.$

Thus, taking $V_{xx} = a, V_{xy} = h, V_{yy} = b, \alpha = g_x, \beta = g_y, u = dx \text{ and} V = dy,$ we conclude that the stationary point corresponds to a maxima (or minima) if the sign of the determinant $\begin{vmatrix} 0 & g_x & g_y \\ g_x & V_{xx} & V_{xy} \\ g_y & V_{xy} & V_{yy} \end{vmatrix}$ is positive (or negative).

The Multivariable Case

When the objective function takes the form

$Z = f\left(x_1, x_2, \ldots\ldots x_n\right)$ *s to* $g\left(x_1, \ldots\ldots x_n\right) = C$

The SOC still hinges on the sign of d^2z. The positive or negative definiteness of d^2 z again involve a bordered Hessian. But, this time this condition must be expressed in terms of the bordered principal minors of the Hessian. The bordered Hessian is given by:

$$\left|\overline{H}_2\right| = \begin{vmatrix} 0 & g_1 & g_2 & . & . & . & . & . & g_n \\ g_1 & z_{11} & z_{12} & . & . & . & . & . & z_{1n} \\ g_2 & z_{21} & z_{22} & . & . & . & . & . & z_{2n} \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ - & - & - & - & - & - & - & - & - \\ g_n & z_{n1} & z_{n2} & . & . & . & . & . & z_{nn} \end{vmatrix}$$

Its successive border principal minors can be defined as:-

$$\left|\overline{H}_2\right| \equiv \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & z_{11} & z_{12} \\ g_2 & z_{21} & z_{22} \end{vmatrix} \qquad \left|\overline{H}_2\right| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & z_{11} & z_{12} & z_{13} \\ g_2 & z_{21} & z_{22} & z_{23} \\ g_3 & z_{31} & z_{32} & z_{33} \end{vmatrix} \qquad etc$$

$$\left|\overline{H}_n\right| = \left|\overline{H}\right|$$

$\left|\overline{H}_2\right|$ – implies the second principal minor of the Hessian.

Remarks: Given the objective function $z = f(x_1, x_2 \ldots\ldots x_n)$ and the constraint $g(x_1, x_2, \ldots\ldots x_n) = 0$, the second order conditions are:

i) If $\left|\overline{H_2}\right|, \left|\overline{H_3}\right|, \ldots\ldots, \left|\overline{H_n}\right| < 0$, then the stationary point is a minima.

ii) If $\left|\overline{H_2}\right| > 0$; $\left|\overline{H_3}\right| < 0$, $\left|\overline{H_4}\right| > 0$, $\ldots\ldots\ldots, (-1)^n \left|\overline{H_n}\right| > 0$, the stationary point is a maxima, where

$$\left|\overline{H_2}\right| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & V_{11} & V_{12} \\ g_2 & V_{21} & V_{22} \end{vmatrix}, \quad \left|\overline{H_3}\right| = \begin{vmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & V_{11} & V_{12} & V_{13} \\ g_2 & V_{21} & V_{22} & V_{23} \\ g_3 & V_{31} & V_{32} & V_{33} \end{vmatrix}, \text{ and so on}$$

$$\left|\overline{H_n}\right| = \begin{vmatrix} 0 & g_1 & g_2 & . & . & . & g_n \\ g_1 & V_{11} & V_{12} & . & . & . & V_{1n} \\ g_2 & V_{21} & V_{22} & . & . & . & V_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ g_n & V_{n1} & V_{n2} & . & . & . & V_{nn} \end{vmatrix}$$

Condition	Maximum	Minimum
Foc	$Z_1 = Z_2 = ... = Z_n = Z\lambda = o$	$Z\lambda = Z_1 = ... Z_n = 0$
Soc	$\left\|\overline{H}_2\right\| > 0 : \left\|\overline{H}_3\right\| < 0 : \left\|\overline{H}_4\right\| > 0 :$	$\left\|\overline{H}_2\right\| = \left\|\underline{H}_3\right\| = ... \left\|\overline{H}_n\right\| < 0$
	..... $(-1)^n \left\|\overline{H}_n\right\| > 0$	

Sufficient condition for a maximum with one constraint:

If the determinant of the border Hessian of that lagrangian function evaluated at the stationary point has the same sign $(-1)^n$ and the largest n -1 leading principal minors alternate in sigh, then d^2 z is negative definite and the stationary point represent maximum.

Sufficient condition for a minimum with one constraint:

If all of the largest n – 1 leading principal minors of its bordered Hessian evaluated at the stationary point are negative, including the determinant of bordered Hessian it self, then d^2z is positive definite on the constraint and the stationary point represents a minimum.

Example 1:

Find the maxima or minima of the function, $z = 5x^2 + 6y^2 - xy$ subject to 24 = x + 2y.

Solution

While forming the Lagrange function, it should be kept in mind that the constraint should be first expressed in implicit form i.e. $g(x, y) = 0$ form. We can write

$$V = 5x^2 + 6y^2 - xy + \lambda(24 - x - 2y)$$

First order conditions:

$$\frac{\partial V}{\partial x} = 10x - y - \lambda = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{\partial V}{\partial y} = 12y - x - 2\lambda = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial V}{\partial \lambda} = 24 - x - 2y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Eliminating λ from (1) and (2): $12y - x = 20x - 2y$ or $14y = 21x$ or $2y = 3x$

Substituting this value of y in equation (3), we get $24 - x - 3x = 0$ or $x = 6$

Also $y = \frac{3}{2}x = \frac{3}{2}(6) = 9$. Thus the stationary point is $(x, y) = (6,9)$.

Second order condition:

$$V_{xx} = \frac{\partial^2 V}{\partial x^2} = 10, \quad V_{xy} = \frac{\partial^2 V}{\partial x \partial y} = -1, \quad V_{yy} = \frac{\partial^2 V}{\partial y^2} = 12$$

Taking differential of the constraint, we can write $0 = dx + 2dy$.

$$\therefore \; g_x = 1 \; and \; g_y = 2$$

Thus, $\left|\overline{H_2}\right| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 10 & -1 \\ 2 & -1 & 12 \end{vmatrix} = -56 < 0$. Thus, the function has minima at (6, 9).

The minimum value of the function [calculated at (6, 9)] is 612.

Example 2:

A firm produces radio sets at two difficult locations. If x_1 and x_2 are the no of radio sets produced in location I and II respectively. The joint cost function of the firm is given by

$$C = 0.1x_1^2 + 0.2x_2^2 + 0.2x_1x_2 + 180x_1 + 60x_2 + 25{,}000$$

If the firm has to supply an order of 1000 radio sets, how many sets should it produce at each location to minimize cost? Find the minimum cost and marginal cost?

Solution:

The constraint: $1000 = x_1 + x_2$

$$L = 0.1x_1^2 + 0.2x_2^2 + 0.2x_1x_2 + 180x_1 + 60x_2 + 25{,}000 + \lambda(1000 - x_1 - x_2)$$

FOC

$$\frac{\partial L}{\partial x_1} = 0.2x_1 + 0.2x_2 + 180 - \lambda = 0$$

$$\lambda = 0.2x_1 + 0.2x_2 + 180 = \lambda \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{\partial L}{\partial x_2} = 0.4x_2 + 0.2x_1 + 60 - \lambda = 0 : \lambda = 0.4x_2 + 0.2x_1 + 60 = \lambda$$

$$0.2x_1x_1 + 0.2x_2 + 180 = 0.4x_2 + 0.2x_1 + 60$$

$$120 = 0.2x_2, \quad x_2^* = {}^{120}\!/_{0.2} = \underline{\underline{600}}$$

$x_1 + x_2 = 1000 \quad , \quad x_1^* = \underline{\underline{400}}$

SOC

$C x_1 = 0.2 x_1 + 0.2 x_2 + 180 \qquad C x_2 = 0.4 x_2 + 0.2 x_1 + 60$

$C x_{11} = 0.2 \qquad\qquad C x_{22} = 0.4$

$C x_{21} = C x_{12} = 0.2$ (Young's Theorem)

$g x_1 = 1 = g x_2$

$$\left|\overline{H}_2\right| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0.2 & 0.2 \\ 1 & 0.2 & 0.4 \end{vmatrix} = -0.2 < 0$$

$\therefore$ The stationary point is a minima.

Activity 1: Find the maxima of $f(x, y, z) = xyz$ subject to $x + 2y + 2z = 60$

And Check your progress below

Check Your Progress

Writing the Lagrange function as

$$V = xyz + \lambda (60 - x - 2y - 2z)$$

First order conditions:

$$\frac{\partial V}{\partial x} = yz - \lambda = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{\partial V}{\partial y} = xz - 2\lambda = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial V}{\partial z} = xy - 2\lambda = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\frac{\partial V}{\partial \lambda} = 60 - x - 2y - 2z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

From (1) and (2) we have $2yz = xz \Rightarrow y = \frac{x}{2}$

From (1) and (3) we have $2yz = xy \Rightarrow z = \frac{x}{2}$

Substituting these values of y and z in (4) we have

$$3x = 60 \Rightarrow x = 20$$

$$\therefore\ y = 10 \ and \ z = 10$$

Second order condition:

We note that

$$g_x = 1\ ,\ g_y = g_z = 2,$$

$$V_{xx} = 0\ ,\ V_{xy} = z = 10,\ V_{xz} = 10,$$

$$V_{yy} = 0\ \ V_{yz} = x = 20,\ V_{zz} = 0.$$

$$\left|\overline{H_2}\right| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & V_{xx} & V_{xy} \\ g_y & V_{xy} & V_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 10 \\ 2 & 10 & 0 \end{vmatrix} = 40 > 0$$

$$\text{And } \left|\overline{H_3}\right| = \begin{vmatrix} 0 & g_x & g_y & g_z \\ g_x & V_{xx} & V_{xy} & V_{xz} \\ g_y & V_{xy} & V_{yy} & V_{yz} \\ g_z & V_{xz} & V_{yz} & V_{zz} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 10 & 10 \\ 2 & 10 & 0 & 20 \\ 2 & 10 & 20 & 0 \end{vmatrix} = -1200 < 0$$

Hence, the stationary point is maxima (see the Remarks given below).

Activity 2:

1. Find the maxima or minima of the function $Z = 5x^2 + 6y^2 - xy$

subject to 24 = x + 2y and Check your progress below

Check Your Progress

$$L = 5x^2 + 6y^2 - xy + \lambda\left(24 - x - 2y\right)$$

$$\frac{\partial L}{\partial x} = 10x - y - \lambda = 0\ /\ \frac{\partial L}{\partial y} = 12y - x - 2\lambda = 0\ /\ \frac{\partial L}{\partial \lambda} = 24 - x - 2y = 0$$

$$\lambda = 10x - y \ \ldots\ldots\ldots (1) \qquad \lambda = 6y - \tfrac{1}{2}x \qquad 24 = x + 2y$$

$10x - y = 6y - \frac{1}{2}x \quad / \quad 20x - 2y = 12y - x \qquad 21x = 14y \quad / \; \underline{\underline{3x = 2y}}$

$24 = x + (3x) \quad , \quad \underline{\underline{x^* = 6}} \quad , \quad 18 = 2y \quad , \; y^* = 9 \quad , \; \lambda^* = 7$

Stationary point : (6 , 9 , 7) = (x* , y* , λ*)

SOC

$f(x,y) = 5x^2 + 6y^2 - xy \;\; / \;\; f_x = 10x - y \;\; , \;\; f_y = 12y - x$

$f_{xx} = 10 \qquad f_{yy} = 12 \qquad f_{xy} = f_{yx} = -1$

$g_x = 1 \qquad g_y = 2$

$$\left|\overline{H}_2\right| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 10 & -1 \\ 2 & -1 & 12 \end{vmatrix} = -56 < 0$$

∴ The function has minima at (6 , 9) and the minimum value is 2 (6 , 9) = 612.

b) Jacobian Determinant (J)

A Jacobian determinant permits testing for functional dependence for both linear and non-linear case. It is composed of all the first order partial derivation of a system equation arranged in ordered sequence.

If the equitation's are linear the Jacobian determinant is same as Hessian determinant. However, for non-linear equations the normal Hessian determinate is not applicable.

Let $Y_1=f_1 (X_1,X_2,X_3)$, $Y_2=f_2(X_1,X_2,X_3)$

Therefore, the Jacobian determinant is for partial derivatives is as follows

$$/J/= \begin{bmatrix} \frac{\partial Y1}{\partial X1} & \frac{\partial Y1}{\partial X2} & \frac{\partial Y1}{\partial X3} \\ \frac{\partial Y2}{\partial X1} & \frac{\partial Y2}{\partial X2} & \frac{\partial Y2}{\partial X3} \\ \frac{\partial Y3}{\partial X1} & \frac{\partial Y3}{\partial X2} & \frac{\partial Y3}{\partial X3} \end{bmatrix}$$

Example1: Linear case

$Y_1=6X_1+4X_2$

$Y_2=4X_1+5X_2$

Solution

The Jacobian determinant form is give as,/J/ $\begin{bmatrix} 6 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} X1 \\ X2 \end{bmatrix}=\begin{bmatrix} Y1 \\ Y2 \end{bmatrix}$

Thus, /J/=/H/=30-16=14≠ 0, which implies there is unique solution for X and Y, so you can find the solution by applying cramer's rule.

Example 2: Non- linear case

$Y_1=4X_1-X_2$

$Y_2=16X_1^2+8X_1X_2+x_2^2$

/J/=$\begin{bmatrix} Y11 & Y12 \\ Y21 & Y22 \end{bmatrix}=\begin{bmatrix} 4 & -1 \\ 32x_1+8x_2 & 8x_1+2x_2 \end{bmatrix}$, implies /J/≠0

Activity: Optimize $Y=3X_1^2-5X_1-X_1X_2+6X_2^2-4X_2+2X_2X_3+4X_3^2+2X_3-3X_1X_3$ using

A. Jacobian test and cramer's rule for the First Order Condition.

B. Hessian test for the Second Order Condition.

Check your progress

FOC

$Y_1=6X_1-5-X_2-3X_3=0 \Longrightarrow 6X_1-X_2-3X_3=5$

$Y_2=-X_1+12X_2-4+2X_3=0 \Longrightarrow X_1+12X_2+2X_3=4$

$Y_3=2X_2+8X_3+2-3X_1=0 \Longrightarrow -3X_1+2X_2+8X_3=-2$

$$\begin{bmatrix} Y1X1 & Y1X2 & Y1X3 \\ Y2X1 & Y2X2 & Y2X3 \\ Y3X1 & Y3X2 & Y3X3 \end{bmatrix}\begin{bmatrix} X1 \\ X2 \\ X3 \end{bmatrix}=\begin{bmatrix} C1 \\ C2 \\ C3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -1 & -3 \\ -1 & 12 & 2 \\ -3 & 2 & 8 \end{bmatrix}\begin{bmatrix} X1 \\ X2 \\ X3 \end{bmatrix}=\begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$$

$/J/=/J_3/=/H_3/=6(96-4)+1(-8+6)-(-2+36)$

=552-104

=448≠0, We have unique solution

$$X1=\frac{/A1/}{/J/}=A1=\begin{bmatrix}5 & -1 & -3\\ 4 & 12 & 2\\ -2 & 2 & 8\end{bmatrix}$$

/A1/=400 , X1=400/448=0.89

$$X2=\frac{/A2/}{/J/}=A2=\begin{bmatrix}6 & 5 & -3\\ -1 & 4 & 2\\ -3 & -2 & 8\end{bmatrix}$$

/A2/=184,X2=184/448=0.41

$$X3=\frac{/A3/}{/J/}=A3=\begin{bmatrix}6 & -1 & 5\\ -1 & 12 & 4\\ -3 & 2 & -2\end{bmatrix}$$

/A3/= 2, X3=2/448= 0.005

B) Hessian determinant test for SOC

$$\begin{bmatrix}Y11 & Y12 & Y13\\ Y21 & Y22 & Y23\\ Y31 & Y32 & Y33\end{bmatrix} \qquad \begin{bmatrix}6 & -1 & 5\\ -1 & 12 & 4\\ -3 & 2 & -2\end{bmatrix}$$

/H1/=6,

$$/H2/=\begin{bmatrix}6 & -1\\ -1 & 12\end{bmatrix}$$

= 71 > 0

$$/H3/=\begin{bmatrix}6 & -1 & -3\\ -1 & 12 & 2\\ -3 & 2 & 8\end{bmatrix}$$

=448 > 0

Since /H1/>0, /H2/>0, and /H3/>0, the function is at its minimum when evaluated at optimal values of X calculated above.

The second order differential (dy^2) is positive definiteness.

5.1.4. Constrained Optimization and Envelope theorem

Let $Z = f(x, y, d)$ be the objective function and $g(x, y, d) = 0$ be the constraint the Lagrange function is $L = f(x, y, a) + \lambda\, g(x, y, a)$

Foc

$$\frac{\partial L}{\partial x} = f_x + \lambda g_x = 0$$

$$\frac{\partial L}{\partial y} = f_y + \lambda g_y = 0$$

$$\frac{\partial L}{\partial \lambda} = g(x, y, a) = 0$$

Solving the equations, we get the stationary values of

$x^* = x(a)$, $y^* = y(a)$ *and* $\lambda^* = \lambda(d)$ and the optimum value

$z^* = f(x(a), y(a), \lambda(a))$

Differentiating with respect d

$$\frac{dz^*}{d_a} = f_x \cdot \frac{d_x}{d_a} + f_y \cdot \frac{d_y}{d_a} + f_a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

At optimum : $g(x(a), y(a), d) = 0$ and differentiating with respect d.

$$g_x \frac{d_x}{d_a} + g_y \frac{d_y}{d_a} + g_a = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Multiplying equation (2) by "λ" and add to equation (1). Then.

$$\frac{dz^*}{d_a} = f_x \cdot \frac{d_x}{d_a} + f_y \cdot \frac{d_y}{d_a} + f_a + \underbrace{\left[\lambda g_x \frac{d_x}{d_a} + \lambda\, g_y \frac{d_y}{d_a} + \lambda\, g_a\right]}_{=0}$$

$$\frac{dz^*}{d_a} = \underbrace{\left(f_x + \lambda\, g_x\right)\frac{d_x}{d_a}}_{=0} + \left(f_y + \lambda g_y\right)\frac{d_y}{d_a} + f_a + \lambda\, g_a$$

$$\frac{dz^*}{d_a} = f_a + \lambda\, g_a = \frac{\partial L}{\partial a}$$

$x = x(a)$

$y = y(a)$

$\lambda = \lambda(a)$

Example:

Let $U = f(x_1, x_2)$ be utility function of a consumer and $M = x_1 P_1 + x_2 P_2$ be his budget constraint, where M, P_1, P_2 are parameters. Determine the rate of change of optimal utility with respect each of the three parameters.

Solution:-

$L = f(x_1, x_2) + \lambda [M - x_1 P_1 - x_2 P_2]$

$\frac{du^*}{dM} = \lambda \Rightarrow U^*$ increase by λ units with increase in M.

$\frac{dU^*}{dP^1} = -\lambda x_1 \Rightarrow U^*$ increase by λx_1 units with increase in P_2

5.2. Constrained Optimization with Inequality Constraints

Sometimes optimization problems may have constraints that take the form of inequalities rather than equalities. Constrains may be more naturally expressed as inequalities rather than equalities. For example, consumption of a good is non – negative.

Maximization problems

Let us maximize $\pi = f(x_1, x_2, ... x_n)$ subject to the constraints $g_i (x_1, x_2, ..., x_n) \leq C_i$, where C_is are constants. This can be rewritten clearly as below.

Maximize $\pi = f(x_1, x_2, ... x_n)$

Subject to $g_1 (x_1, x_2, ...x_n) \leq C_1$

$g_2 (x_1, x_2, ...x_n) \leq C_2$

.

.

.

$g_n (x_1, ...x_n) \leq C_m$

[A case with m constraints, n choice variable non negativity restriction, $x_1, x_2, ...x_n \geq 0$]

Thus, as usual we have 3 ingredients; the objective function, the subject function and the non-negativity constraints.

Dear students! What if you have a constraint in $\geq$ sign?

If we encounter optimization problem with $\geq$ sign, we can simply multiply both sides of it by (-1) and change the sign in to $\leq$ inequality. We then proceed as usual.

5.2.1. The Kuhn – Tucker Condition

The Kuhn Tucker condition, unlike the classical optimization, may not be the necessary condition. If we can make adjustments, it can be both necessary and sufficient conditions. The three conditions to solve inequalities using Kuhn-Tucker approach are;

1. Non- negativity constraint
2. Inequalities constraint
3. Complimentary slackness condition

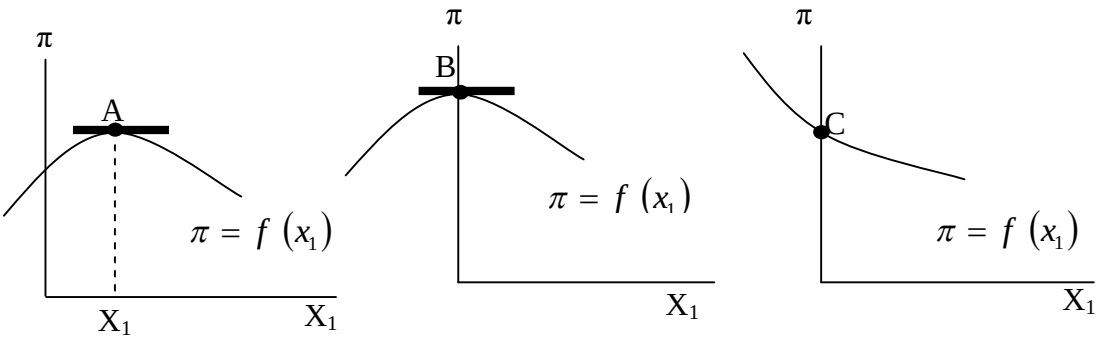
Each step will be discussed below;

1. Effect of Non – negativity Restriction

Assume the objective function with only one variable as

$$Max\ \pi = f(x_1) \text{ Subject to the non – negativity restrictions } x_1 \geq 0$$

There are 3 possibilities for maximum



At point A;
- π is at maximum
- $f'(x_1) = 0$ *and* $x_1 > 0$
- The solution is called interior solution.

At point B;
- π is at maximum
- $f'(x_1) = 0$ *and* $x_1 = 0$
- The solution is called border solution.

At point C;
- π is at maximum
- $f'(x_1) < 0$ *and* $x_1 = 0$
- π has higher values to the left of $x_1 = 0$ but it is out of the domain. Hence, π is at maximum at point C.

Thus from the above discussion, the condition for maximiz

$$f'(x_1) \leq 0 \quad and \quad x_1 \geq 0 \qquad \Rightarrow \qquad x_1 \cdot f(x_1) = 0$$

Generalization:-

$$f_i \leq 0 \quad , \quad x_i \; f_i = 0 \quad where \quad f_i = \frac{\partial \pi}{\partial x_i} \quad i = 1 \; , \ldots . \; n$$

$$\pi = f\left(x_1 \, , \ldots . \, x_n \quad s \;\; to \;\; x_1 \; , \, x_2 \quad \ldots \ldots \quad x_n \geq 0\right)$$

2. Effect of Inequality Constraints:

The maximization problem with inequality constraints and non- negativity restrictions on the choice variables will be clear under.

Let us optimize $\pi = f\left(x_1 \; , \; x_2 \; , \; x_3\right)$

subject to the constraint $g_1 \left(x_1 \; , \; x_2 \; , \; x_3\right) \leq c_1$

$$g_1 \left(x_1 \; , \, x_2 \; , \, x_3\right) \leq c_2$$

$$g_1 \; x_1 \; , \; x_2 \; , \; x_3 \geq 0$$

Step 1: Change the inequality constraint in to equality constraint. To do so, dummy (slack) variables should be introduced to the constraint function. Because, the value in he left hand side is less than the value in the right side.

The appropriate dummy variable for maximization problem is slack variable.

$$g_1 \left(x_1 \; , \; x_2 \; , \; x_3\right) + s_1 = c_1$$

$$g_2 \left(x_1 \; , \; x_2 \; , \; x_3\right) + \; s_2 = c_2$$

Example: Maximize $\pi=5X_1+3X_2$

s.t $6X_1+2X_2\leq 36$

$5X_1+5X_2\leq 40$

$2X_1+4X_2\leq 28$

$X_1, X_2\geq 0$

Change inequality to equality by adding slack variables

$6X_1+2X_2+S1=36$

$5X_1+5X_2+S2=40$

$2X_1+4X_2+S3= 28$

Step -2 write the Lagrangian function:-

$$L = f\left(x_1 \; , x_2 \; , \; x_3\right) + \lambda_1 \; \left(c_1 - g_1 \left(x_1 \; , \; x_2 \; , \; x_3\right) - \; s_1\right) + \lambda_2 \left(c_2 - g_2 \left(x_1 \; , \; x_2 \; , \; x_3\right) - \; s_2\right)$$

Step-3 set a partial first order derivates with respect to the choice variables less than or equal to zero (≤ 0) and those with respect to the lagrangian multipliers≥ 0

Note that for minimization counter part deduct the surplus variable from the constraints and apply all above steps.

Example: Maximize $Y= 10X_1-X_1^2+180X_2-X_2^2$

s.t. $X_1+X_2\leq 80$

$X_1, X_2 \geq 0$

Solution

$L=10X_1-X_1^2+180X_2-X_2^2+\lambda(80-X_1-X_2)$

$LX_1=10-2X_1-\lambda=0$, $\lambda=10-2X_1$ ……(1)

$LX_2=180-2X_2-\lambda=0$, $\lambda=180-2X_2$…… (2)

$L\lambda=80-X_1-X_2=0$, $80= X_1+X_2$ ……(3)

From equations 1 and 2 $\lambda=\lambda$

$10-2X_1=180-2X_2$

$X1=X2-85$……..……(4)

Substitute Eq 4 in Eq 3

$80=(-85+X_2)+X_2$

$80+85= 2X_2$

$165=2X_2$

$X_2=82.5$ ……(5)

Substitute Eq 5 in Eq 4

$X_1=82.5-85$

$=-2.5$, which is not meaningful in economics. so $X_1 = 0$……(6)

As a result the value of X_2 also changed. Hence, $80=0+X_2$, $X_2=80$ ……(7)

To find the value of λ

Substitute Eq 7 in Eq 2 $\lambda = 180-2(80)$

$\lambda = 20$

Check for the Kuhn – Tucker three Conditions

1. Non-negativity condition

$X_1\geq 0$ $(X_1=0)$, $X_2\geq 0$ $(X_2=80)$, $\lambda \geq 0$ $(\lambda = 20)$ condition one is satisfied

2. Inequality condition

$X_1+X_2 \leq 80$

$0+80 \leq 80$ Condition two also satisfied

3. Complementary slackness condition

$X_1 L_{X1} = 0$

$0(10-2(0)-20) = 0$ -satisfied

$X_1 L_{X2}=0$

$80(180-2(80)-20) = 0$ –satisfied

λ L λ=20(80-0-80)=0-satisfied

Interpretation: since the Kuhn-tucker three condition are satisfied, the values $X_1= 0$, $X_2=80$ and $\lambda = 20$ are optimal values.

Self Test Exercise 5

1. Find the extremum of the function $z = xy$ subject to the constraint that $x + y = 6$.

2. Find the extremum of $Z = x_1^2 + x_2^2$ *subject to* $x_1 + 4x_2 = 2$

3. Find the maxima of $f(x, y, z) = xyz$ subject to $x + 2y + 2z = 60$

Model Examination

1. A monopolistic firm produces two related goods with the demand functions $Q_1=100-3P_1+2P_2$, Q2=75+0.5P1-P2, and The total cost function ($TC=Q_1^2+2Q_1Q^2+Q_2^2$)
   a) Find the inverse demand functions.
   b) Using cramer's rules for the F.O.C calculate the value of Q_1 &Q_2.
   c) Using the Hessian Test for the S.O.C, check whether the profit of the firm maximized or not.

2. Given the firms total cost, $TC=45X^2+90XY+90Y^2$ when the firm has to meet a production quota 'g' equal to 2x+3y=60
   a) Find the critical values
   b) Using the Boarder Hessian test the SOC.

CHAPTER SIX

COMPARATIVE STATIC ANALYSIS

Chapter Contents

Dear students, this chapter comprises of the following major sub-topics

- The Nature of Comparative Statics
- Differentiations and its Application to Comparative Static Analysis
- Jacobean and Hessian Determinants
- Comparative Static and General Function Model
- Limitations of Comparative Static Analysis

Learning Objectives

After covering the above sub-topic s, learner will be able to:

- ✓ distinguish between the meaning, types and nature of comparative statics in economics;
- ✓ make use of differentiation as a tool in comparative statics.
- ✓ explain the importance of Jacobian and Hessian determinants in comparative statics.
- ✓ describe the techniques used in finding comparative statics for general functional models.
- ✓ realize the basic limitations of comparative static analysis in economics.

6.1 The Nature of Comparative Statics

Dear learners, what do you think about the term Comparative statics? Can you say something?

Use this space to reflect your idea and compare your reflection with the following clarification.

_____

_____

_____

Comparative Statics analysis is analysis concerned with the comparison of different equilibrium states that are associated with different sets of values of parameters or exogenous variables.

We have two classes of variables.

1. **Choice variables**- are also called *endogenous variables* are variables that are determined within the model. They are under the control of decision makers or firms.
2. **Exogenous Variables or Parameters** - are variables outside the model. They are out of the control of the firms (decision makers). They are determined by factors outside the model.

In comparative statics, there is a need for point of reference or starting point so as to make the comparison clear and visible. For example, initial equilibrium states are most commonly used as starting points.

Example: In the theory of demand and supply price is exogenous: $Q_d = f(p)$ & $Q_s = f(p)$

In both cases p is exogenous in the sense that it is not influenced by the actions of decision makers. It is rather determined by factors outside the control of both consumers and suppliers in the market. The *initial (pre-change) equilibrium* may be represented by equilibrium, $\bar{P}$, and the corresponding quantity, $\bar{Q}$. If P changes, quantity demanded and quantity supplied can change and hence the initial equilibrium will be upset (disturbed). If the new equilibrium state relevant to the new price can be defined and attained, the question to be posed in comparative statics is 'how would one compare the new equilibrium with the old one?'

In comparative statics, we disregard the process of adjustment of the variables, i.e., we simply compare the pre-change and the post-change equilibrium states and the way through which the change has come is out of the concern of the field.

Hence, comparative statics is essentially concerned with finding the rate of change of the equilibrium value of the choice or endogenous variables with respect to the change in a particular parameter or exogenous variable.

From example; what happens to the quantity demanded of a good when the price of the good, the price of relative commodities or income changes is a representative question in comparative statics. A comparative static analysis can be either qualitative or quantitative in nature. If the interest is to known the direction of change of the endogenous variables as a result of the change in the parameters, the analysis is *qualitative type*. However, if the concern is both on direction and magnitude of the change, the analysis will be *quantitative*.

Note that in economics, theories are commonly tested on the basis of changes in the variables which may or may not result in relation of assumptions

6.2. Differentiations and its Application to Comparative Static Analysis

In comparative statics we make use of derivatives in assessing the rate of change of one endogenous variable as a result of the change in one or more parameters or exogenous variables.

We can consider different models to illustrate this concept. Let R(x) = total revenue as a function of output, x. C(x) = total cost as a function of output, x. tx = total tax paid and t is a per unit tax as a parameter beyond the control of the firm. Here, while x is an endogenous variable, t is exogenous.

Let's define the profit function,

$$\pi = R(x) - C(x) - tx$$

Foc: $\pi' = R'(x) - C'(x) - t = 0$

If the firm is in perfect competitive market, $P = MR$; but from the above,

$MR = MC + t \Rightarrow P = MC + t$. This implies that the firm chooses the level of output such that $MR = MC + t$.

SOC: $\pi'' = R''(x) - C''(x) < 0 \Rightarrow R''(x) < C''(x)$ i.e., the slope of the marginal revenue should be less than that of marginal cost. Any way X is a function of t. A change in t change the optimal level of the output produced $\Rightarrow X = X^*(t) or\ X = f(t)$. If this is the case, we can insert this definition in the FOC.

$$\frac{\partial \pi'(x)}{\partial t} = \frac{d(R'(X^*))}{dx^*}.\frac{dx^*}{dt} - \frac{d(C'(X^*))}{dx^*}.\frac{dx^*}{dt} - \frac{dt}{dt} = 0$$

$$R''(x^*)\frac{dx^*}{dt} - C''(x^*)\frac{dx^*}{dt} = 1$$

$$\frac{dx^*}{dt}[R''(X^*) - C''(X^*)] = 1$$

$\frac{dx^*}{dt} = \frac{1}{R''(x^*) - C''(x^*)}$, but look at the SOC and for π to be maximized $R''(X^*) - C''(X^*) < 0$

$$\Rightarrow \frac{dx^*}{dt} = \frac{1}{R''(X^*) - C''(X^*)} < 0$$

$\frac{dx^*}{dt} < 0$, implies optimal level of output and tax rate are negatively related, i.e., output of the firm decreases as the tax rate of the firm faces increases and vice versa.

Conclusion: A prediction about the size and direction of the change of the choice variable (output in the above case) can be made by looking at the change in the parameter facing the decision maker-a goal in comparative statics.

Example:

Suppose that the monopolist has a demand curve $p = 180 - 2q$ and a total-cost function $C = q^2$. Find the effect of an increase in the output tax on equilibrium output.

Solution

Profit is $\pi = (180 - t)q - 3q^2$, and optimal output is

$$q^* = 30 - \frac{t}{6}$$

and the comparative-static effect is

$$\frac{dq^*}{dt} = -\frac{1}{6}$$

1. Maximize $\pi(x) = R(x) - C(x) = Px - C(x)$

Here the assumption is that P is an exogenous variable, i.e., P is taken as a parameter beyond the control of decision maker.

FOC: $\pi'(x) = P - C'(x) = 0 \ldots\ldots (1)$

SOC: $\pi''(x) = -C''(x) \leq 0 \ldots\ldots (2)$, $C''(X) > 0$, the MC must be upward slopping.

But, we know the optimal level of x (x*) is a function of P, i.e., $x = x^*$(P) and from (1)

$C'(x) = MC = P$

$x = x^*$(P) shows the supply function of the firm starting from the point where MC=P.

$x = x^*$(P) tells us how much the firm offers to the market for every market determined level of price. Inserting $x = x^*$(P) in FOC: $P - C'(x^*(P)) = 0$ and applying the SOC:

$$\frac{dp}{dp}\frac{dC'(x^*)}{dx^*}.\frac{dx^*}{dp} = 0$$

$1 - C''(X^*).\frac{dx^*}{dp} = 0 \Rightarrow \frac{dx^*}{dp}\frac{1}{C''(X^*)}$ and $C''(x^*) > 0$

$$\Rightarrow \frac{dx^*}{dp} > 0$$

This shows there is a positive relationship between the parameter P and the endogenous variable output. In other words the supply function of a competitive firm is up ward slopping.i.e, $x = x^*(p)$ has a graph of the slope.

2. **Market model:** let us consider a market model for a single commodity.

$Q_d = a - bP \ (a,b > 0)$

$Q_s = -c + dP \ (c,d > 0)$

We have two endogenous variables (P & Q) and four parameters (a, b, c & d). At equilibrium,

$$Q_d = Q_s \Rightarrow -c + dP = a - bP \Rightarrow \bar{P} = \frac{a+c}{b+d}$$

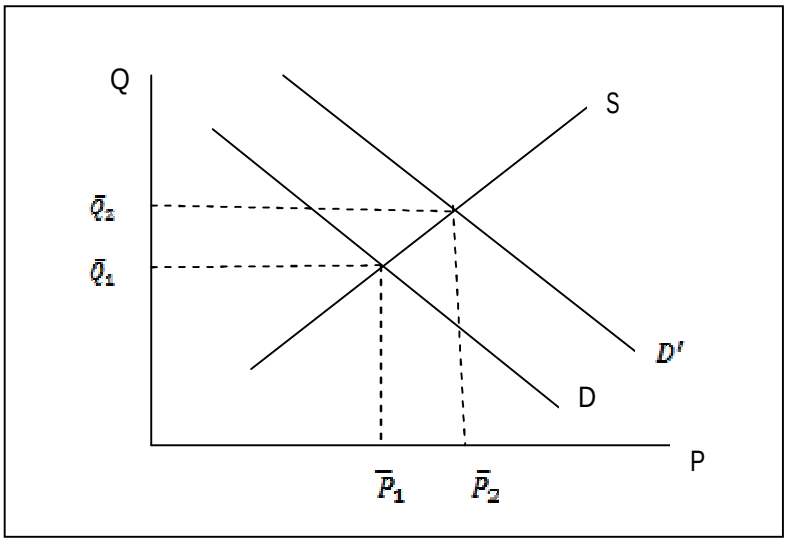
Assume $\bar{P}$ and $\bar{Q}$ are initial equilibrium levels of the P and Q, comparative statics helps us to compute the change in the initial equilibrium values.

$$\frac{\partial \bar{P}}{\partial a} = \frac{1}{b+d} > 0 \qquad \frac{\partial \bar{P}}{\partial b} = \frac{-(a+c)}{(b+d)^2} < 0$$

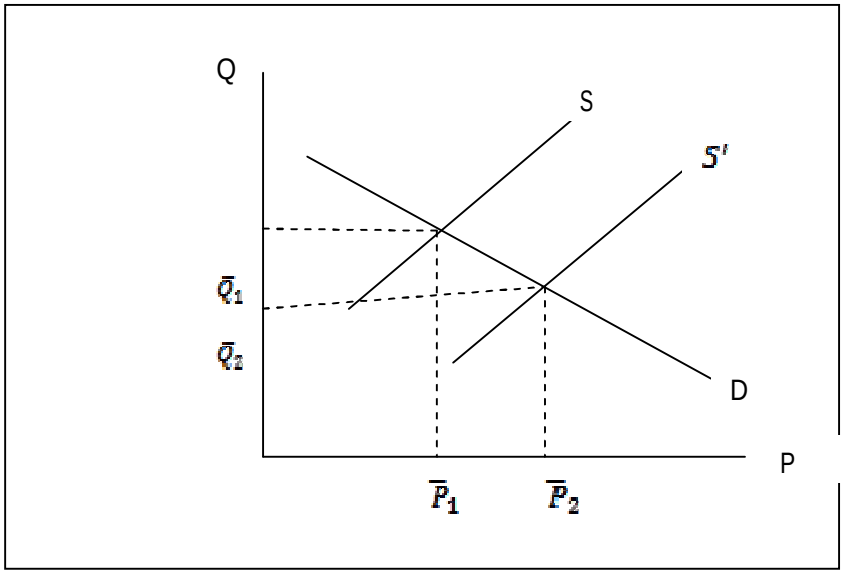
$$\frac{\partial \bar{P}}{\partial c} = \frac{1}{b+d} > 0 \qquad \frac{\partial \bar{P}}{\partial d} = \frac{-(a+c)}{(b+d)^2} < 0$$

NB: There is an inverse relationship between the price of the commodity and the slopes of demand and supply.

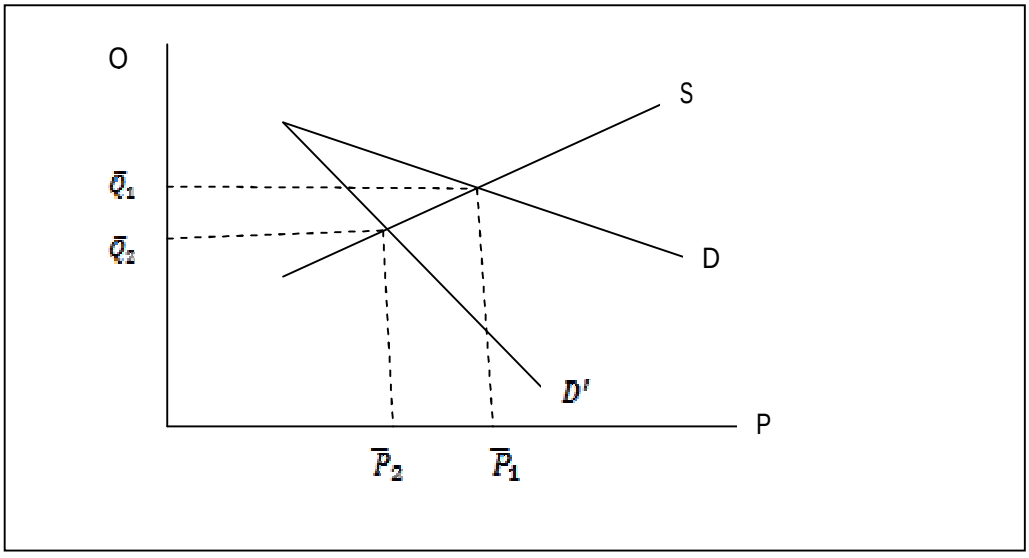
i) An increase in 'a' indicates an upward shift of the demand curve, that is, as $a \uparrow \Rightarrow \bar{P} \uparrow$.



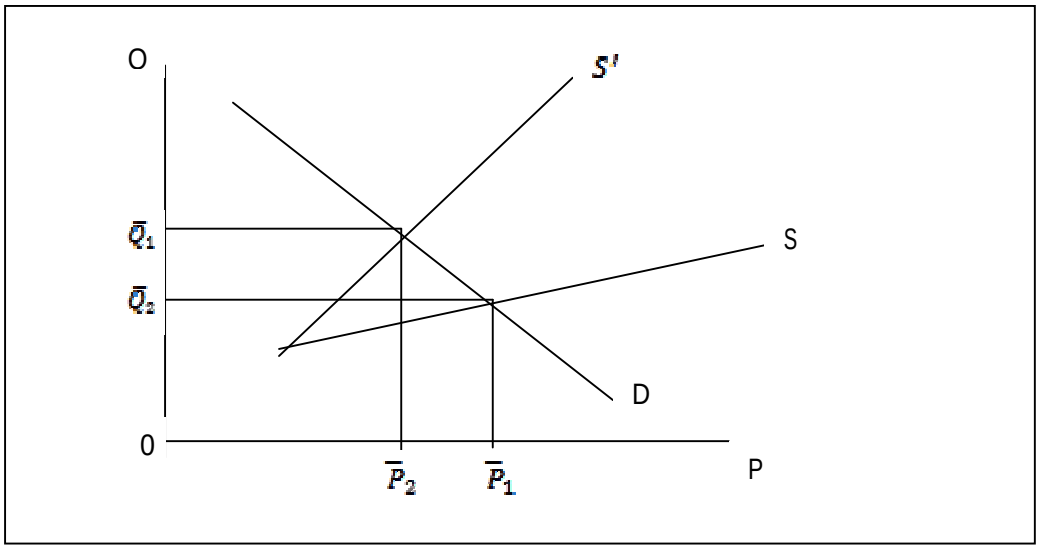
ii) Similarly, an increase in 'c' shifts the supply curve to the right. That is, as $c \uparrow \Rightarrow \bar{P} \uparrow$.



iii) An increase in 'b' increases the slope of the demand curve. That is, as $b \uparrow \Rightarrow \bar{P} \downarrow$.



iv) An increase in'd' increases the slope of the demand curve. That is, $d \uparrow \Rightarrow \bar{P} \downarrow$.



Activity: Express the relationship between the parameters and the equilibrium level of the output both graphically and algebraically.

3. **National Income Model:** The discussion of comparative static analysis can also be made on national income model. We have

$$Y = C + I_0 + G_0$$

$$C = \alpha + \beta(Y - T); (\alpha > 0; 0 < \beta < 1)$$

$$T = \gamma + \delta Y; (\gamma > 0; 0 < \delta < 1)$$

Where,

α – *shows the autonomous consumption level.*

β – *measures the marginal proponesity to consumption (MPC).*

γ – *indicates the non – income tax rate.*

δ – *income tax rate.*

And, Y-T is usually referred to as disposable income (that is, income after tax). Y and C stand for the endogenous variables national income and consumption expenditure respectively. I_0 *and* G_0 are exogenously determined investment and government expenditures.

The first equation is an equilibrium condition (National Income = Total Expenditure), while the second and third are behavioral equations, that is, consumption and tax functions, respectively. Moreover, the equations show that the model is of closed type because the trade terms are not incorporated.

NB: The equations are neither functionally dependent nor inconsistent with each other.

Thus, we can determine the equilibrium levels of the endogenous variables, Y, C and T in terms of the exogenous variables. I_0 *and* G_0 and the parameters α, β, γ *and* δ.

Substitution of the third equation in the second and then the second in the first, we have

$$C = \alpha + \beta\big(Y - (\gamma + \delta Y)\big) = \alpha + \beta(Y - \gamma - \delta Y) = \alpha - \gamma\beta + (1 - \delta)Y$$

$$Y = \alpha + \beta Y - \gamma\beta - \beta\delta Y + I_0 + G_0$$

$$Y + \beta\delta Y - \beta Y = \alpha - \gamma\beta + I_0 + G_0$$

$$Y(1 + \beta\delta - \beta) = \alpha - \gamma\beta + I_0 + G_0$$

$$\bar{Y} = \left[\frac{\alpha - \gamma\beta + I_0 + G_0}{1 + \beta\delta - \beta}\right]$$

$$\bar{T} = \gamma + \delta\left[\frac{\alpha - \gamma\beta + I_0 + G_0}{1 + \beta\delta - \beta}\right]$$

The interest in comparative static is to see the effect of the change in one of the exogenous variable on the endogenous variables. To do so we make first order derivatives.

For example:

$\frac{\partial Y}{\partial G} = \frac{1}{1+\beta\delta-\beta} > 0$, indicate the government expenditure multiplier.

$$\frac{\partial \bar{Y}}{\partial \delta} = -(1 + \beta\delta - \beta)^{-2}(\alpha - \beta\gamma + I_0 + G_0)\beta$$

$$= \frac{-\beta(\alpha - \beta\gamma + I_0 + G_0)}{(1 + \beta\delta - \beta)(1 + \beta\delta - \beta)}$$

$$= \frac{-\beta\bar{Y}}{(1 + \beta\delta - \beta)} < 0$$

This result indicates income tax multiplier.

Activity: Find $\frac{\partial Y}{\partial \alpha}$ *and* $\frac{\partial^2 Y}{\partial \alpha_3 \gamma}$, and determine their signs.

6.3. Jacobian and Hessian Determinants

Both follow the same procedure in their computation. That is, we take the coefficients of first order derivatives as elements of the determinant of both the Jacobian and Hessian. Or, simply we evaluate the second order derivative of the Lagrangian function with respect to the choice variables and take them as elements of the determinants.

The two differ in their purpose, while the Jacobians are used to check whether a given system has a solution before an attempt, the Hessians are used as sufficient conditions to determine a given point as a relative maximum or minimum point.

Note: The detail concepts and mathematical application of Jacobian and Hessian determinants addressed in the previous chapter. The relevance of this sub topic here is that in order to remind the students to relate the concepts with comparative statistics application and we use it in our further analysis.

6.4 Comparative Static and General Function Model

In all the above models the equilibrium values of the endogenous variables could be explicitly expressed in terms of the exogenous variables. Accordingly the technique of simple partial differentiation was all we needed to obtain the desired comparative static information.

However, when a model contains functions expressed in general form, explicit solutions are not available, i.e., the equilibrium values of the endogenous variables can not be expressed explicitly in terms of parameters and/or exogenous variables. In such cases a new technique must be adopted that makes use of concepts such as total differentials, implicit function rule e.t.c.

Let's try to illustrate the point with a market model. Consider a market model where Q_d is a function of both price and an exogenously determined income Y_0, and Q_s is a function of price alone.

$$Q_d = D(P, Y_0) \text{ where } \frac{\partial D}{\partial P} < 0 \text{ and } \frac{\partial D}{\partial Y_0} > 0.$$

$$Q_s = S(P) \quad \text{where } \frac{dS}{dP} > 0.$$

And, the equilibrium position of the market is defined by $Q_d = Q_s$.

This implies

$D(P,Y_0) = S(P)$

$D(P,Y_0) - S(P) = 0.$

Even though the above equation can not be solved explicitly for the equilibrium price, $\bar{P}$, we assume that there does exist a static equilibrium before and after the change in the exogeneous variable Y_0. Say we have obtained $\bar{P}$, if income of consumers changes, the whole equilibrium will be upset. This indicates that the optimal value of $\bar{P}$ which sets the market at equilibrium is a function of the exogenous variable Y_0. That is, $\bar{P} = \bar{P}(Y_0)$.

The change in income upsets the equilibrium by causing a shift in the demand function implying that every value of Y_0 yields a unique value of $\bar{P}$.

Since $D(P,Y_0) - S(P) = 0$ is an implicit function, we can taket it as an identity: $F(\bar{P},Y_0) = 0$.

The comparative static analysis of such a model is concerned with how a change in Y will affect the equilibrium position of the model. Thus, we can raise two equations:

(1) What is the effect of the change of Y_0 on $\overline{P}$ $\left(\frac{d\bar{P}}{dY_0}\right)$?

(2) What is the effect of the change of Y_0 on $\overline{Q}$ $\left(\frac{d\bar{Q}}{dY_0}\right)$?

From$D(\bar{P},Y_0) - S(\bar{P}) = 0$, we can answer question number one.

Applying the implicit function rule:

$$\frac{d\bar{P}}{dY_0} = \frac{-\partial F/\partial Y_0}{\partial F/\partial \bar{P}} = \frac{-\frac{\partial D}{\partial Y_0}}{\frac{\partial D}{\partial \bar{P}} - \frac{dS}{d\bar{P}}} = \frac{negative(Posttive)}{Negative - Negative} > 0.$$

Hence, $\frac{d\bar{P}}{dY_0} > 0$. This indicates that an increase in income results in an increase in equilibrium price and vice versa. To answer the second question, that is $\frac{d\bar{Q}}{dY_0}$: At equilibrium, $Q_d = Q_s = \bar{Q}$.

We know that $Q_d = Q_s = S(\bar{P})$ but $\bar{P} = \bar{P}(Y_0)$. Thus, substituting we get: $\bar{Q} = S(\bar{P}(Y_0))$

Applying the chain rule, we have

$$\frac{d\bar{Q}}{dY_0} = \frac{dS}{d\bar{P}} \cdot \frac{d\bar{P}}{dY_0} \quad where \ \frac{dS}{d\bar{P}} > 0 \ and \ \frac{d\bar{P}}{dY_0}.$$

Therefore, $\frac{dQ}{dY_0} > 0$- implying that an increase in income increases the equilibrium level of output.

Generally, the comparative static results convey the proposition that an upward shift of the demand curve (due to a rise in Y), results in a higher equilibrium price and equilibrium quantity. The above derivation of the relationship between $\bar{P}$ and $\bar{Q}$ and the exogenous variable Y_0 can be carried out by having a simultaneous equation approach.

$$Q_d = D(P, Y_0); \ Q_s = S(P)$$

At equilibrium, $Q_d = Q_s = Q \Rightarrow Q_d - Q = 0 \ and \ Q_s - Q = 0$

Writing these in general form:

$$D(P, Y_0) - Q = 0$$

$$S(P) - Q = 0$$

Converting these equations in to an identity form:

$$F^1(P, Q, Y_0) = D(P, Y_0) - Q = 0$$

$$F^2(P, Q, Y_0) = S(P) - Q = 0$$

In the above two equations, we have two endogenous variables, P and Q and one exogenous variable, Y_0.

Let's apply the partial differentiation rule on F^1 and F^2, we can form a Jacobian, one row the F^1 and the other for F^2:

$$|J| = \begin{vmatrix} \frac{\partial F^1}{\partial P} & \frac{\partial F^1}{\partial Q} \\ \frac{\partial F^2}{\partial P} & \frac{\partial F^2}{\partial Q} \end{vmatrix} = \begin{vmatrix} \frac{\partial D}{\partial P} & -1 \\ \frac{dS}{dP} & -1 \end{vmatrix} = \frac{dS}{dP} - \frac{\partial D}{\partial P} > 0 \text{ because } \frac{dS}{dP} > 0 \text{ and } \frac{\partial D}{\partial P} < 0.$$

Since $|J| \neq 0$, we can obtain equilibrium P and Q, that is, $\bar{P} = \bar{P}(Y_0)$ and $\bar{Q} = \bar{Q}(Y_0)$.

Thus, the equilibrium condition can be written in the form of the identity $D(\bar{P}, Y_0) - \bar{Q} = 0$ and $S(\bar{P}) - \bar{Q} = 0$.

Taking the total differential of these equations with respect to Y_0, we have

$$\frac{\partial D}{\partial \bar{P}} . d\bar{P} + \frac{\partial D}{\partial Y_0} . dY_0 - d\bar{Q} = 0$$

$$\frac{dS}{d\bar{P}} . d\bar{P} - d\bar{Q} = 0$$

Dividing both sides of both equations by dY_0 yields:

$$\frac{\partial D}{\partial \bar{P}} . \frac{d\bar{P}}{dY_0} - \frac{d\bar{Q}}{dY_0} = -\frac{\partial D}{\partial Y_0}$$

$$\frac{dS}{d\bar{P}} . \frac{d\bar{P}}{dY_0} - \frac{d\bar{Q}}{dY_0} = 0$$

Now, taking $\frac{d\bar{P}}{dY_0}$ and $\frac{d\bar{Q}}{dY_0}$ as variables and writing the above equations in matrix form:

$$\begin{bmatrix} \frac{\partial D}{\partial \bar{P}} & -1 \\ \frac{dS}{d\bar{P}} & -1 \end{bmatrix} \begin{bmatrix} \frac{d\bar{P}}{dY_0} \\ \frac{d\bar{Q}}{dY_0} \end{bmatrix} = \begin{bmatrix} -\frac{\partial D}{\partial Y_0} \\ 0 \end{bmatrix}$$

In order to solve for the variables we can apply the Cramer's rule, but before that let us check $|J| \neq 0$.

$$|J| = \begin{vmatrix} \frac{\partial D}{\partial \bar{P}} & -1 \\ \frac{dS}{d\bar{P}} & -1 \end{vmatrix} = \frac{dS}{d\bar{P}} - \frac{\partial D}{\partial \bar{P}} > 0 \text{ which is different from zero.}$$

Hence, it is possible to continue find the solutions of the problem.

$$\frac{d\bar{P}}{dY_0} = \frac{\begin{vmatrix} -\frac{\partial D}{\partial Y_0} & -1 \\ 0 & -1 \end{vmatrix}}{|J|} = \frac{\frac{\partial D}{\partial Y_0}}{\frac{dS}{d\bar{P}} - \frac{\partial D}{\partial \bar{P}}} > 0$$

$$\frac{d\bar{Q}}{dY_0} = \frac{\begin{vmatrix} \frac{\partial D}{\partial \bar{P}} & -\frac{\partial D}{\partial Y_0} \\ \frac{dS}{d\bar{P}} & 0 \end{vmatrix}}{|J|} = \frac{0 - \left(\frac{\partial D}{\partial Y_0}\right) \cdot \frac{dS}{d\bar{P}}}{\frac{dS}{d\bar{P}} - \frac{\partial D}{\partial \bar{P}}} = \frac{\left(\frac{\partial D}{\partial Y_0}\right) \cdot \frac{dS}{d\bar{P}}}{\frac{dS}{d\bar{P}} - \frac{\partial D}{\partial \bar{P}}} > 0$$

Therefore, both $\frac{d\bar{P}}{dY_0}$ and $\frac{d\bar{Q}}{dY_0}$ are greater than zero, identical with results obtained before.

6.5. Comparative Statistics with several endogenous and exogenous variables

We have seen that we can extend the comparative-statics method readily to problems in which only general functional specifications are given. We now go further and consider the generalization to problems with more than one endogenous and one exogenous variable.

We begin by considering the case of just two variables of each type. As we will see, we require one equilibrium or first-order condition—an f-function—for each *endogenous* variable. Therefore in a model with two endogenous and two exogenous variables, the solution will be given by the conditions

$$f^1(x_1^*, x_2^*, \alpha_1, \alpha_2) = 0$$
$$f^2(x_1^*, x_2^*, \alpha_1, \alpha_2) = 0$$

Now the equilibrium solutions for the endogenous variables will depend on both exogenous variables. We want to derive and sign, if possible, the four derivatives $\partial x_i^*/\partial \alpha_j, i, j = 1, 2$.

Assume that it is possible to solve these equations for x_1^* and x_2^* as differentiable functions of α_1 and α_2, $x_1^*(\alpha_1, \alpha_2)$ and $x_2^*(\alpha_1, \alpha_2)$. Then inserting these into the above conditions gives

$$f^1(x_1^*(\alpha_1, \alpha_2), x_2^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2) = 0$$
$$f^2(x_1^*(\alpha_1, \alpha_2), x_2^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2) = 0$$

Regarding these now as identities, we differentiate through with respect to α_1, to obtain

$$f_1^1 \frac{\partial x_1^*}{\partial \alpha_1} + f_2^1 \frac{\partial x_2^*}{\partial \alpha_1} + f_{\alpha_1}^1 = 0$$
$$f_1^2 \frac{\partial x_1^*}{\partial \alpha_1} + f_2^2 \frac{\partial x_2^*}{\partial \alpha_1} + f_{\alpha_1}^2 = 0$$

where $f_{\alpha_1}^i \equiv \partial f^i/\partial \alpha_1, i = 1, 2$. Again, the partial derivatives are all evaluated at the given equilibrium point and so can be regarded as given numbers. We can

therefore write these equations as the linear system

$$\begin{bmatrix} f_1^1 & f_2^1 \\ f_1^2 & f_2^2 \end{bmatrix} \begin{bmatrix} \partial x_1^*/\partial \alpha_1 \\ \partial x_2^*/\partial \alpha_1 \end{bmatrix} = \begin{bmatrix} -f_{\alpha_1}^1 \\ -f_{\alpha_1}^2 \end{bmatrix}$$

In order to solve this system, we require that the determinant

$$|D| = f_1^1 f_2^2 - f_1^2 f_2^1 \neq 0$$

Assuming that this condition holds and applying Cramer's rule, we have the solutions

$$\frac{\partial x_1^*}{\partial \alpha_1} = \frac{\begin{vmatrix} -f_{\alpha_1}^1 & f_2^1 \\ -f_{\alpha_1}^2 & f_2^2 \end{vmatrix}}{|D|} = \frac{-(f_{\alpha_1}^1 f_2^2 - f_{\alpha_1}^2 f_2^1)}{|D|}$$

$$\frac{\partial x_2^*}{\partial \alpha_1} = \frac{\begin{vmatrix} f_1^1 & -f_{\alpha_1}^1 \\ f_1^2 & -f_{\alpha_1}^2 \end{vmatrix}}{|D|} = \frac{-(f_1^1 f_{\alpha_1}^2 - f_1^2 f_{\alpha_1}^1)}{|D|}$$

To sign these derivatives, we then need to know the signs of all the partial derivatives involved. Moreover notice that the numerators and denominators involve differences between two terms, and so we may also have to know or assume something about the relative magnitudes of these terms in order to put signs to the solutions.

In exactly the same way, we can derive the comparative-statics effects of changes in α_2, though the details of this are left to the reader. We now consider some economic applications.

The IS-LM Model

Consider now an IS-LM model in which aggregate expenditure E is a function of aggregate income Y and the interest rate R. We will assume that there is an exogenous component to expenditure, $\bar{E}$, so we write

$$E = \bar{E} + E(Y, R), \qquad 0 < E_Y < 1, E_R < 0$$

where the partial derivatives indicate that we expect expenditure to increase when income increases and fall when the interest rate rises.

The equilibrium condition is that the aggregate supply and demand for goods and services must be equal

We have two endogenous variables, Y and R, with one exogenous variable $\bar{E}$, but only one equation, and so we cannot solve for the equilibrium values of the endogenous variables. What this means is that given a value for the rate of interest, we can find a value of income that equates the demand and supply of goods and services, or conversely, given a level of income, we can find a rate of interest that achieves this equilibrium. But the single condition itself is insufficient to determine unique equilibrium levels of both variables.

$$Y - \bar{E} - E(Y, R) = 0$$

Then, from the rule for differentiating implicit functions, we have

$$\frac{dR}{dY} = \frac{1 - E_Y}{E_R} < 0$$

The sign follows from the assumptions on the partial derivatives. An increase in $\bar{E}$ shifts the IS curve upward. Thus, if we hold Y fixed and treat R and $\bar{E}$ as variable, then we have

$$\frac{dr}{d\bar{E}} = -\frac{1}{E_R} > 0$$

The LM curve represents equilibrium in the money market in which the demand for money is

$$L = L(Y, R), \qquad L_Y > 0, \ L_R < 0$$

Equilibrium is where the demand for money is equal to money supply, $\bar{M}$, an exogenous constant, giving

$$L(Y, r) - \bar{M} = 0$$

By differentiation, we have then:

$$\frac{dR}{dY} = -\frac{L_Y}{L_R} > 0$$

where the sign is implied by the assumptions on the partial derivatives.

The exogenous money supply is a parameter in the LM relationship. To show how changes in $\bar{M}$ affect the LM curve, treat Y as fixed and R and $\bar{M}$ as variable in the money-market equilibrium condition. Then differentiating gives

$$\frac{dR}{d\bar{M}} = \frac{1}{L_R} < 0$$

Therefore an increase in the money supply reduces the interest rate required for money-market equilibrium at each level of income.

In the comparative-statics analysis, we want to find the effects of changes in the two exogenous variables, $\bar{E}$ and $\bar{M}$, on the equilibrium values of the endogenous variables Y^* and R^*.

Then, applying the general results for the case of a change in $\bar{E}$, we have the linear system

$$\begin{bmatrix} (1 - E_Y) & -(E_R) \\ L_Y & L_R \end{bmatrix} \begin{bmatrix} \partial Y^*/\partial \bar{E} \\ \partial R^*/\partial \bar{E} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We denote, by $|D|$, the determinant

$$|D| \equiv (1 - E_Y)L_R + L_Y(E_R) < 0$$

The sign follows unambiguously from the restrictions on the partial derivatives.

The comparative-statics effects are

$$\frac{\partial Y^*}{\partial \bar{E}} = \frac{\begin{vmatrix} 1 & -(E_R) \\ 0 & L_R \end{vmatrix}}{|D|} = \frac{L_R}{|D|} > 0$$

$$\frac{\partial R^*}{\partial \bar{E}} = \frac{\begin{vmatrix} (1 - E_Y) & 1 \\ L_Y & 0 \end{vmatrix}}{D} = -\frac{L_Y}{D} > 0$$

Thus an increase in investment causes an increase in both national income and the rate of interest. The increase in demand for investment goods increases income directly; this increases the demand for money and so raises the rate of interest with the money supply fixed

The comparative-statics effects of a change in $\bar{M}$ are derived in a similar way, and can also be shown to be unambiguously determined on the given assumptions. This is left as an exercise (see question 1 of the exercises at the end of this section).

A Linear IS-LM Model

The expenditure function is $E = \bar{E} + 0.7Y - 100R$. The demand for money is $L = 30 + 0.2Y - 10.5R$. The money supply is $\bar{M}$. Solve for equilibrium Y and R as functions of autonomous investment $\bar{E}$ and the money supply $\bar{M}$. Show the effects on the equilibrium values of changes in $\bar{E}$ and $\bar{M}$.

Solution

The equilibrium conditions are

$$Y - (0.7Y - 100R) - \bar{E} = 0$$
$$30 + 0.2Y - 10.5R - \bar{M} = 0$$

Gathering terms, we can write these as the linear system

$$\begin{bmatrix} 0.3 & 100 \\ 0.2 & -10.5 \end{bmatrix} \begin{bmatrix} Y \\ R \end{bmatrix} = \begin{bmatrix} \bar{E} \\ \bar{M} - 30 \end{bmatrix}$$

The determinant of the square matrix is $-10.5(0.3) - 100(0.2) = -23.15$. Using

Cramer's rule, we have as solutions

$$Y^* = \frac{\begin{vmatrix} \bar{E} & 100 \\ \bar{M} - 30 & -10.5 \end{vmatrix}}{-23.15} = \frac{-(10.5\bar{E} + 100\bar{M} - 3{,}000)}{-23.15}$$
$$= 0.45\bar{E} + 4.32M^0 - 129.59$$

$$R^* = \frac{\begin{vmatrix} 0.3 & \bar{E} \\ 0.2 & \bar{M} - 30 \end{vmatrix}}{-23.15} = \frac{0.3\bar{M} - 0.2\bar{E} - 9}{-23.15}$$
$$= 0.39 - 0.013\bar{M} + 0.009\bar{E}$$

It follows that

$$\frac{\partial Y^*}{\partial \bar{E}} = 0.45 \quad \frac{\partial Y^*}{\partial \bar{M}} = 4.32 \quad \frac{\partial R^*}{\partial \bar{E}} = 0.009 \quad \frac{\partial R^*}{\partial \bar{M}} = -0.013$$

Competitive Firm's Input Demands

A firm sells its output into a perfectly competitive market and so faces a fixed price p. It also hires labor in a competitive labor market at a wage w and rents capital on a competitive capital market at rental rate r. It has the production function $f(L, K)$, which is strictly concave, and it seeks to maximize profit

$$\pi = pf(L, K) - wL - rK$$

Its first-order conditions are

$$pf_L(L^*, K^*) - w = 0$$
$$pf_K(L^*, K^*) - r = 0$$

We wish to determine the effects on input demands, L^* and K^*, of changes in the input prices. Applying the earlier general results for the effects of a change in the wage w gives the linear system

$$\begin{bmatrix} pf_{LL} & pf_{LK} \\ pf_{KL} & pf_{KK} \end{bmatrix} \begin{bmatrix} \partial L^*/\partial w \\ \partial K^*/\partial w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Defining the determinant

$$|D| = p^2(f_{LL}f_{KK} - f_{KL}f_{LK})$$

we note that if the second-order sufficient conditions for the problem are satisfied, then $D > 0$. That is, the determinant D is precisely the Hessian determinant we obtain when we formulate the second-order conditions for the profit-maximization problem. This is an important point that we will take up further below. We then have the comparative-statics results

$$\frac{\partial L^*}{\partial w} = \frac{\begin{vmatrix} 1 & pf_{LK} \\ 0 & pf_{KK} \end{vmatrix}}{D} = \frac{pf_{KK}}{|D|} < 0$$

$$\frac{\partial K^*}{\partial w} = \frac{\begin{vmatrix} pf_{LL} & 1 \\ pf_{KL} & 0 \end{vmatrix}}{D} = -\frac{pf_{KL}}{|D|}$$

The sign of the first of these derivatives follows from $f_{KK} < 0$. This in turn follows from the second-order conditions. Thus the demand curve for labor has a negative slope. However, in order to sign the effect of a change in the wage rate on the demand for capital, we need to know the sign of f_{KL}, the effect of a change in the labor input on the marginal product of capital. It is plausible to assume that this is positive (though it may not be) and so an increase in the wage rate would also decrease the demand for capital. We can derive the effects of a change in the rental rate of capital in a similar way, and this is left as an exercise.

Example:

Use the Cobb-Douglas production function $y = pL^{\alpha}K^{\beta}$ to find the comparative-static effects $\partial L^*/\partial w$ and $\partial K^*/\partial w$.

Solution

The Hessian is

$$|D| = p^2\alpha\beta L^{2\alpha-2}K^{2\beta-2}(1-\alpha-\beta) > 0$$

for $1 > \alpha + \beta$. Since $f_{KK} = \beta(\beta-1)L^{\alpha}K^{\beta-2} < 0$, and $f_{KL} = \alpha\beta L^{\alpha-1}K^{\beta-1} > 0$, we have

$$\frac{\partial L^*}{\partial w} = p\frac{\beta(\beta-1)L^{\alpha}K^{\beta-2}}{|D|} < 0$$

$$\frac{\partial K^*}{\partial w} = -p\frac{\alpha\beta L^{\alpha-1}K^{\beta-1}}{|D|} < 0$$

Clearly, in the case of a Cobb-Douglas production function, an increase in the wage unambiguously reduces the demand for capital. ■

6.6. Limitations of Comparative Static Analysis

Comparative static is helpful in finding out how a dis-equilibriating change in a parameter, will affect the equilibrium state of a model. However, by its very nature, it has the following limitations.

1. It ignores the process of adjustment from the old equilibrium to the new one.
2. It neglects the time element involved in adjustment process from one equilibrium to another.
3. It assumes always a change in a parameter and/or exogenous variable results in a new equilibrium. It disregards the possibility that new equilibrium may be attained even because of the inherent instability of the model.

Self-Test Exercise 6

1. Suppose that there is a government in the simple Keynesian Model. The government buys goods and services in the amount G and raises revenue through a tax on income. Let the rate of income tax be t, so that disposal income is (1-t) Y. if t=0.2, c=0.8 and I=500, find the value of the government expenditure multiplier.
2. Suppose that the monopolist has a demand curve p=180-2q and a total –cost function $C=q^2$. Find the effect of an increase in the output tax on equilibrium output.

Model Examination

1. Two relations $f(x,y,z) = 0$ and $g(x,y,z) = 0$ define y and z as function of x.

   Obtain expressions f or $\frac{dy}{dx}$ and $\frac{dz}{dx}$ in terms of the partial derivatives of f and g. Two goods x and y are produced jointly with the aid of two factors K and L. Technical conditions give two production functions $X = f(K,L)$ and $Y = g(K,L)$.

Find $\left(\frac{dK}{dN}\right)$Y constant, $\left(\frac{dL}{dY}\right)$X constant, and interpret these derivatives.

2. Use Jacobian determinants to test the existence of functional dependence between the functions paired below:

(a) $y_1 = 3x_1^2 + x_2$

$y_2 = 9x_1^4 + 6x_1^2(x_2 + 4) + x_2(x_2 + 8) + 12$

(b) $y_1 = 3x_1^2 + 2x_2^2$

$y_2 = 5x_1 + 1$

3. Assuming that the equation $F(U, x_1, x_2, \ldots, x_n) = 0$ implicitly defines a utility function $U = f(x_1, x_2, \ldots, x_n)$:

A. Find the expressions for $\frac{\partial U}{\partial x_2}, \frac{\partial U}{\partial x_n}, \frac{\partial x_3}{\partial x_2}$, and $\frac{\partial x_4}{\partial x_n}$.

B. Interpret their respective economic meanings.

4. If the equation $F(x, y, z) = 0$ implicitly defines each of the three variables as a function of the other two variables, and if all the derivatives in question exist, find the value of $\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z}$

5.

The demand function for coffee is given by

$$D_c = 100 - 2p_c + 0.5p_t$$

and that for tea is given by

$$D_t = 120 - p_t + 0.75p_c$$

where p_c is the price of coffee and p_t is the price of tea. The respective supply functions are

$$S_c = 10 + p_c + 5w_c$$
$$S_t = 5 + 2p_t + 2w_t$$

where w_c and w_t are the indexes of weather conditions affecting production of coffee and tea respectively. Interpret these supply and demand functions. Give the comparative-statics effects on equilibrium prices of changes in the weather conditions variables.

CHAPTER SEVEN

DYNAMIC OPTIMIZATION

Chapter Contents

Dear students, this chapter covers the following major sub-topics

- Definitions and concepts on dynamic analysis.
- Differential and Difference Equations
- Dynamic Stability of Equilibrium
- Economic Applications of Dynamic Optimization
- Optimal Control Theory

Learning Objectives

After going through this chapter, students are expected to

- ✓ distinguish static analysis vs dynamic analysis
- ✓ define basic concepts used in dynamic optimization
- ✓ differentiate continuous time vs discrete time case
- ✓ demonstrate how a time lag can affect the pattern of adjustment to equilibrium in some basic economic models.
- ✓ set up and solve linear first-order difference equations
- ✓ familiarize yourselves about the application of linear differential and difference equations in economics.
- ✓ apply the difference equation solution method to the cobweb and Keynesian
- ✓ identify the stability conditions in the above models.
- ✓ sketch phase diagrams differential as well as difference applications
- ✓ describe the application of optimum control theory

Introduction

In earlier chapters much of the economic analysis used has been static and comparative statics.

In *static analysis* an economic variable is assumed to be a function of another variable in the same time period. *Comparative statistics* entails the comparison of different (static) equilibrium situations, with no mention of the mechanism by which price and quantity adjust to their new equilibrium values. For example, 1) Consumption is the function of income for the same time period: $C = f(y)$. 2) Supply is the function of price for the same time period: $S = f(P)$
But, dynamic analysis introduces the explicit consideration of time in to the picture. For example 1) Current consumption may depend on past incomes: $Ct = f(Yt-1)$ *or* $C(t) = f(y(t)-1)$

2) Quantity supplied in one period may depend on the price of the previous period. A good example is *agricultural food supply (Cobweb model)*.

Thus, the branch of economics that looks at how variables adjust between equilibrium values is known as '***dynamics***', and this chapter gives an introduction to some simple dynamic economic models. For instance, the ways in which markets adjust over time vary tremendously. In commodity exchanges, prices are changed by the minute and adjustments to new equilibrium prices are almost instantaneous. In other markets the adjustment process may be a slow trial and error process over several years, in some cases so slow that price and quantity hardly ever reach their proper equilibrium values because supply and demand schedules shift before equilibrium has been reached. There is therefore no one economic model that can explain the dynamic adjustment process in all markets.

The simple dynamic adjustment models explained here will give you an idea of how adjustments can take place between equilibria and how mathematics can be used to calculate the values of variables at different points in time during the adjustment process. Models that assume a process of continual adjustment are considered, using *'differential equations*'.
In this chapter, time is considered an important *variable* and the dynamic adjustment process between equilibria is seen as a *step-by-step process*. This enables us to calculate different values of the variables that are adjusting to new equilibrium levels.

7.1 Definitions and Concepts on dynamic analysis

i) **Discrete-time case:** Where time can be considered as a discrete variable, in which case the variable undergoes a change only once within a period of time. This utilizes the methods of difference equations.

Example: Interest compounding per year, month etc. The curve is not smooth here.

ii) **Continuous-time case:** Where time is considered as a continuous variable, in which case something is happening to the variable at each point of time. Here, integral calculus and differential equations are used. Example, population growth.

iii) **Difference Equation:** An equation that expresses the relation between a dependent variable and a lagged independent variable or variables which changes at discrete intervals of time.

iv) **Differential Equation:** An equation that expresses the value of dependent variable in time as a function of independent variable of time itself and of given values of dependent variables at one or more moments in time.

v) **Ordinary differential equations:** are equations involving a single independent variable.

vi) **Autonomous equation:** is a differential equation where time is not as explicit and a separate argument. Example: $dy(t)/dt = ay(t) + b(t)$

Note: 1) The equation $ay(t) + b(t)$, has two parts, homogenous and non homogenous: $ay(t)$ = homogenous part, $b(t)$ = non homogenous part because the term may be time varying. This shows time path $y(t)$ from its initial value to its value at some terminal time as a function of time path $b(t)$, time itself and a given value of $y(t)$ at one moment in time $\{y(t)\}_t^k = 0$

2) The solution for $y^*(t) = ay(t) + b$, the component $y(t)$ is the *steady state value* which is called *equilibrium value* or *long run value* or *stationary value* .

vii) **Steady state solution** is that point where the graph crosses the horizontal axis and at that point $y^* = 0$

viii) **Non-autonomous differential equation:** Here time included as separate argument

Example: $y(t) = ay(t) + b(t) + ct$

viiii) **Control variables**: are variables that take the form of flows, such as the flow of consumption or the flow of labor services.

x) **State variables:** are stocks, such as the stock of capital, the stock of available oil, or the stock of gold.

xi) **Complementary function (CF):** is a function that tells us how the variable in question varies from the equilibrium solution as time changes.

xii) **Particular solution (PS):** is a constant value about which adjustments in the variable in question take place over time.

xiii) **General solution (GS):** to a difference equation is the full solution, which is the summation of the above two elements. That is, GS = PS+CF

xiiii) **Phase diagram:** is a diagram that offers qualitative information about stability of an equation. It determines whether equation will converge or diverge to or from inter temporal equilibrium level or steady state value.

7.2. Differential and Difference Equations

Differential equations are equations involving derivatives (differentials). They express the rates of change of continuous functions over time. The objective in working with differential equations is to find a function, without a derivative (differential) which satisfies the differential equation. Such a function is called the solution or integral of the equation.

The difference between differential and difference equation is the nature of time. Time is considered continuous in differential but discrete in difference equations. Otherwise, they serve the same purpose.

7.2.1. Differential Equations

The order of a differential equation is the order of the highest derivative in the equation. The degree of a differential equation is the highest power to which the derivative of the highest order is raised. The time path of a dependent variable from an initial moment is written as t = 0 to some terminal moment as t = k is written as $\{y(t)\}_t^k = 0$.

Differential equations expresses relation of an implicit or explicit between a function y= f(t)and one or more of its derivatives.

Example: 1) $\frac{dy}{dx} = 2x + 6$: First order, first degree

2) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + x^2 = 0$: Second order, first degree

3) $\left(\frac{d^2y}{dx^2}\right)^7 + \left(\frac{d^3y}{dx^3}\right)^5 = 75y$: Third order, fifth degree

General form: $\frac{dy}{dt} + VY = Z$ where V and Z are constants or functions of time.

When constants⇒ $\frac{dy}{dt} + 2Y = 3$ and when functions of time ⇒ $\frac{dy}{dt} + V(t)Y = Z(t)$.

Note: For a first order linear differential equation $\frac{dy}{dt}$ and y must be not higher than the first degree and no product $y\frac{dy}{dt}$ may occur. General solution: $y(t) = Ae^{-vt} + \frac{Z}{V}$

Where, A ≡is an arbitrary constant, $Ae^{-vt} \equiv Yc \Rightarrow$ the complementary solution,

$\frac{Z}{V} = Yp \Rightarrow$Particular integral

Proof: Given the general linear differential equation:

$$\frac{dy}{dt} + VY = Z \Rightarrow \frac{dy}{dt} = Z - VY$$

Separating the variables: $\frac{dy}{Z-VY} = dt$

Integrating both sides: $\int \frac{dy}{Z-VY} = \int dt$

$$\frac{-1}{V}\ln(Z - VY) = t + C_1$$

$$\ln(Z - VY) = -Vt + (-VC_1)$$

$$ln(Z - VY) = -Vt + C2 \qquad \text{Where, } C_2\text{=-V}C_1$$

$$Z - VY = e^{-vt+c2} \quad \text{Since if } \ln y = x \Rightarrow = e^x$$

$$Z - VY = e^{-vt}e^{c2} = e^{-vt}c \qquad \text{Where c=}e^{c2}$$

$$-VY = ce^{-vt} - Z$$

$$Y = \frac{-C}{V}e^{-vt} + \frac{Z}{V} \qquad \text{Where} \quad A = \frac{-C}{V}$$

Hence, for the above general linear differential equation, the general solution is given as $Y = Ae^{-vt} + \frac{Z}{V}$.

This general solution works for both V and Z as a constants and functions of time.

7.2.2. Finding a Definite solution for a Differential Equation

Given a general solution of a differential equation, if an initial condition (boundary value) is given, A can be specified, in which case a definite solution is possible. Hence, the arbitrary constant in the general solution (i.e. A) can be definitized by means of an initial condition.

Given $Y(t) = Ae^{-vt} + \frac{Z}{V}$

At $t = 0, y(0) = Ae^{-v(0)} + \frac{Z}{V} = A + \frac{Z}{V} \Rightarrow A = Y(0) - \frac{Z}{V}$

Thus, the definite solution is formed by replacing 'A' by $Y(0) - \frac{Z}{V}$ in the general solution as follows. $Y(t) = \left[Y(0) - \frac{Z}{V}\right]e^{-vt} + \frac{Z}{V}$

Note: The particular integral (Yp) represents the intertemporal equilibrium level of $Y(t)$. The complementary solution Yc denotes the deviation of the time path from the equilibrium level. For, $Y(t)$ to be dynamically stable, Yc must approach zero as t approaches infinity (∞).

Example: Find the general solution for a differential equation $\frac{dy}{dt} + 4y = 12$

Method 1: since V=4 and Z=12, the general solution will be $Y(t) = Ae^{4t} + 3$

Method 2: using the equation in separated form, an explicit solution can be found as follows:

$\frac{dy}{dt} + 4y = 12 \Rightarrow \frac{dy}{dt} = 12 - 4y$

Separating the variables: $dy/_{12 - 4y} = dt$

Integrating both sides: $\int {}^{dy}/_{12-4y} = \int dt$

$$\frac{-1}{4}\ln(12-4y) = t + C_1$$

$$\ln(12-4y) = -4t - 4C_{1\Rightarrow}) = 12 - 4y = e^{-4t-4c_1}$$

$$-4y = e^{-4t}e^{-4c_1} - 12 \Rightarrow y = e^{-4t}\frac{e^{-4c_1}}{-4} + 3$$

The general solution is given by: $y = Ae^{-4t} + 3$ where $A = \frac{-1}{4}e^{-4c_1}$

7.2.3. First Order Linear Difference Equations

First order linear difference equations are used to analyze changes with respect to time when what happens in one period depends up on what happened in the previous period. A difference equation expresses a relationship between a dependent variable and a lagged independent variable which changes at discrete intervals of time.

Example: Consumption in one period depends on the previous income. That is,

$c_t = f(y_{t-1})$ A_t = for discrete case, $A_{(t)}$ = for continous case

Note: The order of a difference equation is determined by the greatest number of periods lagged. A first order difference equation expresses a time lag of one period; second order difference equation a two period time lag and so on.

Example: The change in y as t changes from t to $t + 1$ is called the first difference of y

$$i.e.\frac{\Delta y}{\Delta t} = \Delta yt = y_{t+1} - y_t \quad \text{[first order]}$$

Most of the time delta (Δ)is omitted, and we write like this

$I_t = a(y_{t-1} - t_{t-2})$ [second order]

$Q_t = a + bp_{t-1}$ [first order]

$y_{t+3} - 9y_{t+2} + 2y_{t+1} + 6y_t = 8$ [third order]

The solution of a difference equation defines y for every value of t and does not contain a difference expression.

The general form of a linear difference equation is given by

$y_{t+1} - by_t = c$ or in some cases $y_t = by_{t-1} + c$, $where\ b$ and c are constants.

7.2.4. Methods of Solving Difference Equations

A. Iterative method

The first order difference equation describes the pattern of change of y between two consecutive periods only. Hence, once the difference equation is specified and an initial value y_0 is given, it is possible to find y, from the equation. Similarly, once y_1 is found, y_2 will be immediately obtained, and so on, b y repeated application (iteration) of the pattern of change specified in the difference equation. The results of iteration will enable us to inter a time path or the variable under consideration.

Examples:

1. Find the solution of a difference equation $\Delta y_t = 2$ assuming an initial value of $y = 15$.

solution:

$y_t = y_{t+1} - y_t$, $\Delta y_t = 2 \Rightarrow y_{t+1} = y_t + 2$

since Δ Then by successive substitutions of $t = 0,1,2,3, e.t.c$ we obtain

$t = 0: y_1 = y_0 + 2$

$t = 1: y_2 = y_1 + 2 = (y_0 + 2) + 2 = y_0 + 4 = y_0 + 2(2)$

$t = 2: y_3 = y_2 + 2 = [y_0 + 2(2)] + 2 = y_0 + 6 = y_0 + 3(2)$

$t = 3: y_4 = y_3 + 2 = [y_0 + 3(2)] + 2 = y_0 + 8 = y_0 + 4(2)$

. . .

. . .

. . .

For any period $t = y_t = y_0 + t(2)$

$\Rightarrow y_t = 15 + 2t$ (this is the time path of the variable).

2. Find the solution of the difference equation $y_{t+1} = by_t$ given that the initial value of y is y_0.

Solution:

Given $y_{t+1} = by_t$

At $t = 0: y_1 = by_0$

At $t = 1: y_2 = by_1 = b(by_0) = b^2y_0$

At $t = 2: y_3 = by_2 = b(b^2y_0) = b^3y_0$

. . .

. . .

At $t = t: y_t = b^t y_0$ (The solution of the difference equation)

The solution of the differential equation:

$y(t) = Ae^{-vt}$ (it is natural logarithm expreses constant rate of continuous growth)

The solution of the difference equation:

$y_t = Ab^t$ (exponential equation expreses constant rates of discrete growth)

B. General Method:

Given the first order difference equation: $y_{t+1} - by_t = c$ its general solution will consist of the sum of a complimentary function (y_c) and a particular integral (y_p).

Definition: y_p equals the intertemporal equilibrium level of *y* and y_c represents the deviation of the time path from that equilibrium level. For the time path to be dynamically stable, y_c must approach zero as t approaches infinity.

y_c-is the general solution of the reduced (homogeneous) equation.

$y_{t+1} - by_t = 0$

y_p-is the general solution of the complete or non-homogeneous equation.

$y_{t+1} - by_t = c$

a) First find y_c for the homogeneous equation;

Given: $y_{t+1} - by_t = 0 \Rightarrow y_{t+1} = by_t$

The general solution will be of the form $y_t = Ab^t$ (see example 2 above in iterative method) where, $Ab^t \neq 0$, for otherwise y_t becomes a horizontal straight line on t-axis. In that case the complimentary solution will be $y_c = Ab^t$.

b) Next, find the particular integral for the non-homogeneous equation:

Given $y_{t+1} - by_t = c$, we can choose any solution particular integral (y_p). Thus, we take the simplest form of the trial solution: $y_t = k$ (a constant).

Then, since y maintains the same constant value over time, we must also have $y_{t+1} = k$. Then, substitute these values in to the complete equation:

$$y_{t+1} - by_t = c \Rightarrow k - bk = c \Rightarrow k(1-b) = c \Rightarrow k = \frac{c}{1-b}$$

Hence, the particular solution becomes $y_p = k \Rightarrow y_p = \frac{c}{1-b}, b \neq 1$.

Note that since $\frac{c}{1-b}$ is a constant, a stationary equilibrium is indicated in this case.

Since y_p is undefined at b=1, we need to find some other solution for the non-homogeneous equation. So, let's try a solution of the form $y_t = kt$ which indicates a moving equilibrium, $y_t = kt$ implies $y_{t+1} = k(t+1)$.

Substituting these values in the complete equation, we obtain

$y_{t+1} - by_t = c \Rightarrow k(t+1) - b(kt) = c$

$\Rightarrow k(t+1) - kt = c$, since $b = 1$

$\Rightarrow k(t+1-t) = c \Rightarrow k = c$ thus, $y_p = kt \Rightarrow y_p = ct$, for $b = 1$

c) Finally, adding y_p *and* y_c we arrive at the general solution

$y_t = y_c + y_p \Rightarrow y_t = Ab^t + \frac{c}{1-b}$ ……. (1) General solution when $b \neq 1$.

$y_t = Ab^t + ct \Rightarrow y_t = A + ct$ …… (2) General solution when $b = 1$.

Eliminating the arbitrary constant, the definite solution will be written as:

From (1) at $t = 0, y_0 = Ab^0 + \frac{c}{1-b} \Rightarrow A = y_0 - \frac{c}{1-b}$

$$y_t = \left(y_0 - \frac{c}{1-b}\right) b^t + \frac{c}{1-b} \quad (1') \text{ [Definite solution, when } b \neq 1]$$

From (2) at $t = 0, y_0 = A + c(0) = A$

$$y_t = y_0 + ct \quad (2') \text{ [Definite slution, when } b = 1]$$

Example: Solve the first order difference equation $y_{t+1} - 5y_t = 1, (y_0 = {}^7/_4)$.

Solution: Using the definite solution formula ($b \neq 1$)

$$y_t = \left(y_0 - \frac{c}{1-b}\right) b^t + \frac{c}{1-b}$$

$$\text{Since } b = 5, c = 1 \Rightarrow y_t = \left({}^7/_4 - \frac{1}{1-5}\right)(5)^t - \frac{1}{4}$$

$$y_t = 2(5)^t - \frac{1}{4}$$

Activity: Try the following and check your progress

Given the difference equation $y_t = -7y_{t-1} + 16$ and $y_0 = 5$, find the time path

Check your progress:

Given $y_t = -7y_{t-1} + 16$, to change it in to the general form of difference equation, shift the equation by one year: $y_{t+1} = -7y_{t-1+1} + 16$

$$y_{t+1} + 7y_{t-1+1} = 16$$

Since b=-7 and c=16, $y_t = (y_0 - \frac{c}{1-b})\, b^t + \frac{c}{1-b}$

$$y_t = \left(5 - \frac{16}{1+7}\right)(-7)^t + \frac{16}{1+7} = 3(-7)^t + 2$$

To check the answer, substitute *t=0* and *t=1* in the solution.

$$at\ t = 0, y_0 = 3(-7)^0 + 2 = 5$$
$$at\ t = 1, y_1 = 3(-7)^1 + 2 = -19$$

Then, substitute $y_1 = -19, for\ y_t\ and\ y_0 = 5$ for y_{t-1} in the original equation

$$y_t = -7y_{t-1} + 16$$

$$-19 = -7(5) + 16 \Rightarrow -19 = -19 \text{ (compare)}$$

7.3. The Dynamic Stability of Equilibrium

In the continuous- time case, the dynamic stability of equilibrium could be done using stability conditions.

A. Monotonic difference equation

The dynamics or changes generated by the equation is ***monotonic*** if and only if the sign $(y_t - y_{t-1})$ is the same for any t.

Example: $y_t = ½\ y_{t-1} + 300$

Sequence of y can determine by repeated use of difference equation.

- **If $y_0 = 400$**
- $y_1 = ½\ x_0 + c = ½\ 400 + 300 = 500$
- $y_2 = 1/2\ \ y_1 + c = ½\ 500 + 300 = 550$
- $y_3 = ½\ y_2 + c = ½\ 550 + 300 = 575$
- $y_4 = ½\ y_3 + c = ½\ 575 + 300 = 587\ ½$
-

B. Oscillatory difference equation

The value of y changes $(y_t - y_{t-1})$ alters from one period to another is called oscillatory.

Consider the equation: $y_t = -\ ½\ y_{t-1} + 300$,

- **Assume $y_0 = 400$**
- $y_1 = -\ ½\ y_0 + c = -\ ½\ 400 + 300 = 100$
- $y_2 = -\ 1/2\ \ y_1 + c = -\ ½\ 100 + 300 = 250$
- $y_3 = -\ ½\ y_2 + c = -\ ½\ 250 + 300 = 175$
- $y_4 = -\ ½\ \ y_3 + c = -\ ½\ 175 + 300 = 212\ ½$

C. Steady state

One part of the solution of a difference equation is the steady state value of the endogenous variable for a given value of exogenous variables.

The steady state value condition satisfies: $y_\infty = ay_\infty + c$ *or* $y_\infty = y_\infty = \frac{1}{1-ac}\, y_t = ay_{t-1} + c$,

The steady state value y_∞ exists if the sequence (y_t) converges. If there is no unique real number y_∞, then the sequence diverges.

Example: If $y_t = ½\ y_{t-1} + 300$

The steady state value of the difference equation is:

$y_\infty = (1/1-½)300 = 600$ where, a = ½, c = 300

Since the first five numbers of the sequence converge towards the steady value of $y_\infty = 600$, $(y_t - y_{t-1})$ decreases with increase in t, it is ***stable***.

$y_\infty = 1/\ 1- (-½) 300 = 200$

When $(y_t - y_{t-1})$ increases with increase in t, the sequence generated will diverges as time progress.

Example: If $y_t = 2y_{t-1} - 300$, assume $y_0 = 400$

The *steady value* for this equation is$x_\infty =$ 1/1-(2) (-300) = 300

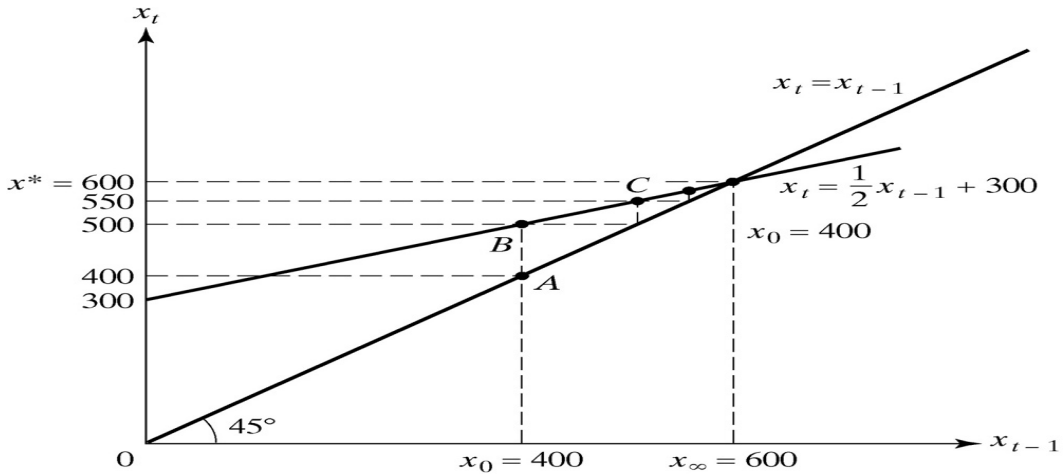
- $y_1 = 2\ y_0 + c = 2(\ 400) - 300 = 500$
- $y_2 = 2y_1 + c = 2(\ 500) - 300 = 700$
- $y_3 = 2y_2 + c = 2(700) -\ 300 = 1100$
- $y_4 = 2y_3 + c = 2(1100) -\ 300 = 1900$

The sequences are increasing with increase in t- the equation is unstable and diverging.

The Phase Diagram

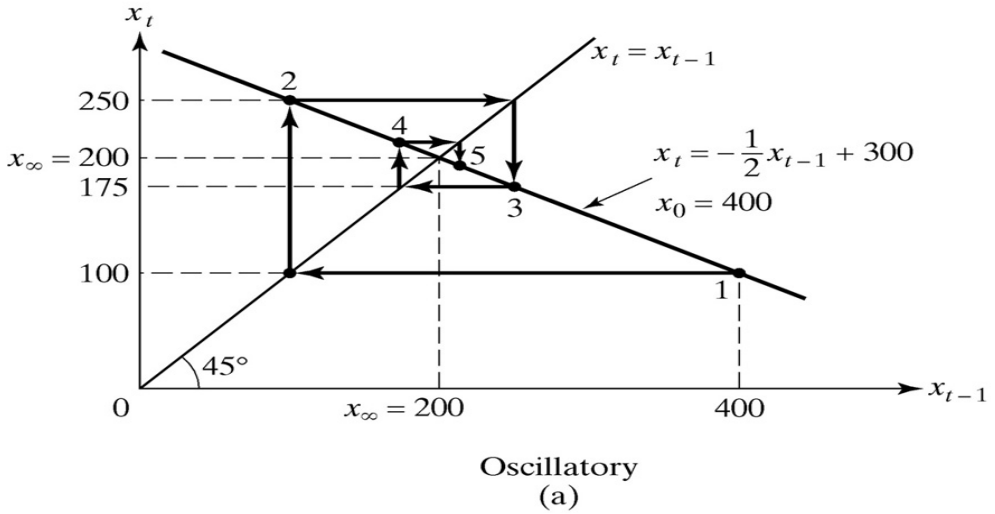
To illustrate the dynamic path of a variable generated by a difference equation is through the use of phase diagram.

Phase diagram, for $y_t = ½\ y_{t-1} + 300$- stable



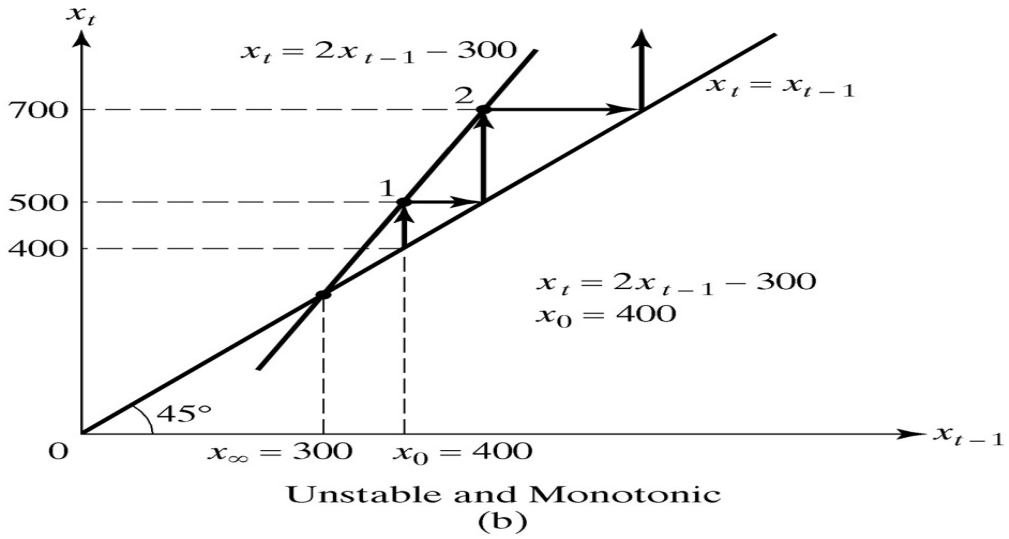
- *A difference equation, in a phase diagram is stable if the graph of the equation is strictly increasing and it crosses the 45 ^{0} line from above when moving from left to right.*
- *A difference equation is unstable in a phase diagram, if the graph of the equation is strictly increasing and crosses the 45 ^{0} line from below when moving from left to right.*

Phase diagram for $y_t = -½\ y_{t-1} + 300$-stable



Oscillatory
(a)

Unstable value $y_t = 2y_{t-1} - 300$



Unstable and Monotonic
(b)

Activity:

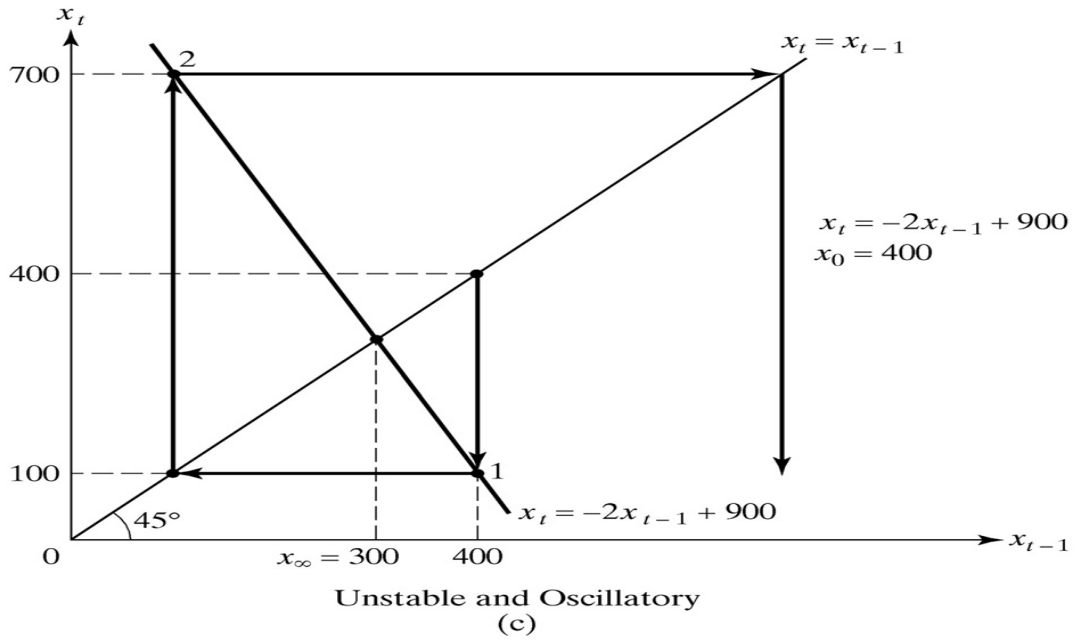
For the difference equation: $y_t = -2y_{t-1} + 900$, draw the phase diagram and comment on stability

Check your progress

The steady value is = 1/1-(-2) (900) = 300

The sequences of the endogenous variable become;

- $y_0 = 400$
- $y_1 = -2\,y_0 + c = -2(400) + 900 = 100$
- $y_2 = -2y_1 + c = -2(100) + 900 = 700$
- $y_3 = -2y_2 + c = -2(700) + 900 = -500$
- $y_4 = -2y_3 + c = -2(-500) + 900 = 1900$



Unstable and Oscillatory
(c)

Summary

- ***Stable:*** the sequence generate by the difference equation $y_t = ay_{t-1} + c$, converges to the steady state value ($y_\infty = \frac{1}{1-ac} y_t$). If –1 < a< 1, the equilibrium solution is stable.
- ***Unstable:*** if the sequence generated through difference equation $y_t = yx_{t-1} + c$ diverges from the steady state value, if a > 1 or a < -1 and the initial value does not equal, the equilibrium solution($y_\infty = \frac{1}{1-ac} y_t$) is unstable.

Stability condition using the general formula for the difference equation:

$y_t = \left(y_0 - \frac{c}{1-b}\right) b^t + \frac{c}{1-b}$, it can be expressed in a more general formula as:

$y_t = Ab^t + a, A = y_0 - \frac{c}{1-b}$ & $a = \frac{c}{1-b}$. The time path y_t will be dynamically stable only if the complimentary function $Ab^t \to 0 \; as \; t \to \infty$

Assuming for the moment $A = 1$ and $a = 0$, the exponential expression b^tgenerates ***seven*** different time paths depending on the value of b (which can range from $-\infty$ to ∞).

1. If $b > 1, b^t$ increases at an increasing rate as t increase. Thus, the time path will explode (diverge) and move farther and farther away from the horizontal line.

Example: If $b = 3$, the value of b^t at different time periods will be

t	0	1	2	3	4
b^t	1	3	9	27	81

2. If $b = 1$, b^t=1 for all values of t, thus the time path is represented by a horizontal line.
3. If $-1 < b < 1$, (i.e. b is a positive fraction), b^t decreases as t increases, thus the time path would be damped and moves towards the equilibrium line (horizontal line).

Example: if $b = 1/3$

T	0	1	2	3	4
b^t	1	$1/3$	$1/9$	$1/27$	$1/81$

4. If $b = 0$, $b^t = 0$ for all value of t.

5. If $-1 < b < 0$, i.e. (b is negative fraction), b^t oscillates between positive and negative values, and the time path will draw closer and closer to the horizontal line (equilibrium).

Example: if $b = -1/3$

T	0	1	2	3	4
b^t	1	$-1/3$	$-1/9$	$-1/27$	$-1/81$

6. If $b = -1$, b^t oscillates between +1 and -1.

Example:

T	0	1	2	3	4	5
b^t	1	-1	1	-1	1	-1

7. If $b < -1$, b^t will oscillate and move farther and farther away from the horizontal line equilibrium.

Example:

T	0	1	2	3	4	5
b^t	1	-2	4	-8	16	-32

Summary:

t

If $|b| > 1$ and $b < -1$, then the time path explodes (diverges).
If $|b| < 1$ *or* $-1 < b < 1$, the time path converges.
If $b > 0$, the time path is non-oscillatory.
If $b > 0$, the time path is oscillatory.

The Roles of A and a

a) The magnitude of A produces a scale effect without changing the configuration of the time path, i.e., the Value of A serves to scale up or down the value of b^t(the time path).
b) The sign of A produces a mirror effect. If $A = -1$, a mirror image of the time path of $b^t$with respect to the horizontal axis will be produced.
c) The magnitude of 'a' affects the vertical intercept of the graph and the graph shifts up or down depending on the sign of 'a'. This in no way affects the convergence or divergence of the time path, but alters the level with respect to which the convergence or divergence is gauged

Activity: Dear students please try the following and check your progress

1. Given a difference equation $y_{t+1} - \frac{1}{3}y_t = 6$ (where $y_0 = 1$)
   (a) Solve the difference equation.
   (b) Check the answer using $t = 0$ and $t = 1$, an
   (c) Comment on the nature of the time path.
2. Given $\Delta g_t = 14$,
   (a) Solve the difference equation
   (b) Check the answer using $t = 0$ and $t = 1$, and
   (c) Comment on the nature of the time path.

Check Your Progress

1.

(a) From the difference equation, since $b = \frac{1}{3}$ and $c = 6$, using the formula for the definite

$$y_t = \left(y_0 - \frac{c}{1-b}\right)b^t + \frac{c}{1-b} = \left(1 - \frac{6}{1-\frac{1}{3}}\right)\frac{1}{3}^t + \frac{6}{1-\frac{1}{3}} = -8\left(\frac{1}{3}\right)^t + 9;\ \text{hence}, y_t = -8\left(\frac{1}{3}\right)^t + 9$$

(b) Given $y_t = -8\left(\frac{1}{3}\right)^t + 9$, at $t = 0$, $y_0 = -8\left(\frac{1}{3}\right)^0 + 9 = 1$

$$\text{at } t = 1, y_1 = -8\left(\frac{1}{3}\right)^1 + 9 = {}^{19}/_{3}$$

Then, substituting $y_1 = {}^{19}/_{3}$ for y_{t+1} and $y_0 = 1$ for y_t in the original equation:

$${}^{19}/_{3} - {}^{1}/_{3}(1) = 6 \Rightarrow \frac{18}{3} = 6 \text{ (compare)}$$

(c) Since the base, $b = \frac{1}{3} > 0$, y_t is non − oscillatory.

$$|b| = \frac{1}{3} < 1, y_t \text{ is convergent.}$$

Therefore, the time path is non−oscillatory and convergent.

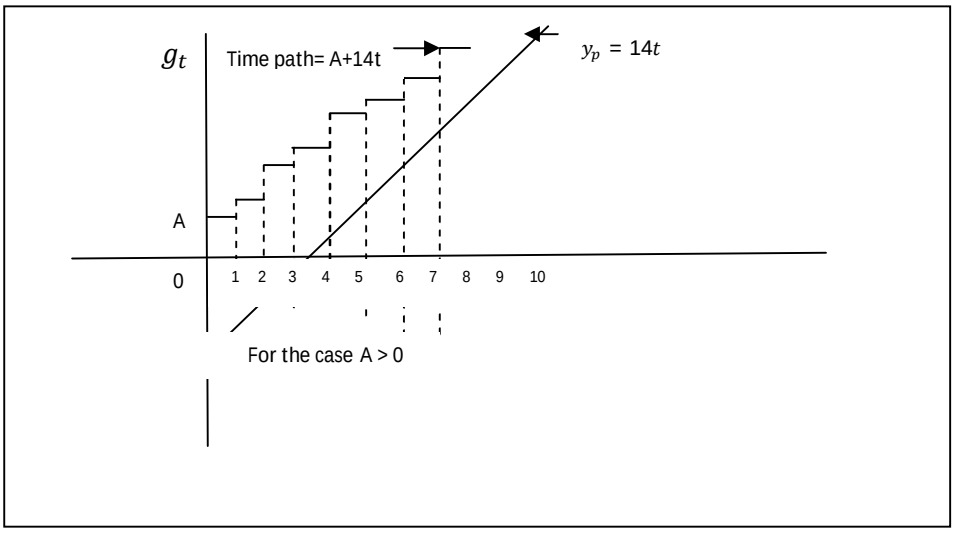
2.

(a) Rewriting the equation: $\Delta g_t = g_{t+1} - g_t = 14$, since $b = 1$ and $c = 14$, the formula for the general solution gives:

$$g_t = A + ct = A + 14t \Rightarrow g_t = A + 14t$$

(b) At $t = 0, g_0 = A + 14(0) = A$ and at $t = 1, g_1 = A + 14(1) = A + 14$
Substituting in the original equation: $g_{t+1} - g_t = 14 \Rightarrow A + 14 - A = 14$ (compare)

(c) Since $b = 1 > 0, g_t$ is non-oscillatory. And if $A \neq 0, |b| = 1, g_t$ is neither divergent nor convergent because it is a moving equilibrium (it changes from time to time).



7.4. Economic Applications of Dynamic Optimization

7.4.1. Uses of Differential Equations in Economics and their applications

Differential equations are used to determine the conditions for dynamic stability in micro-economic models of market equilibria and to trace the time path of growth under various conditions in macro-models. Given the growth rate of a function, differential equations enable the economist to find the function whose growth is described. Furthermore, from point elasticity it enables us to estimate the demand function.

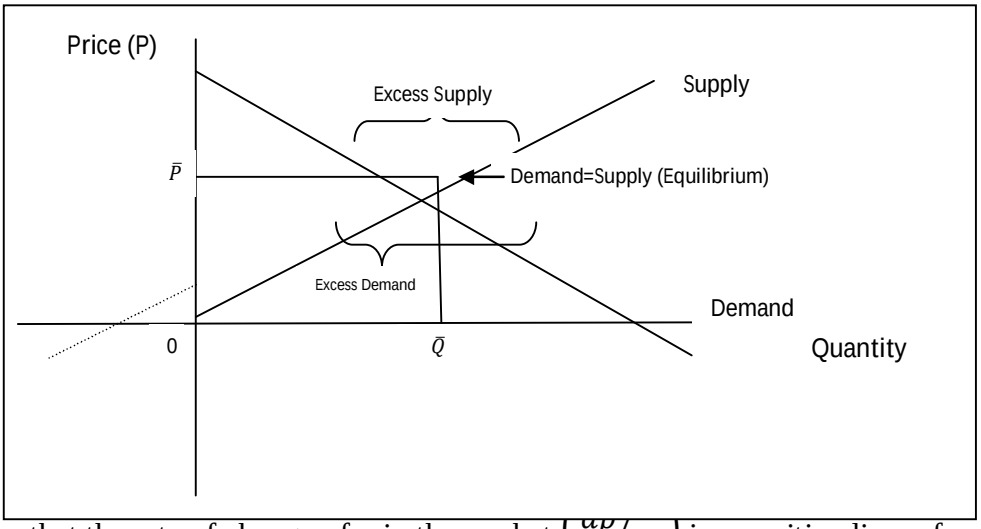
Examples:

1. Given the demand function $Q_d = c + bP$ and the supply function $Q_s = g + hP$ determine the condition for price stability in the market (i.e. under what conditions $p\ (t)$ will converge to $\bar{p}$ (equilibrium price) as time$\rightarrow \infty$).

Either $\begin{Bmatrix} Q_s = g + hP, g < 0 \; and \; h > 0 \\ Q_s = -g + hP, g, h > 0 \end{Bmatrix}$ $\begin{Bmatrix} Q_d = c + bP\,, c > 0, b < 0 \\ Q_d = c - bP\,, c, b > 0 \end{Bmatrix}$ are correct.

Solution:

First find equilibrium price ($\bar{p}$): $Q_d = Q_s \Rightarrow c + bP = g + hP \Rightarrow (h-b)p = c - g \Rightarrow \bar{p} = \frac{c-g}{h-b}$



Assume that the rate of change of p in the market $\left({}^{dp}/_{dt}\right)$ is a positive linear function of excess demand$(Q_d - Q_s)$:

${}^{dp}/_{dt} = m(Q_d - Q_s), where\, m > 0$

$m \equiv$ adjustment coefficient substituting the values (parameters)of Q_d and Q_s ; we get:

${}^{dp}/_{dt} = m(c + bp - g - hp) = m(c - g + (b-h)p)$

Rearranging to fit the general format of differential equation:$\left({}^{dy}/_{dt} + vy = z\right)$

$${}^{dp}/_{dt} + m(h-b)p = m(c-g)$$

Then, since v= $m(h-b)p$, and$z = m(c-g)$, the general solution becomes

$(y(t) = Ae^{-vt} + \frac{z}{v})$: $p(t) = Ae^{-m(h-b)t} + \frac{c-g}{h-b}$

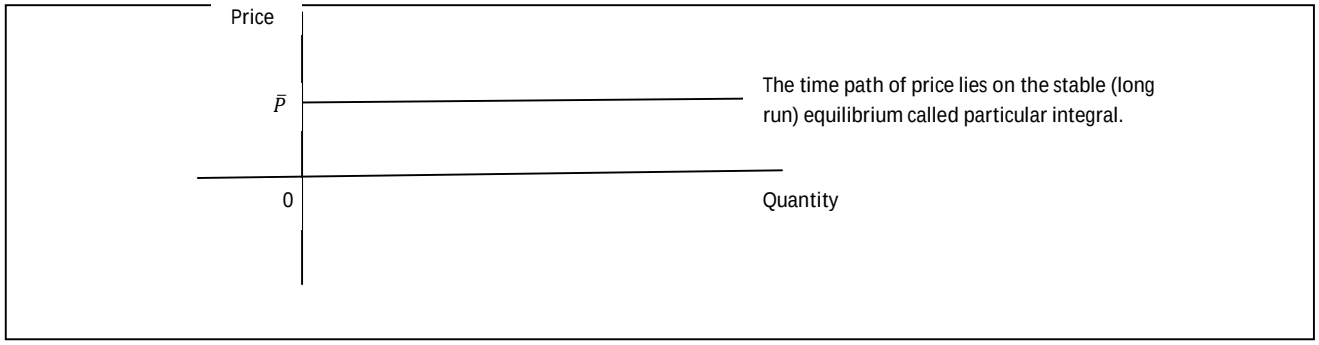
The definite solution will be: $p(t) = (p(0) - \frac{c-g}{h-b})e^{-m(h-b)t} + \frac{c-g}{h-b}$

Using the relation $\bar{p} = \frac{c-g}{h-b}$: $p(t) = (p(0) - \bar{p})e^{-m(h-b)t} + \bar{p}$

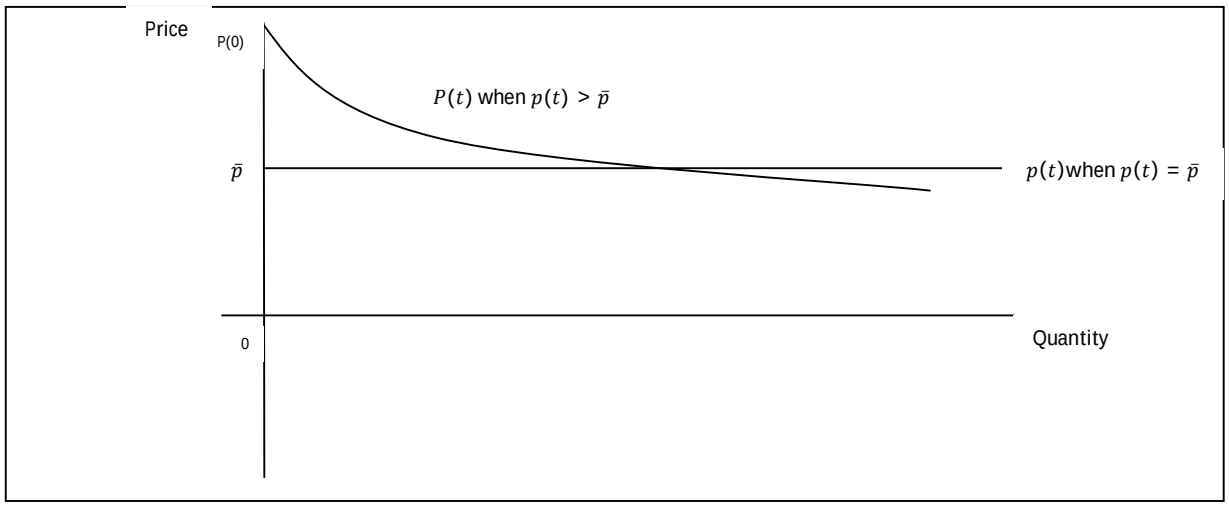
Let $k = m(h-b) \Rightarrow p(t) = (p(0) - \bar{p})e^{-kt} + \bar{p}$

Whether $p(t)$ tends to $\bar{p}$ as $t \to \infty$ depends on whether the exponential expression $e^{-kt} \to 0$ as $t \to \infty$. since $p(0)$ and $\bar{p}$ are constants and $k > 0$ $(k = m(h-b) > 0)$, $e^{-kt} = \left(\frac{1}{e^{-kt}}\right) \to 0 \; as \; t \to \infty$. Consequently, the time path will indeed lead the price towards the equilibrium position. In this case, the equilibrium is said to be dynamically stable. But, depending on the relative magnitude of $p(0)$ and $\bar{p}$, the above solution yields three possible time paths.

Case 1: If $p(0) = \bar{p}$ (the initial price is equal to the equilibrium price), the first term on the right disappears and $p(t) = \bar{p}$. The time path is horizontal and adjustment is immediate.

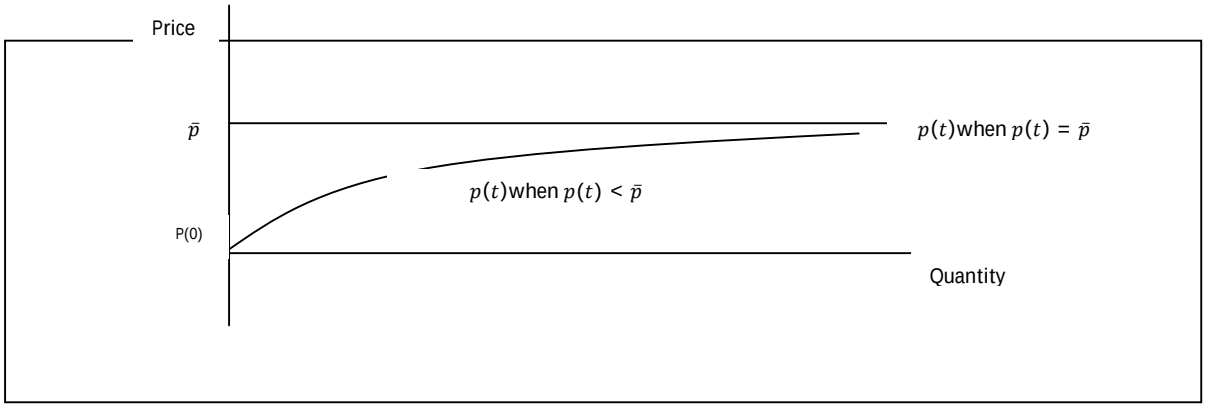


Case2: If $p(0) > \bar{p}, p(0) - \bar{p} > 0. thus \; the \; time \; path \; p(t) > \bar{p}$ and (t) approaches $\bar{p} \; from \; above \; as \; t \to \infty [since \; (p(0) - \bar{p})e^{-kt} \to 0 \; as \; t \to \infty]$



Case 3: If $p(0) < \bar{p}$, $p(0) - \bar{p} < 0$. Thus, $p(t) < \bar{p}$ and approaches it from below as $t \to \infty$.

In general, to have dynamic stability, the derivation of the time path from equilibrium must either be identically zero or steadily decreasing with time. Or, dynamic stability requires asymptotic vanishing of the complimentary function as t becomes infinite.



Activity: Dear students please try and check your progress

Find the demand function $Q = f(p)$ if $e = -k$, a constant.

Check Your Progress:

We know that the price elasticity of demand (E) is given by:

$$e = \frac{dQ}{dP}.\frac{Q}{P} = -k \text{ (we are trying to derive } \frac{dy}{dt} + vy = z)$$

$$\frac{dQ}{dP} = -k\frac{Q}{P}$$

Separating the variables: $\frac{dQ}{Q} = -k\frac{dP}{P}$

Integrating both sides: $\int \frac{dQ}{Q} = \int -k\frac{dP}{P} \ln Q = -k \ln p + c$

$\Rightarrow \ln Q + k \ln p = c \Rightarrow \ln Q + \ln p^{k} = c \Rightarrow \ln(Qp^{k}) = c \Rightarrow Qp^{k} = e^{c}$

We can say: $Qp^{k} = c_1$ where $c_1 = e^{c} \Rightarrow c_1 p^{-k}$

Example:

Consider a differential equation, $u^*(t) = - ½ u(t) + 2$

Where, the initial value $u(0) = 1$. The steady state value of this equation is the point where $u^*(t) = 0$ which is $u_\infty = - 2/- ½ = 4$.

The definite solution is: $u(t) = \left(1 + \frac{2}{-\frac{1}{2}}\right) \cdot e^{-½t} - \frac{2}{-\frac{1}{2}}$

$= - 3e^{-½ t} + 4$

This equation is **stable** since, for any initial value $u(t) \rightarrow 4$ as $t \rightarrow \infty$, over the time it converges to the steady state value.

Applying this solution to the linear differential equation: $V^*(t) = 2v(t) – 4$

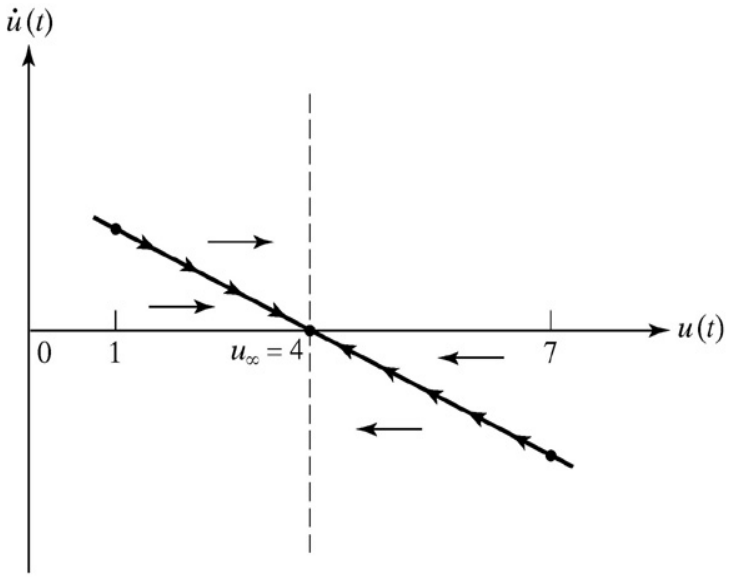
With the initial value $v(0) = 1$, we obtain, $v(t) = \left(1 + -\frac{4}{2}\right) \cdot e^{2t} - -\frac{4}{2}$

$= - e^{2t} + 2,$

This solution is unstable since v(t) diverges further and further from its steady state value of 2 as $t \rightarrow \infty$. In particular $v(t) \rightarrow - \infty$ as $t \rightarrow \infty$, since $v(0) < 2$ for any $v(0) > 2$.

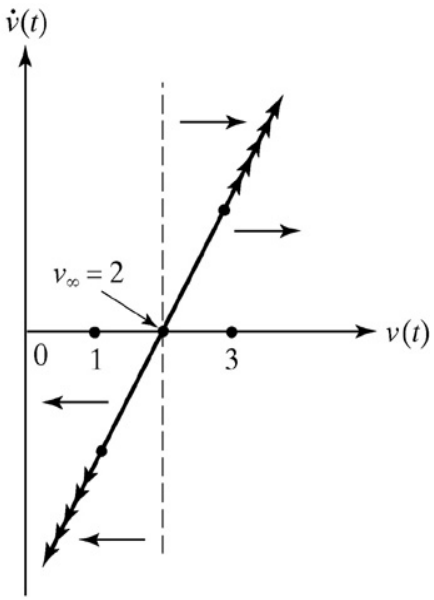
If instead for any $v(0) > 2$, $v(t) \rightarrow \infty$ as $t \rightarrow \infty$. The only situation where $\{v(t)\}$ is bounded occurs when $v(0) = 2$ in which case $v(t) = 2$ for all t. In each case the graph of the differential equation crosses the horizontal axis at the steady state value of the equation $u_\infty = 4$ in fig.(a) and $v_\infty = 2$ in fig(b).

Analyzing a differential equation with a constant term using the phase diagram: Qualitative solution using phase diagram for linear differential equation



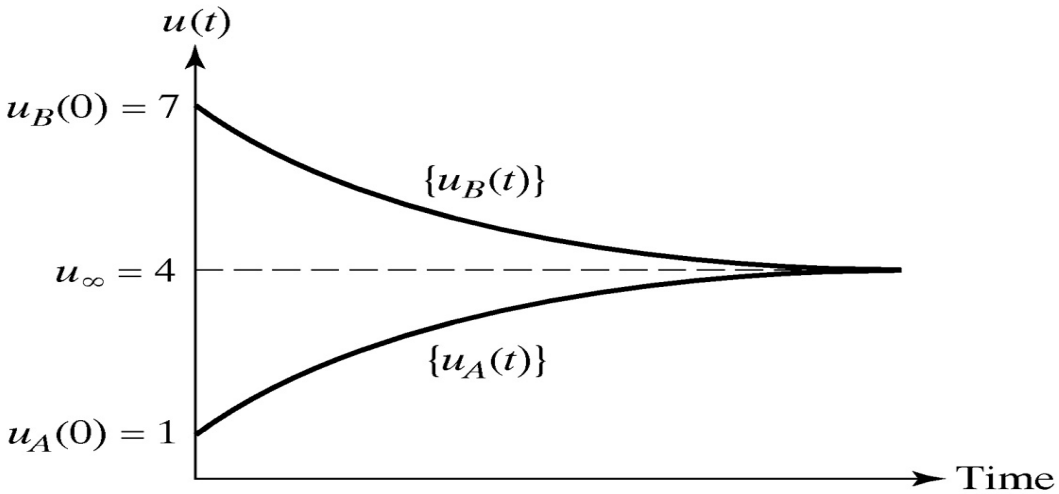
$$\dot{u}(t) = -\frac{1}{2}u(t) + 2$$

(a)



$$\dot{v}(t) = 2\,v(t) - 4$$

(b)



(a)

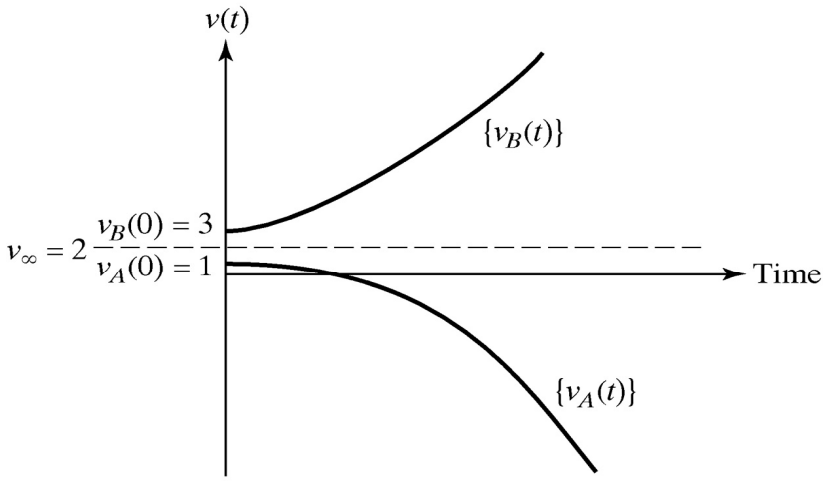
It is clear from the diagram that the motion of u(t) from the initial value u_A (0) = 1, is towards u_∞. The arrows of motion point towards the right for any value of u(t) less than the steady state since , for any u(t) < 4. $u^*(t) = -\frac{1}{2}u(t) + 2 > 0$

It can also see that the motion of u(t) from the initial value of $u_B(0) = 7$, is towards u_∞. The arrows of motion point towards the left for any u(t) > 4, in this case $u^*(t) = -\frac{1}{2}u(t) + 2 < 0$.

The arrows of motion in this phase diagram indicates that this differential equation is stable since u(t) moves towards steady state value u_∞, no matter what its initial value.

Unstable solution: $V^*(t) = 2v(t) - 4$

Initial condition $v_A(0) = 1$ and $v_B(0) = 3$, the two time paths are diverging from the steady state value of $v_\infty = 2$



(b)

7.4.1.1. Application of linear first order differential equations

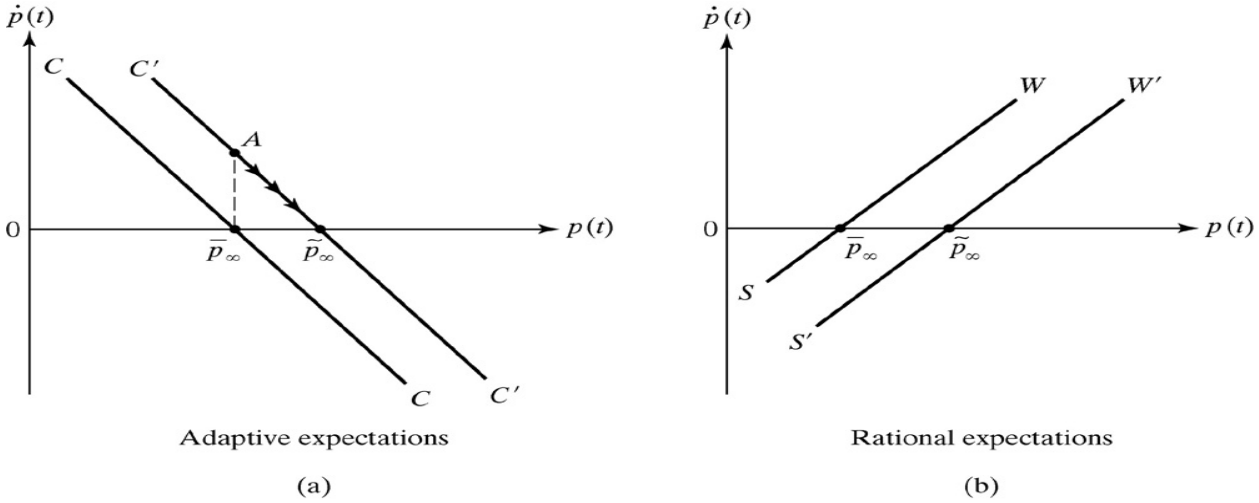
Illustration: Stability in Monetary models expected inflation and actual inflation

The two common expectations in inflation are:

***Adaptive expectations*-** overestimate or underestimate the level of inflation

Rational expectations- is an alternative assumption which assume that people use all available information and guess on average inflation correctly.

The relationship in the monetary model is the money demand equation is: $m(t) - p(t) = -\lambda\pi(t)$ where m(t) is the logarithm of money supply, p(t) is the logarithm of price level and $\pi(t)$ is the logarithm of expected inflation rate , all at moment t and λ is a positive constant. The relation between actual inflation is p(t) and the expected inflation is $\pi(t)$ is as given.



A continuous time equation that reflects the assumption of adaptive expectation is: $\pi^*(t) = \alpha(p^*(t) - \pi(t))$. Where α = the speed with which the expected inflation responds to the difference between actual and the expected inflation ($\alpha > 0$). This equation shows that if actual inflation $p^*(t)$ is greater than expected inflation $\pi(t)$, then there is an increase in the expected inflation . This equation can be rewritten as: $p^*(t) = 1/\alpha\ \pi^*(t) + \pi(t)$. Differentiating the money demand equation with respect to time: $m^*(t) - p^*(t) = -\lambda\pi^*(t)$

The first order differential equation is: $p^*(t) = -\alpha/1-\alpha\lambda\ p(t) + \alpha/1-\alpha\lambda\ m(t) + 1/1-\alpha\lambda\ m^*(t)$

This is given in the phase diagram as CC which has a negative slope and is stable if $\alpha\lambda < 1$. Price level converges over time to the steady state value labeled as $\bar{p}_\infty$

If the money supply increased from ṁ to ḿ, the differential equation is the new one shown as C'C'. From the original steady state $\bar{p}_\infty$, the inflation rate jumps from zero to A and the price level slowly evolves to its newly long run level of $\tilde{p}_\infty$.

- The money demand equation shows that during the transition from $\bar{p}_\infty$ to $\tilde{p}_\infty$, expected inflation equals $1/\lambda\ (p(t) - \tilde{m})$.
- Expected inflation is always negative during the transition since the steady state of price level is $\tilde{p}_\infty = \tilde{m}$ and up until the steady state is reached , $p(t) < \tilde{m}$.
- Actual inflation is positive during the transition and so people consistently expect a lower inflation rate than the one that occurs

It gives a perfect foresight about the future value of a variable exactly equals the actual outcome. Assumption is that $\pi(t) = p^*(t)$. Substituting this equation for expected inflation into the money demand equation we get the differential equation as: $p^*(t) = 1/\lambda\ p(t) - 1/\lambda\ m(t)$

This is shown in the diagram the line SW which assumes that m(t) = $\bar{m}$. This line SW crosses the horizontal axis at the steady state value of the differential equation $\bar{p}_{\infty} = \bar{m}$. Positive slope indicates that it is an unstable difference equation

If the money supply increases to $\hat{m}$, the differential equation line shifts to line S'W', which has the steady state value of the price level $\bar{p}_{\infty} = \hat{m}$. There is a permanent and increasing deflation as we move down the S'W' schedule, even though the money supply is constant. The price level jumps from $\bar{p}_{\infty}$ to $\hat{p}_{\infty}$, but there are no dynamics

The ***perfect foresight model*** with a constant money supply m is $p(t) = (p(0) - m)\, e^{1/\lambda} + m$ which has the positive coefficient $1/\lambda$. If p(0) does not equal to m , then the price level either goes to positive infinity if p(0) > m or negative infinity if p(0) <m, even though the money supply is constant. Price level jumps such that p(t) = m at any moment.

The perfect foresight model at time t equals $p(t) = 1/\lambda \int_{s-t}^{\infty} m(s) e^{t-s/\lambda}\, ds$. This shows that the price at any moment represents the discounted value of the time path of the expected future money supply. Any expected future increase in the money supply will increase the current price level. The effect of an expected increase in the money supply on the current price level is larger, closer to present it occurs.

An application of the backward solution to the differential equation for the price level is the adaptive expectations where the relevant period extends far into the past give us to the solution to the adaptive expectation model.

7.4.1.2. Non Linear differential equation and phase diagram

In some equations there may be more than one crossing and hence more than one solution. Diagrammatically the stability of the solution is indicated by arrows of motion, if arrows point towards steady state, the solution is stable. A phase diagram of differential equation is expressed as y'. A phase diagram can be used to study the properties of non linear differential equations with respect to stability.

Any steady state point where the differential equation crosses the horizontal axis from above moving from left to right is stable and any steady state point where the differential equation crosses the horizontal axis from below is unstable.

NB: If dy'/dy < 0 equation is stable and if it is > 0 , it is unstable.

Example: $y' = 8y - 2y^2$.

a. Find the steady state solution

b. Determine the function is at maximum or minimum

c. Sketch the phase diagram

d. Find the slope of phase diagram

Solution:

a. Set $y' = 0$ and solve algebraically

$y' = 8y - 2y^2 = 0$

$= 2y(4- y) = 0$ hence $y^* = 0$, $y^* = 4$ = steady state solutions

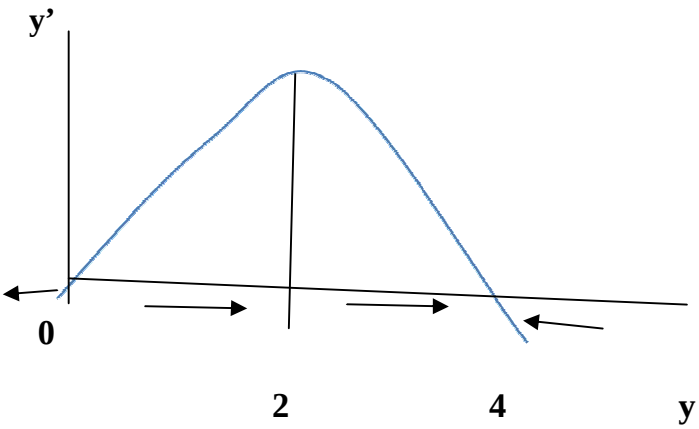
The phase diagram will pass through horizontal axis at $y^* = 0$ and $y^* = 4$

b. To determine the function is at maximum or minimum

$dy'/dy = 8- 4y = 0$ so that $y = 2$ critical value

$d^2 y/dy^2 = -4 < 0$, concave and maximum

c. Phase diagram



Where the graph lies above horizontal axis, $y^* > 0$ and arrows point to the right and graph lies below horizontal axis, $y^* < 0$ and arrows point to left. The arrows point away from the inter-temporal equilibrium at $y^* = 0$ is unstable and $y^* = 4$ is stable.

d. Slope of the phase diagram is positive at $y^* = 0$ and negative at $y^* = 4$ and hence stable at this point. Evaluate the steady state level at $y_1^* = 0$, $y_2^* = 4$

- Dy'/dy= 8- 4(0)=4 > 0 , y_1^* - is not stable
- Dy'/dy= 8- 4(4) = -8 < 0 , y_2^* = 4 -is stable

Activity: Repeat steps (a-d) for the non-linear differential equation: $\dot{Y}= 3y^2 – 18y$

Check your progress

a. Set $\dot{Y}= 0$, $3y^2 – 18y = 0$ which means

3y(y-6)=0 so that $\bar{Y}_1= 0$ and $\bar{Y}_2 = 6$, the inter temporal equi; solution where phase diagram crosses the horizontal axis.

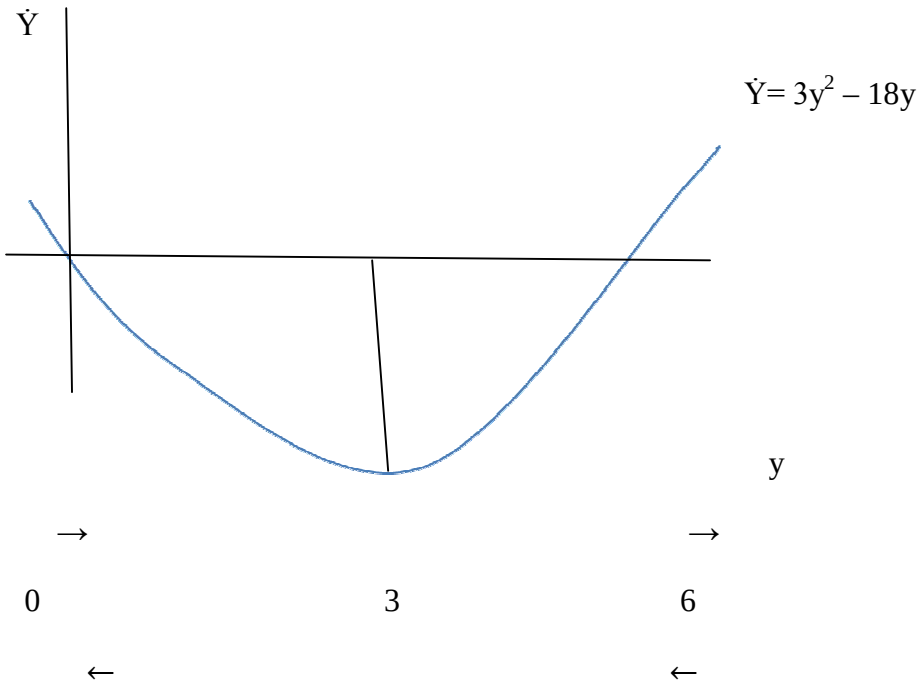
b. Find the critical value by taking the derivative and solve.

$\dot{Y}= 3y^2 – 18y$

d'y/dy = 6y- 18 so that y = 3 the critical value and

$d^2y/dy^2 = 6 > 0$, the minimum and convex

The arrows of motion to $\bar{Y}_1= 0$ and away from $\bar{Y}_2 = 6$. , $\bar{Y}_1= 0$ stable and $\bar{Y}_2 = 6$ is unstable. Phase diagram with negative slope is at $\bar{Y}_1= 0$ must be stable and $\bar{Y}_2 = 6$ with slope is unstable



- $d\dot{Y}/dy = 6y - 18$, $d\dot{Y}/dy(0) = 6(0)-18 = -18 < 0$ stable = $\bar{y}_1$
- $d\dot{Y}/dy(6) = 6(6) - 18 = 18 > 0$ unstable = $\bar{y}_2$

7.4.2. Uses of Difference Equations in Economics

7.4.2.1. Cob Web Model

The change from continuous time to discrete time produces no effect on the fundamental nature of dynamic analysis, although the formulation of the problem must be altered. Basically, our dynamic problem is still to find a time path for some given pattern of change of a variable y over time, where t now allowed to taking only *integer values*.

To illustrate the use of difference equations in economic analysis, we use a market model of a single commodity (cobweb model). In some markets, particularly agricultural markets, supply cannot immediately expand to meet increased demand. Crops have to be planted and grown and livestock takes time to raise.

Some manufactured products can also take a while to produce when orders suddenly increase. The cobweb model takes into account this delayed response on the supply side of a market by assuming that quantity supplied now (Q_t^s) depends on the ruling price in the previous time period (P_{t-1}) i.e. $Q_t^s = f(P_{t-1})$.

Consumer demand for the same product (Q_t^d), however, is assumed to depend on the current price, i.e. $Q_t^d = f(P_t)$

This is a reasonable picture of many agricultural markets. The quantity offered for sale this year depends on what was planted at the start of the growing season, which in turn depends on last year's price. Consumers look at current prices, though, when deciding what to buy.
When such a supply function interacts with a demand function of the form: $Q_{d,t} = f(p_t)$ interesting dynamic price patterns will result.

The cobweb model also assumes that: *the market is perfectly competitive and supply and demand are both linear schedules.*

Before we go any further, it must be stressed that this model does *not* explain how price adjusts in all competitive markets, or even in all perfectly competitive agricultural markets. It is a simple model with some highly restrictive assumptions that can only explain how price adjusts in these particular circumstances.

Some markets may have a more complex lag structure, e.g. $Q_t^S = f(P_{t-1}, P_{t-2}, P_{t-3})$, or may not have linear demand and supply and hence the cobweb assumptions do not apply. Having said all this, the cobweb model can still give a fair idea of how price and quantity adjust in many markets with a delayed supply.

Using the linear version of the lagged supply and un-lagged demand function we get the market model with the following equations:

$Q_{dt} = c + bp_t \ (c > 0, b < 0)$

$Q_{st} = g + hp_{t-1} \ (g < 0, h > 0)$

$Q_{dt} = Q_{st}$ [equilibrium conditions]

Using the last equation, the model can reduced to a single first order difference equation:

$Q_{dt} = Q_{st}$

$c + bp_t = g + hp_{t-1} \Rightarrow bp_t = (g - c) + hp_{t-1}$

$$p_t = \frac{g-c}{b} + \frac{h}{b}p_{t-1} \Rightarrow p_t - \frac{h}{b}p_{t-1} = \frac{g-c}{b}$$

This is what is known as a *'linear first-order difference equation'*. A difference equation expresses the value of a variable in one time period as a function of its value in earlier periods; in this case: $P_t = \mathrm{f}(P_{t-1})$

Shifting the periods forward by one period: $p_{t+1} - \frac{h}{b}p_t = \frac{g-c}{b}$

The definite solution is given as $\left[p_t = \left(p_0 - \frac{c}{1-b}\right)b^t + \frac{c}{1-b}\right]$:

$$p_t = \left(p_0 - \frac{g - c/b}{1 - h/b}\right)(h/b)^t + \frac{g - c/b}{1 - h/b}$$

$$p_t = \left(p_0 - \frac{g-c}{b-h}\right)\left(\frac{h}{b}\right)^t + \frac{g-c}{b-h} \quad let\ \bar{p} = \frac{g-c}{b-h}$$

Hence, $p_t = (p_0 - \bar{p})\left(\frac{h}{b}\right)^t + \bar{p}$

Two points may be observed in regard to this time path:

i. The significance of the expression $(p_0 - \bar{p})$: (same as A in Ab^t)
   a) Its sign tells us whether the time path commences (starts) above or below equilibrium (mirror effect)
   b) Its magnitude helps us to determine how far above or below the time path is (scale effect)

ii. Under 'normal' demand and supply condition $(b < 0 \; and \; h > 0)$ so that $\frac{h}{b} < 0$, indicating that the time path is oscillatory.

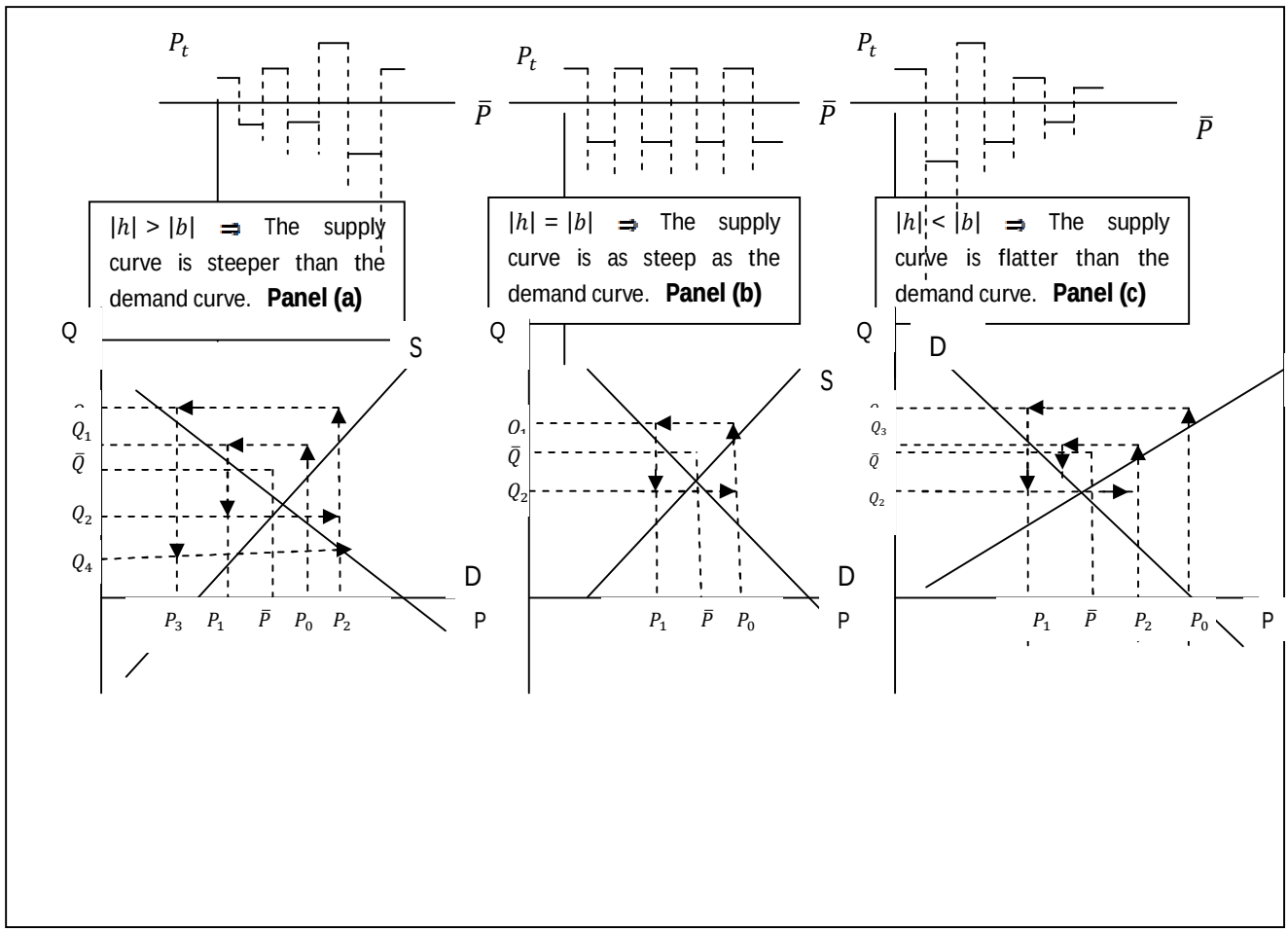
a. $if \; |h| > |b|, \left|\frac{h}{b}\right| > 1,$ and p_t explodes (diverges).

b. $if \; |h| = |b|, \left|\frac{h}{b}\right| = -1,$ and p_t oscillates uniformly.

c. $if \; |h| < |b|, \left|\frac{h}{b}\right| < 1,$ and p_t converges and approaches $\bar{p}$.

In short, for stability the supply curve must be flatter than the demand curve. (or, $h<b$ in magnitude). Assuming $p_0 > \bar{p}$, the time path of price and quantity can be illustrated as follows.

Graphic Illustration



Given an initial price(p_0) (where $p_0 > \bar{p}$), quantity supplied in the next period (period 1) will be Q_1. But, at this point (Q_1), supply exceeds demand, so that there is a tendency for price to fall. Consequently, the market clears at p_1.

-Assuming p_1 to prevail, period 2's supply will be limited to $Q_2(Q_2 < Q_1)$.

-At Q_2 , demand exceeds supply, so there is a tendency for p to rise. As a result the market clears at p_2.

- Again, assuming p_2 to prevail, the supply in period 3 will be Q_3(where$Q_3 > Q_2$). At Q_3 , since supply>demand, the market clears at p_3 . Replacing this reasoning we can trace out the prices and quantities in subsequent periods by following the arrowheads, thereby, spinning a cob web around the demand and supply curves.

The time path of price is oscillatory and explosive when *S* is steeper than *D or (h>b).*

Fig (b): p_t oscillates uniformly so that equilibrium is unstable. p_t Neither diverges nor converts.

Fig (c): p_t is oscillatory and convergent when *S* is flatter than *D (h<b)* so that equilibrium of time path (p_t) will be dynamically stable.

Although in theoretical models of unstable markets price 'explodes' and the market collapses, this may not happen in reality if:

- producers learn from experience and do not simply base production plans for the next period on the current price,
- supply and demand schedules are not linear along their entire length,
- government intervention takes place to support production.

Examples:

1. Given the demand function
   $Q_t = 20 - 2\,p_t$ and the supply function $Q_t = -5 + 3\,p_{t-1}$,
   a. Find p_t *and* Q_t *when* $p_0 = 4$
   b. Find $\bar{p}$ and $\bar{Q}$
   c. Find p_1, Q_1 , p_2, Q_2 *..e.t.c.*
   d. Using cobwebs comment on the stability of the system.

Solution: $\bar{p} = \frac{g-c}{b-h} = \frac{-5-20}{-2-3} = 5$

Hence, $p_t = (p_0 - \bar{p})(\frac{h}{b})^t + \bar{p} \Rightarrow \frac{3}{2} > 1$

a. From $Q_{dt} = Q_{st} \Rightarrow 20 - 2\,p_t == -5 + 3\,p_{t-1} \Rightarrow p_t = \frac{25}{2} - \frac{3}{2}\,p_{t-1}$

Shifting the time periods forward one period and rearranging:

$$p_{t+1} + \frac{3}{2}\,p_t = \frac{25}{2}$$

Definite solution:

$$p_t = \left(4 - \frac{25/2}{1+3/2}\right)(-3/2)^t + \frac{25/2}{1+3/2} = (4\text{-}5)\,(-3/2)^t + 5$$

$$p_t = -(-3/2)^t + 5 \; or \; p_t = 5 - (-3/2)^t$$

Substation in the demand function yields:

$$Q_t = 20 - 2\left(5 - (-3/2)^t\right) = 10 + 2(-3/2)^t \; or \; Q_t = 2(-3/2)^t + 10$$

b. If there is an equilibrium (i.e., if the price is constant at any period meaning $P_t = P_{t-1} = P_{t-2} = \cdots = P_{t-n}$), then it follows that

$P_t = P_{t-1} = \bar{P}$ and $Q_t = \bar{Q}$

Then, from $Q_{dt} = Q_{st}$

$$20 - 2\bar{P} = -5 + 3\bar{P} \Rightarrow \bar{P} = 5$$

From the demand function, $\bar{Q} = 20 - 2(5) \Rightarrow \bar{Q} = 10$

c. Using $P_t = 5 - (-\frac{3}{2})^t \; and \; Q_t = 2(-\frac{3}{2})^t + 10$

$$P_0 = 5 - \left(-\frac{3}{2}\right)^0 = 4 \qquad Q_0 = 2(-\frac{3}{2})^0 + 10 = 12$$

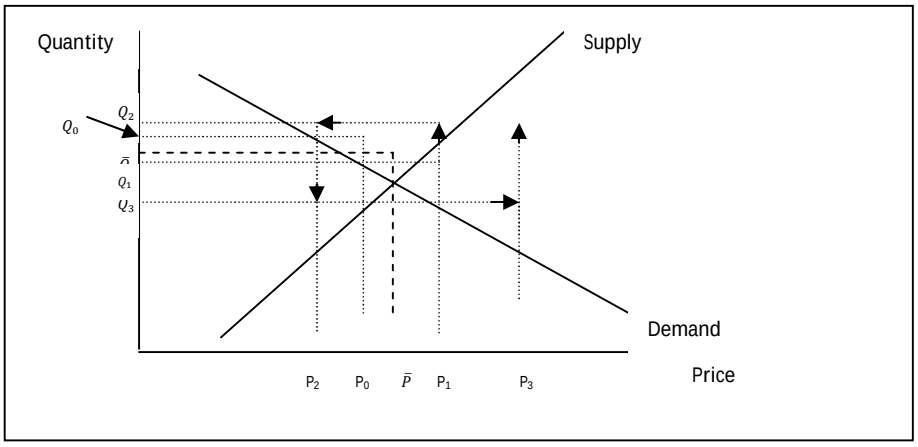
$$P_1 = 5 - \left(-\frac{3}{2}\right)^1 = 6.5 \qquad Q_1 = 2(-\frac{3}{2})^1 + 10 = 7$$

$$P_2 = 5 - \left(-\frac{3}{2}\right)^2 = 2.75 \qquad Q_2 = 2(-\frac{3}{2})^2 + 10 = 14.5$$

$$P_3 = 5 - \left(-\frac{3}{2}\right)^3 = 8.38 \qquad Q_3 = 2(-\frac{3}{2})^3 + 10 = 3.25$$

We use these points to plot the diagram.

d. Using the Cobwebs



The equilibrium is unstable because the time path is oscillatory and divergent.

2. In an agricultural market where the assumptions of the cobweb model apply, the demand and supply schedules are: Q_t^d = 400 - $20P_t$ and Q_t^s = -50 + $10P_{t-1}$

A long-run equilibrium has been established for several years but then one year there is an unexpectedly good crop and output rises to 160. Explain how price will behave over the next few years following this one-off 'shock' to the market.

Solution

In long-run equilibrium, price and quantity will remain unchanged each time period. This means that: the long-run equilibrium price, P* = P_t = P_{t-1} and the long-run equilibrium quantity Q* = Q_t^d = Q_t^s

Therefore, when the market is in equilibrium, Q* = 400 - $20P^*$ and Q* = -50 + $10P^*$

Equating to solve for P^* and Q* gives

$400-20P^*=-50+10P^*$

$450 = 30P^*$

$15 = p^*$

Q* = 400 - $20P^*$ = 400 - 300 = 100

These values correspond to the point where the supply and demand schedules intersect

If an unexpectedly good crop causes an amount of 160 to be supplied onto the market one year, then this means that the short-run supply schedule effectively becomes the vertical line. To sell this amount the price has to be reduced to some P_o, corresponding to the point, where S_o cuts the demand schedule.

Producers will then plan production for the next time period on the assumption that P_o is the ruling price. The amount supplied will therefore be Q_1. However, in the next time period when this reduced supply quantity Q_1 is put onto the market it will sell for price P_1.

These trace out a cobweb pattern (hence the 'cobweb' name) which converges on the long-run equilibrium where the supply and demand schedules intersect.

Quantity supplied in each time period is calculated by simply entering the previously ruling price into the market's supply function

$$Q_t^s = -50 + 10P_{t-1}$$

but how is this price calculated?

There are two ways:

(a) from first principles, using the given supply and demand schedules, and

(b) using a difference equation, in the format (1) derived earlier.

(a) The demand function

$Q_t^d = 400 - 20P_t$, can be rearranged to give the inverse demand function

$$P_t = 20 - 0.05Q_t^d$$

The model assumes that a fixed quantity arrives on the market each time period and then price adjusts until $Q_t^d = Q_t^s$. Thus, P_t, can be found by inserting the current quantity supplied, Q_t^s, into the function for P_t.

Assuming that the initial disturbance to the system when Q^S rises to 160 occurs in time period 0, the values of P and Q over the next three time periods can be calculated as follows: $Q_t^s = 160$ (initial given value, inserted into inverse demand function)

$P_o = 20 - 0.05Q_0^S = 20 - 0.05(160) = 20 - 8 = 12$

This price in period 0 then determines quantity supplied in period 1, which is

$Q_t^s = -50 + 10P_o = -50 + 10(12) = -50 + 120 = 70$

This quantity then determines the market-clearing price, which is

$P1 = 20 - 0.05\ Q_1^s = 20 - 0.05(70) = 20 - 3.5 = 16.5$

The same adjustment process then continues for future time periods, giving

$Q_2^s = -50 + 10P_1 = -50 + 10(16.5) = -50 + 165 = 115$

$P_2 = 20 - 0.05\ Q_2^s = 20 - 0.05(115) = 20 - 5.75 = 14.25$

$Q_3^s = -50 + 10P_2 = -50 + 10(14.25) = -50 + 142.5 = 92.5$

$P_3 = 20 - 0.05\ Q_3^s = 20 - 0.05(92.5) = 20 - 4.625 = 15.375$

The pattern of price adjustment is therefore 12, 16.5, 14.25, 15.375, etc., corresponding to the cobweb graph. Price initially falls below its long-run equilibrium value of 15 and then converges back towards this equilibrium, alternating above and below it but with the magnitude of the difference becoming smaller each period.

(b) The same pattern of price adjustment can be obtained by using the difference equation

$$P_t = \frac{c-a}{b} + \frac{d}{b} P_{t-1} \qquad (1)$$

and substituting in the given values of *a, b, c* and *d* to get

$$P_t = \frac{(-50)-400}{-20} + \frac{10}{-20} P_{t-1}$$

$$P_t = 22.5 - 0.5\, P_{t-1} \qquad (2)$$

The original price P_o still has to be derived by inserting the shock quantity 160 into the demand function, as already explained, which gives

$$P_o = 20 - 0.05(160) = 12$$

Then subsequent prices can be determined using the difference equation (2), giving

$P_1 = 22.5 - 0.5P_o = 22.5 - 0.5(12) = 16.5$

$P_2 = 22.5 - 0.5P_1 = 22.5 - 0.5(16.5) = 14.25$

$P_3 = 22.5 - 0.5P_2 = 22.5 - 0.5(14.25) = 15.375$ etc.

These prices are the same as those calculated by method (a), as expected.

Activity: In a market where the assumptions of the cobweb model hold

$$Q_t^d = 200 - 8P_t \quad \text{and} \quad Q_t^s = -43 + 8.2P_{t-1}$$

The long-run equilibrium is disturbed when quantity suddenly changes to 90. What happens to price in the following time periods?

Check your progress

In long-run equilibrium

$Q^* = Q_t^d = Q_t^s$ and $P^* = P_t = P_{t-1}$

Substituting these equilibrium values and equating demand and supply we can find the new equilibrium price. Thus

$$200 - 8P^* = Q^* = -43 + 8.2P^*$$

$$243 = 16.2P^*$$

$$15 = P^*$$

This will be an unstable equilibrium as

$$\left|\frac{d}{b}\right| = \left|\frac{8.2}{-8}\right| = 10.025 > 1$$

The difference equation that describes the relationship between price in one period and the next will take the usual cobweb model format

$$P_t = \frac{c-a}{b} + \frac{d}{b} P_{t-1} \qquad (1)$$

where $a = 200$, $b = -8$, c = -43 and $d = 8.2$, giving

$$P_t = \frac{-43-200}{-8} + \frac{8.2}{-8} P_{t-1}$$
$$= 30.375 - 1.025P_{t-1}$$

Using the formula derived earlier, the solution to this difference equation will therefore be

$$P_t = \frac{a-c}{d-b} + A\left(\frac{d}{b}\right)^t$$
$$= \frac{200-(-43)}{8.2-(-8)} + A\left(\frac{8.2}{-8}\right)^t$$
$$= \frac{243}{16.2} + A(-11.025)^t$$
$$= 15 + A(-1.025)^t$$

The first part of this solution is of course the equilibrium value of price which has already been calculated above. To derive the value of A, we need to find price in period 0.

The quantity supplied is 90 in period 0 and so, to find the price that this quantity will sell for, this value is substituted into the demand function. Thus

$$Q_0^d = 90 = 200 - 8P_0$$
$$8P^0 = 110$$
$$P_0 = 13.75$$

Substituting this value into the general solution (2) we get

$$P_o = 13.75 = 15 + A(-1.025)^0$$
$$13.75 = 15 + A$$
$$-1.25 = A$$

Note that, the value of parameter A is the difference between the equilibrium value of price and the value it initially takes when quantity is disturbed from its equilibrium level, i.e.

$$A = P_o - p^* = 13.75 - 15 = -1.25$$

Putting this value of A into the general solution (2), the specific solution to the difference equation in this example now becomes $P_t = 15 - 1.25(-1.025)^t$

Using this formula to calculate the first few values of P_t gives

$$P_o = 15 - 1.25(1.025)^0 = 13.75$$
$$P_1 = 15 + 1.25(1.025)^1 = 16.2S$$
$$P_2 = 15 - 1.25(1.025)^2 = 13.69$$
$$P_3 = 15 + 1.25(1.025)^3 = 16.35$$

We can see that, although price is gradually moving away from its long-run equilibrium value of 15, it is a very slow process. By period 10, price is still above 13.00, as

$$P_{10} = 15 - 1.25(1.025)^{10} = 13.40$$

and it takes until time period 102 before price becomes negative, as the figures below show:

$$P_{100} = 15 - 1.25(1.025)^{100} = 0.23$$
$$P_{101} = 15 + 1.25(1.025)^{101} = 30.14$$
$$P^{102} = 15 - 1.25(1.025)^{102} = -0.51$$

7.4.2.2. The Lagged Keynesian macroeconomic model

In the basic Keynesian model of the determination of national income, if foreign trade and government taxation and expenditure are excluded, the model reduces to the accounting identity:

$$Y = C + I \qquad (1)$$

and the consumption function

$$C = a + bY \qquad (2)$$

To determine the equilibrium level of national income y^* we substitute (2) into (1), giving

$$Y^* = a + bY^* + I$$

$$Y^*(1 - b) = a + I$$

$$Y^* = \frac{a + I}{1 - b}$$

This can be evaluated for given values of parameters a and b and exogenously determined investment **I.** If there is a disturbance from this equilibrium, say exogenous investment **I** alters, then the adjustment to a new equilibrium will not be instantaneous.

This is the basis of the well-known multiplier effect. An initial injection of expenditure will become income for another sector of the economy. A proportion of this will be passed on as a further round of expenditure, and so on until the 'ripple effect' dies away.

Because consumer expenditure may not adjust instantaneously to new levels of income, a lagged effect may be introduced. If it is assumed that consumers' expenditure in one time period depends on the income that they received in the previous time period, then the consumption function becomes

$$C_t = a + bY_{t-1} \qquad (3)$$

where the subscripts denote the time period.

National income, however, will still be determined by the sum of all expenditure within the current time period. Therefore the accounting identity (1), when time subscripts are introduced, can be written as

$$Y_t = C_t + I_t \qquad (4)$$

From (3) and (4) we can derive a difference equation that explains how Y_t depends on Y_{t-1} Substituting (3) into (4) we get

$$Y_t = (a+bY_{t-1}) + I_t$$
$$Y_t = bY_{t-1} + a + I_t \quad (5)$$

Let us illustrate how this lagged effect works using a numerical example.

Example:

In a basic Keynesian macroeconomic model it is assumed that initially $Y_t = C_t + I_t$, where $I_t = 134$ is exogenously determined, and $C_t = 40 + 0.6Y_{t-1}$

The level of investment I_t then falls to 110 and remains at this level each time period. Trace out the pattern of adjustment to the new equilibrium value of Y, assuming that the model was initially in equilibrium.

Solution

If the system is initially in equilibrium then income in one time period is equal to expenditure in the previous time period, and income is the same each time period. Thus

$$Y_t = Y_{t-1} = Y^*$$

where Y^* is the equilibrium level of Y. Therefore, when the original value of I_t of 134 is inserted into the accounting identity the model becomes

$$Y^* = C_t + 134 \quad (1)$$
$$C_t = 40 + 0.6Y^* \quad (2)$$

By substitution of (2) into (1)

$$Y^* = (40 + 0.6Y^*) + 134$$
$$Y^*(1 - 0.6) = 40 + 134$$
$$0.4Y^* = 174$$
$$Y^* = 435$$

This is the initial equilibrium value of Y before the change in I.

Assume time period 0 is the one in which the drop in I to 110 occurs. Consumption in time period 0 will be based on income earned the previous time period, i.e. when Y was still at the old equilibrium level of 435.

Thus, $C_0 = 40 + 0.6(435) = 40 + 261 = 301$

Therefore, $Y_0 = C_0 + I_0 = 301 + 110 = 411$

In the next time period, the lagged consumption function means that C_1 will be based on Y_o. Thus

$$Y_1 = C_1 + 110$$
$$= (40 + 0.6Y_o) + 110$$
$$= 40 + 0.6(411) + 110$$
$$= 40 + 246.6 + 110 = 396.6$$

The value of *Y* for other time periods can be calculated in a similar fashion:

$$Y_2 = C_2 + I_2$$
$$= (40 + 0.6Y_1) + 110$$
$$= 40 + 0.9(396.6) + 110$$
$$= 387.96$$
$$Y_3 = C_3 + 1_3$$
$$= (40 + 0.6Y_2) + 110$$
$$= 40 + 0.6(387.96) + 110$$
$$= 382.776, \quad \text{and so on.}$$

It can be seen that in each time period *Y* decreases by smaller and smaller amounts as it readjusts towards the new equilibrium value. This new equilibrium value can easily be calculated using the same method as that used above to work out the initial equilibrium.

When $1 = 110$ and $Y_t = Y_{t-1} = Y^*$ then the model becomes

$$Y^* = C_t + 1 = C_t + 110$$
$$C_t = 40 + 0.6Y^*$$

By substitution

$$Y^* = (40 + 0.6Y) + 110$$
$$(1 - 0.6)Y^* = 150$$
$$Y^* = \frac{150}{0.4} = 375$$

Unlike the cobweb model described earlier, the adjustment in this Keynesian model is always in the same direction, instead of alternating on either side of the final equilibrium. Successive values of *Y* just approach the equilibrium by smaller and smaller increments because the ratio in the complementary function to the difference equation is not negative as it was in the cobweb model. If the initial equilibrium had been below the new equilibrium then, of course, *Y* would have approached its new equilibrium from below instead of from above.

Difference equation solution

The solution is split into two components: the *equilibrium* or *particular solution* and the *complementary function*. We first need to find the particular solution, which will be the new equilibrium value of *Y**. When this is equilibrium achieved

$$Y_t = Y_{t-1} = Y^* \qquad (1)$$

In equilibrium, the single lag Keynesian model

$$C_t = a + bY_{t-1} \qquad (2)$$

and

$$Y_t = C_t + I_t \qquad (3)$$

can therefore be written as

$$C_t = a + bY^*$$
$$Y^* = C_t + I_t$$

By substitution

$$Y^* = a + b\,y^* + I_t$$
$$(1 - b)\,Y^* = a + I_t$$
$$Y^* = \frac{a + I_t}{1-b} \qquad (4)$$

If the given values of a, b and It are put into (4) then the equilibrium value of Y is determined. This is the first part of the difference equation solution.

Returning to the difference equation (1) which we are trying to solve

$$Y_t = bY_{t-1} + a + I_t$$

If the two constant terms a and $\boldsymbol{I_t}$ are removed then this becomes

$$Y_t = bY_{t-1} \qquad (5)$$

To find the complementary function we use the standard method and assume that this solution is in the format $Y_t = Ak^t$ (6)

where A and k are unknown parameters. This means that

$$Y_{t-1} = Ak^{t-1} \qquad (7)$$

Substituting (6) and (7) into (5) gives

$$Ak^t = bAk^{t-1}$$
$$k = b$$

Thus the complementary function is

$$Y_t = Ab^t \qquad (8)$$

The general solution to the difference equation is the sum of the particular solution (4) and the complementary function (8). Hence

$$Y_t = \frac{a + I_t}{1-b} + Ab^t \qquad (9)$$

If t is increased, then the value of b^t in the general solution (9) will diminish as long as $|b| < 1$. This condition will be met since b is the marginal propensity to consume which has been estimated to lie between 0 and 1 in empirical studies. Therefore Y_t will always head towards its new equilibrium value. The value of the constant A can be determined if an initial value Y_0 is known. Substituting into (9), this gives: $Y_0 = \frac{a+I_t}{1-b} + Ab^0$

Remembering that $b^0 = 1$, this means that

$$A = Y_0 - \frac{a+I_t}{1-b} \qquad (10)$$

Thus A is the value of the difference between the initial level of income Y_0, immediately after the shock, and its final equilibrium value Y^*.

Putting this result into (9) above, the general solution to our difference equation becomes

$$Y_t = \frac{a+I_t}{1-b} + \left(Y_0 - \frac{a+I_t}{1-b}\right) b^t \qquad (11)$$

This may seem to be a rather cumbersome formula but it is straightforward to use. If you remember that: $\frac{a+I_t}{1-b}$ Y^*, is the equilibrium value of Y_t and rewrite (11) as

$$Y_t = Y^* + (Y_o - Y^*)b^t \qquad (12)$$

you will find it easier to work with.

We can now check that this solution to the lagged Keynesian model difference equation works with the numerical Example considered above. This model assumed

$$Y_t = C_t + I_t$$

where It was initially 134 and $C_t = 40 + 0.6Y_{t-1}$, which corresponded to an initial equilibrium of Y_t of 435.

When I_t was exogenously decreased to 110, the adjustment path towards the new equilibrium value of Y of 375 was worked out by an iterative method. Now let us see what values our difference equation will give.

We have to be careful in determining the initial value Y_0, immediately *after* the increase in investment has taken place. This depends on I_0, which will be the *new* level of investment of 110, and C_0. The level of consumption in period 0 depends on the previously existing equilibrium level of Y_I which was 435 in time period 'minus one'. Therefore,

$$C_o = a + bY_{t-1} = 40 + 0.6(435) = 301$$
$$Y_o = C_o + I_0 = 301 + 110 = 411 \qquad (13)$$

This is the same initial value Y_o as that calculated in Example 13.7. The new equilibrium value of income is

$$Y^* = \frac{a+I_t}{1-b} = \frac{40+110}{1-0.6} = \frac{150}{0.4} = 375 \qquad (14)$$

Substituting (13) and (14) into the formula for the general solution to the difference equation derived above

$$Y_t = Y^* + (Y_o - Y^*)b^t \qquad (15)$$

The general solution for this numerical example becomes

$$Y_t = 375 + (411 - 375)0.6^t$$
$$= 375 + 36(0.6)^t$$

The first few values of Y are thus

$$Y_1 = 375 + 36(0.6)^1 = 375 + 21.6 = 396.6$$
$$Y_2 = 375 + 36(0.6)^2 = 375 + 12.96 = 387.96$$
$$Y_3 = 375 + 36(0.6)^3 = 375 + 7.776 = 382.776$$

These are exactly the same as the answers computed by the iterative method so far, which is what one would expect.

This difference equation solution can now be used to calculate Y_t in any given time period.

For example, in time period 9 it will be

$$Y_9 = 375 + 36(0.6)^9 = 375 + 0.3628 = 375.3628$$

As t increases in value, eventually the value of $(0.6)^t$ becomes so small as to make the second term negligible. In the above example we can say that for all intents and purposes Y_t has effectively reached its equilibrium value of 375 by the ninth time period, although theoretically Y_t would never actually reach 375 if infinitesimally small increments were allowed.

Activity:

There is initially an equilibrium in the basic Keynesian model

$$Y_t = C_t + I_t$$
$$C_t = 650 + 0.5Y_{t-1}$$

With I_t remaining at 300. Then I_t suddenly increases to 420 and remains there.

a. What will be the actual level of Y six time periods after this change?

b. How many time periods will it take Y_t to reach 2,130 in the preceding example?

Check your progress

a. The initial equilibrium in period 'minus l' before the change is

$$Y^*_{-1} = \frac{a+I_t}{1-b} = \frac{650+300}{1-0.5} = \frac{950}{0.5} = 1{,}900$$

Therefore the value of C in time period 0 when the increase in I takes place is

$$C_o = 650 + 0.5(1{,}900) = 650 + 950 = 1{,}600$$

and so the value of Y_t immediately after this shock is

$$Y_o = C_o + 1_0 = 1{,}600 + 420 = 2{,}020$$

The new equilibrium level of Y is

$$Y^* = \frac{a+I_t}{1-b} = \frac{650+420}{1-0.5} = \frac{1{,}070}{0.5} = 2{,}140$$

Substituting these values into the general solution for the lagged Keynesian macroeconomic model difference equation, we get the general solution for this example, which is

$$\begin{aligned} Y_t &= Y^* + (Y_o - Y^*)b^t \\ &= 2{,}140 + (2{,}020 - 2{,}140)0.5^t \\ &= 2{,}140 - 120(0.5)^t \end{aligned}$$

Therefore, six time periods after the increase in investment

$$\begin{aligned} Y_6 &= 2{,}140 - 120(0.5)^6 \\ &= 2{,}140 - 1.875 = 2{,}138.125 \end{aligned}$$

b. We know that $Y_t = 2{,}130$ and we wish to find t. Thus substituting this value and the initial value Y_o and the new equilibrium Y^* calculated in Example 13.8, into the Keynesian model general solution formula: $Y_t = Y^* + (Y_o - Y^*)b^t$

we get

$$\begin{aligned} 2{,}130 &= 2{,}140 + (2{,}020 - 2{,}140)0.5^t \\ -10 &= -120(0.5)^t \\ 0.08333 &= (0.5)^t \end{aligned}$$

To get t, put this into the log form, which gives

$$\log 0.08333 = t \log 0.5$$

$$\frac{\log 0.083333}{\log 0.5} = t$$

$$3.585 = t$$

Therefore Y will have exceeded 2,130 by the end of the fourth time period.

7.5. Optimal Control Theory

Optimal control theory allows consideration of the optimal time path of a set of variables rather than just the identification of a stationary equilibrium. This theory shows how to solve continuous-time maximization problems in which the objective function includes an integral and the constraints include a differential equation.

In. this section we build on the intuition developed in the discrete-time problems discussed in the previous section to present the main results of optimal control theory. The continuous-time version of the problem facing the managers of the firm discussed in the previous section involves the optimal selection of a time path of the control variable C(t), which represents the flow of expenditures on labor. In general, the path of the control variable in an optimal control problem must be **piecewise continuous.** This condition requires that the path of the control variable is continuous but for (possibly) some finite number of jump discontinuities and that the size of the jumps is finite.

This problem also includes a constraint on the accumulation of the state variable capital, k(t) . The state variable in an optimal control problem must be continuous and **piecewise differentiable,** that is, the path can have a finite number of "corners" or points where its derivative is not defined.

The profit of the firm over the time period $t = 0$ to T is a function of both the control variable and the state variable, as well as time itself, and is given by the integral

$$\int_{t=0}^{T} \pi(c(t), k(t), t)dt,$$

Where $\pi(c(t), k(t), t)$ is the instantaneous profit function. This integral is the continuous-time analogue of the discrete-time. The continuous-time analogue to the difference equation that describes the evolution of the capital stock is the differential equation

$$\dot{k}(t) = f(c(t), k(t), t),$$

Where $\dot{k}(t) \equiv \frac{dk(t)}{dt}$. The functions $\pi(c(t), k(t), t) \quad and \quad f(c(t), k(t), t)$ are assumed to be continuous in all their arguments and to have continuous first-order partial derivatives with respect to k(t) and t, but not necessarily with respect to c(t) , The initial condition mirrors with

$$k(0) = k.$$

We consider two possible cases with respect to the terminal condition, the fixed terminal condition case where, $k(T) = \bar{k}$, as well as the case where $k(T)$ is chosen optimally.

The **maximum principle** (also known as **Pontryagin's maximum principle)** provides a set of conditions that are necessary for obtaining the maximum value of the objective function subject to the constraints.

The maximum principle: If c(t) and k(t) maximize $\int_{t=0}^{T} \pi(c(t), k(t), t)dt$

subject to the state equation: $\dot{k}(t) = f(c(t), k(t), t)$,

Where,

- $\pi(c(t), k(t), t)$ –is piecewise continuous,
- $f(c(t), k(t), t)$ –is continuous and piecewise differentiable,

- Both $\pi(c(t),k(t),t)$ *and* $f(c(t),k(t),t)$ have continuous first-order partial derivatives with respect to k(t) and *t,* and the initial condition *k(0)* = $\underline{k}$ holds, then, for the *Hamiltonian function* $H(c(t),k(t),\tau(t),t) = \pi(c(t),k(t),t) + \tau(t).f(c(t),k(t),t)$, the following maximum principle conditions must be met at the optimal values $c^*(t)$ *and* $k^*(t)$:

i) $H(c^*(t),k^*(t),\tau(t),t) \geq H(c(t),k^*(t),\tau(t),t)$ *for all* t

ii) $\frac{\partial H(c^*(t),k^*(t),\tau(t),t)}{\partial k(t)} = -\dot{\tau}(t)$

iii) $\frac{\partial H(c^*(t),k^*(t),\tau(t),t)}{\partial \tau(t)} = \dot{k}(t)$

iv) $k(0) = \underline{k}$.

where the partial derivatives in conditions (ii) and (iii) are evaluated along the optimal path.
One of the following conditions must also be met:

(a) If the terminal value of the state variable must equal $\bar{k}$, then $k(T) = \bar{k}$.

(b) If the terminal value of the state variable is not given, then $\tau(T)k(T) = 0$.

We also note the following.

- If there is a continuous first-order partial derivative of the Hamiltonian with respect to the control variable c(t) and there is a solution to the problem on the open interval (0,t), condition (i) is satisfied by the condition

$$\frac{\partial H(c^*(t),k^*(t),\tau(t),t)}{\partial c(t)} = 0.$$

- We can solve a problem that requires the minimization of the integral of an instantaneous objective function, $u(c(t),k(t),\tau(t),t)$, *v(~(t), k(t), A(t), t),* in the same fashion as the maximum problem by simply defining a function

$$w(c(t),k(t),\tau(t),t) = -u(c(t),k(t),\tau(t),t)$$

and finding the maximum of the problem by using the function $w(c(t),k(t),\tau(t),t)$.
We illustrate the use of the maximum principle with the following example, which has a simple geometric interpretation. Consider the problem of choosing the optimum path of the control c(t) to maximize

$$\int_{t=0}^{10} -\sqrt{1 + c(t)^2}\, dt$$

subject to the constraints

$$\frac{dk(t)}{dt} = (t)$$

K(0) =4

K(10) =24,

where *y(t)* is the state variable. We form the Hamiltonian

$$H\big(c(t),k(t),\tau(t)\big) = -\sqrt{1 + c(t)2} + \tau(t).c(t).$$

The maximum principle shows that the first order-conditions are

$$\frac{\partial H(c(t),k(t),\tau(t))}{\partial c(t)} = -\frac{c(t)}{\sqrt{1+c(t)2}} + \tau(t) = 0 \quad (1)$$

$$\frac{\partial H(c(t),k(t),\tau(t))}{\partial k(t)} = 0 = -\tau(t) \quad (2)$$

$$\frac{\partial H(c(t),k(t),\tau(t))}{\partial \tau(t)} = c(t) = k(t). \quad (3)$$

$$k(0) = 4 \quad (4)$$

$$K(10) = 24. \quad (5)$$

The condition (2) shows that the co-state variable is constant since $\tau(t) = O$. This result, combined with (1), shows that the control variable c*(t)* is also constant, and, therefore, we denote it as e. To solve for the optimal value e, we integrate condition (3) to obtain

$$\int_{t=0}^{10} \frac{dk}{dt}\, dt = \int_{t=0}^{10} c.dt$$

$$k(10) - k(0) = c.10 - c.0 = 10.c.$$

The conditions (4) and (5) show that k(10) –k(0)=24-4=20, and, therefore, the optimal value of the control variable is $c = \frac{20}{10} = 2$.

This problem has the geometric interpretation of finding the minimum distance between the two points (0,4) and (10,24) in a graph of the function k=f(t). The distance between the line segment connecting any two points i and i + *m* on the function f(t), is given by the *Pythagorean theorem* to be:

$$s_i = \sqrt{(\Delta t)^2 + (\Delta k(i))^2},$$

Where $\Delta t = (i + m) - i = m \; and \; \Delta k(i) = k(i + m) - k(i)$. This is equal to

$$s_i = \left(\sqrt{1 + \left(\frac{\Delta k(i)}{\Delta t}\right)^2}\right) \cdot \Delta t.$$

If the distance 0 to 10 is divided up into *m*-sections, then the total distance of the function f(t) can be approximated by

$$\sum_{i=0}^{M} s_i = \sum_{i=0}^{M} \left(\sqrt{1 + \left(\frac{\Delta k(i)}{\Delta t}\right) 2}\right). \Delta t.$$

In the limit, as $\Delta t \to 0$, this sum equals the integral

$$\int_{t=0}^{10} \sqrt{1 + \left(\frac{dk}{dt}\right) 2}\, .dt.$$

Given the state equation $\frac{dk(t)}{dt} = c(t)$, the integrand is $\sqrt{1 + c(t)2}$. The solution to the optimal control problem shows that the optimal value of c is the constant 2. Therefore the function that minimizes the distance is the straight line:

k(t)=2t + A, where *A* is a constant. The terminal condition shows that *k(10)* = 24 and, therefore, *A* = 24 - 20 = 4. Thus the function that minimizes the distance between the points is k(t) = 2t + 4

As you may have suspected, and as the maximum principle confirms, the shortest distance between two points is a straight line.

The conditions that provide for a maximum in the continuous-time case closely resemble those of the discrete-time case.

While a complete proof of these conditions is beyond the scope of this module, the discussion we made so far provides some clues for the maximum principle and we wind-up our analysis here.

Self Test Exercise 7

1. Find the general solution for $\frac{dy}{dt} + 3t^2y = t^2$
2. Given a difference equation $x_t + 3x_{t-1} + 8 = 0$ and $x_0 = 16$,
   (a) Solve the difference equation.
   (b) Check the answer using $t = 0$ and $t = 1$, and
   (c) Comment on the nature of the time path.
3. **In** a market where the assumptions of the cobweb model apply, the demand and supply functions are $Q_t^d = 120 - 4P_t$ and $Q_t^s = -80 + 16P_{t-1}$

If in one time period the long-run equilibrium is disturbed by output unexpectedly rising to a level of 90, explain how price will adjust over the next few time periods.

4. The cobweb model assumptions hold in a market where

$$Q_t^d = 160 - 2P_t \quad \text{and} \quad Q_t^s = -20 + 2P_{t-1}$$

If the previously ruling long-run equilibrium is disturbed by an unexpectedly low output of 50 in one time period, what will happen to price in the following time periods?

Model Examinations

1. Find y_c, y_p, the general solution and the definite solution, given:
   (a) $\frac{dy}{dt} + 4y = 12; y(0) = 2$
   (b) $\frac{dy}{dt} - 2y = 0; y(0) = 9$
   (c) $\frac{dy}{dt} + 10y = 15; y(0) = 0$
   (d) $2\frac{dy}{dt} + 4y = 6; y(0) = 1\frac{1}{2}$
2. Check the validity of your answers to the preceding problem.
3. Let the demand and supply be : $Q_d = \alpha - \beta P - \eta\frac{dP}{dt}, \quad Q_s = \sigma P \quad (\alpha, \beta, \eta, \sigma > 0)$
   (a) Assuming that the rate of change of price over time is directly proportional to the excess demand, find the time path $P(t)$ (general solution).
   (b) What is the intertemporal equilibrium price? What is the market-clearing equilibrium price?
   (c) What restriction on the parameter σwould ensure dynamic stability?

4. Solve the following first-order linear differential equations; if an initial condition is given, definitize the arbitrary constant:
   (a) $\frac{dy}{dt} + 2ty = 0$
   (b) $\frac{dy}{dt} + 2ty = t; y(0) = \frac{3}{2}$
   (c) $\frac{dy}{dt} + t^2y = 5t^2; y(0) = 6$
   (d) $2\frac{dy}{dt} + 12y + 2e^t = 0; y(0) = \frac{6}{7}$
5. Draw the time paths of the differential equations in the above problem where initial condition is given.
6. Solve the following first order difference equations by iteration:
   (a) $\Delta y_t = 7; y(0) = 5$
   (b) $y_{t+1} = y_t + 1;\ y_0 = 10$
   (c) $y_{t+1} = \alpha y_t;\ \ y_0 = \beta$
   (d) $y_{t+1} = \alpha y_t - \beta;\ (y_t = y_0 \text{ when } t = 0)$
7. Find the equations in question (6) based on the general method and find their general and definite solutions. Do your answers check with those obtained by the iterative method?
8. Draw and discuss the nature of the following time paths:
   (a) $y_t = 3^t + 1$
   (b) $y_t = 2\left(\frac{1}{3}\right)^t$
   (c) $y_t = 5\left(-\frac{1}{10}\right)^t + 3$
   (d) $y_t = -3\left(\frac{1}{4}\right)^t + 2$
9. Given $Q_{dt} = 86 - 0.8P_t$ and $Q_{st} = -10 + 0.2P_{t-1}$, and if $P_0 = 100$, determine
   (a) The market price P_t in any time period,
   (b) The equilibrium price $\bar{P}$, and
   (c) Using Cobwebs comment the stability of the time path.

10. A Keynesian macroeconomic model with a single-time-period lag on the consumption function, as described below, is initially in equilibrium with the level of I_t given at 500.

$$Y_t = C_t + I_t$$
$$C_t = 750 + 0.5Y_{t-1}$$

I_t, is then increased to 650. Use difference equation analysis to find the value of Y_I in the fourth time period after this disturbance to the system. Will it then be within 1 % of its new equilibrium level?

Answer Key

Self –Test Exercise 2.1

1. MR=280
2. A) $2\text{-}120/X^2$

   B) $X=\sqrt{60}$ units

3. $\pi(100)=(160)$,which is a loss

Self –Test Exercise 2.2

1. $888
2. 0.465

Self –Test Exercise 2.3

A. Technical coefficient obtained by using the formula aij = $\frac{xij}{xj}$

$$A = \begin{bmatrix} 0.13 & 0.03 & 0.76 & 0.23 \\ 0.33 & 0.08 & 0.23 & 0.12 \\ 0.37 & 0.18 & 0.04 & 0.04 \\ 0.1 & 0.23 & 0.4 & 0.24 \end{bmatrix}$$

B. $$X\text{-}f = \begin{bmatrix} 600 \\ 600 \\ 400 \\ 1000 \end{bmatrix} - \begin{bmatrix} 160 \\ 140 \\ 0 \\ 400 \end{bmatrix} = \begin{bmatrix} 440 \\ 460 \\ 400 \\ 600 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 439.6 \\ 459.6 \\ 400 \\ 599.8 \end{bmatrix}$$

Therefore, AX approximately equals to X-F.

Self –Test Exercise 2.4

First convert the third constraint into single sign: $-7x_1 + 2x_2+x_3 >= -10$

Max $Z^* = 9Y_1 + 5Y_2 - 10Y_3 + 4Y_4 + 3Y_5$

S.t $3Y_1+Y_2-7Y_3+Y_4+2Y_5 \leq 5$

$4Y_1+3Y_2 +2Y_3-2Y_4+5Y_5 \leq -6$

$6Y_1+2Y_2 +Y_3+4Y_4 - 3Y_5 \leq 4$

$Y_1, Y_2, Y_3, Y_4, Y_5 \geq 0$

Self –Test Exercise 3

1. $\frac{1}{9} - \frac{10}{27}x + \frac{75}{81}x^2$
2. E_d=8 & state of demand is inelastic
3. Em=0.99 & the type of good is necessity
4. E_d=b
5. f(0)=6, f'(0)=8, f''(0)=-14, f'''(0)=18, f''''(0)=48. Thus, the Macularian series will be $6+8X-14X^2+18X^3+48X^4$
6. i) MP_K= a Q/k
   ii) MP_L= b Q/ L
   iii) MP_T= c Q/T

Self –Test Exercise 4.1

1. X=250 units
2. X=10 units

Self –Test Exercise 4.2

1. a) i. X=1 & Y=2
      ii. The function is at relative minimum point
   b) i. X=4 & Y=3
      ii. The function is at relative maximum point

2. Optimizing profit to mean Appling FOC & SOC

   i. X=100 &Y=300

   ii. The maximum profit is birr 1100

3. a). i. Q_1=47 &Q_2=37

      ii. P_1=53 &P_2=43

      iii. $\pi = 3578$

   b). i. Q_1=94, Q_2=74, P_1=P_2=P=6

      ii. Discrimination profit i.e $\pi = 3578$ while, non-discrimination profit is zero. Thus, the difference is 3758

Self –Test Exercise 5

1. $x^* = y^* = 3$ and the extremum value is $\underline{\underline{\bar{Z} = 9}}$

2. $x_1^* = 2/17$, $x_2^* = 8/17$ *and* , $\lambda^* = 4/17$ *and* $Z^* = 4/17$

3.

$$L = xyz + \lambda\,(60 - x - 2y - 2z)$$

$$\frac{\partial L}{\partial x} = yz - \lambda = 0\,, \quad \underline{\underline{\lambda = yz}}$$

$$\frac{\partial L}{\partial y} = xz - 2\lambda = 0 \qquad / \quad \frac{\partial L}{\partial z} = xy - 2\lambda = 0$$

$$\lambda = \underline{\underline{xz/2}} \qquad\qquad \lambda = \frac{xy}{2}$$

$$\frac{\partial L}{\partial \lambda} = 60 - x - 2y - 2z = 0 \qquad / \quad x + 2y + 2z = 60$$

$$y/2 + 2y + 2y = 60 \Rightarrow g_y = 120 \qquad y^*$$

$$y^* = z^* = 10\,, \quad x^* = 20$$

Soc :-

$$\left|\overline{H}\right| = \begin{vmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 10 & 10 \\ 2 & 10 & 0 & 20 \\ 2 & 20 & 20 & 0 \end{vmatrix} = -1200 < 0$$

$$\left|\overline{H}_2\right| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 10 \\ 2 & 10 & 0 \end{vmatrix} = +40 > 0 \qquad \left|\overline{H}_3\right| = \begin{vmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 10 & 10 \\ 2 & 10 & 0 & 20 \\ 2 & 10 & 20 & 0 \end{vmatrix} = -1200 < 0$$

$\therefore$ The stationary point is a maxima.

Self –Test Exercise 6

1. Aggregate demand is now C+I+G and in equilibrium

   Y=C+I+G

   =c(I-t)Y+I+C

   =0.8(0.8)Y+500+G

   So that Y*=1.400+2.8G

   The government expenditure multiplier is

   $\frac{dY^*}{dG}$=2.8, P*=10-$\frac{y}{3}$, q*=10-$\frac{y}{3}$

   So the comparative statics we seek are

   $\frac{dp^*}{dy}$= -1/3 $\frac{dq^*}{dy}$ =-1/3

2. Profit is π= (180-t)q-3q^2, and optimal output is q=30-t/6 and the comparative static effect is $\frac{dq^*}{at}$= -1/6

Self Test Exercise 7

1. Rearranging the function to the general form of differential equations, we get:

$$\frac{dy}{dt} = t^2 - 3t^2y = t^2(1-3y)$$

Separating the variables: $^{dy}/_{1-3y} = t^2dt$

Integrating both sides: $\int {}^{dy}/_{1-3y} = \int t^2dt$

$$\frac{-1}{3}\ln(1-3y) = \frac{1}{3}t^3 + C_1 \Rightarrow \ln(1-3y) = -t^3 + C_2 \quad C_2 = -3C_1$$

$$1-3y = e^{-t^3+C_2} \Rightarrow 1-3y = e^{C_2}e^{-t^3}$$

$$-3y = e^{C_2}e^{-t^3} - 1 \Rightarrow Y = \frac{-1}{3}e^{C_2}e^{-t^3} + \frac{1}{3}$$

This is a general solution since it contains a definite integral $Y = Ae^{-t^3} + \frac{1}{3}$ where $A = \frac{-1}{3}e^{C_2}$

2.

(a) Shift the time periods forward one period to conform to the general form:

$$x_{t+1} + 3x_{t-1+1} + 8 = 0 \Rightarrow x_{t+1} + 3x_t = -8$$

Since $b = -3$ and $c = -8$, the definite solution formula gives:

$$x_t = \left(x_0 - \frac{c}{1-b}\right)b^t + \frac{c}{1-b} = \left(16 - \frac{-8}{1-(-3)}\right)(-3)^t + \frac{-8}{1-(-3)} \Rightarrow x_t = 18(-3)^t - 2$$

(b) At $t = 0, x_0 = 18(-3)^0 - 2 = 16$ *and at* $t = 1, x_1 = 18(-3)^1 - 2 = -56$

Substitution in the original equation yields:

$$x_{t+1} + 3x_t = -8 \Rightarrow -56 + 3(16) = -8 \text{ (compare)}$$

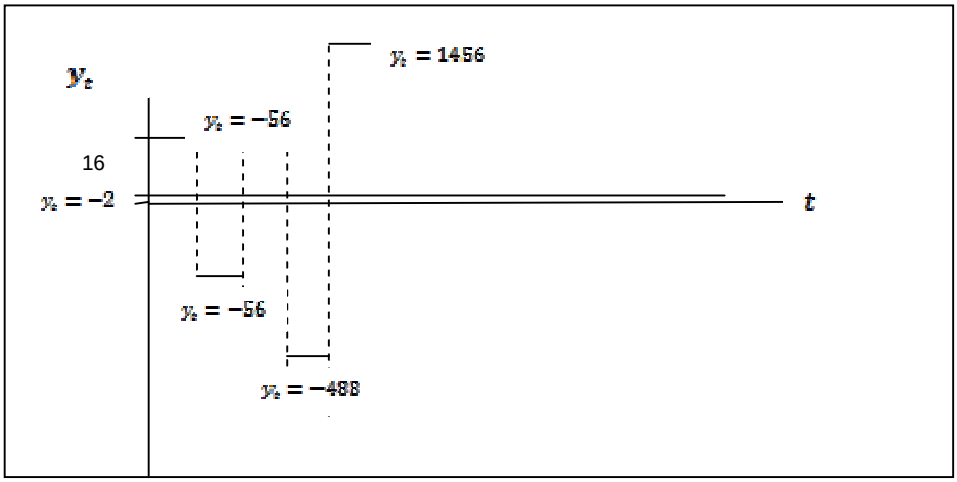
(c) With the base $b = -3 > 0$, **and** $|b| = 3 > 1$, the time path is oscillatory and divergent.

In the concepts of saving and interest, negative values of y_p is feasible, but in other cases it may not be possible.

Graphic Illustration

Given the definite solution of the difference equation as $x_t = 18(-3)^t - 2$, the possible values of t and x_t in graph are tabulated as follows:

T	0	1	2	3	4
x_t	16	-56	160	-488	1456



3. *Solution*

The long-run equilibrium price can be determined from the formula

$$P* = \frac{c-a}{b-d} = \frac{(-80)-120}{-4-16} = \frac{-200}{-20} = 10$$

Thus, the long-run equilibrium quantity is

$$Q^* = 120 - 4P^* = 120 - 4(10) = 80$$

Using the standard cobweb model difference equation

$$P_t = \frac{c-a}{b} + \frac{d}{b} P_{t-1} \qquad (1)$$

and substituting the known values, we get

$$P_t = \frac{(-80)-120}{-4} + \frac{16}{-4} P_{t-1} = 50 - 4P_{t-1} \qquad (2)$$

The initial price P_o can be found by inserting the shock quantity of 90 into the demand function. Thus

$$Q_0^s = 90 = 120 - 4P_o$$
$$4P_o = 30$$
$$P_o = 7.5$$

Putting this value into the difference equation (2) above we get

$$P_1 = 50 - 4P_o = 50 - 4(7.5) = 20$$

$$P_2 = 50 - 4P_1 = 50 - 4(20) = -30$$

There is not much point in going any further with the calculations. Assuming that producers will not pay consumers to take goods off their hands, negative prices cannot exist.

The initial quantity 90 put onto the market causes price to drop to 7.5. Suppliers then reduce supply for the next period to

$$Q_1^s = -80 + 16P_o = -80 + 16(7.5) = 40$$

This sells for price $P_1 = 20$ and so supply for the following period is increased to

$$Q_2^s = -80 + 16P_1 = -80 + 16(20) = 240$$

Consumers would only consume 120 even if price were zero (where the demand schedule hits the axis) and so, when this quantity of 240 is put onto the market, price will collapse to zero and there will still be unsold produce. Producers will not wish to supply anything for the next time period if they expect a price of zero and so no further production will take place.

This is clearly an unstable market. The stability condition depends on the slopes of the supply and demand schedules. If the absolute value of the slope of the demand schedule is less than the absolute value of the slope of the supply schedule then the market is stable, and vice versa. These slopes are inversely related to parameters *b* and *d*, since the vertical axis measures *p* rather than *q*. Thus the stability conditions are

Stable: $|d/b| < 1$ $\quad$ $Unstable: |d/b| > 1$

4.

Solution

Substituting the values $a = 160$, $b = -2$, c =-20 and d = 2 for this market in to the cobweb difference equation general solution

$$P_t = \frac{a-c}{d-b} + A\left(\frac{d}{b}\right)^t \qquad (1)$$

gives,

$$P_t = \frac{160-(-20)}{2-(-2)} + A\left(\frac{2}{-2}\right)^t$$

$$= \frac{180}{4} + A(-1)^t$$

$$= 45 + A(-1)^t \qquad (2)$$

To determine the value of A, first substitute the given value of 50 for Q_0 into the demand function so that

$$160 - 2P_o = 50 = Q_0$$

$$110 = 2P_o$$

$$55 = P_o$$

Now substitute this value for P_o into the general solution (2) above, so that

$$P_o = 55 = 45 + A(-1)^0$$

$$55 = 45 + A$$

$$A = 10$$

The specific solution to the difference equation for this example is therefore

$$P_t = 45 + 10(-1)^t$$

Using this formula to calculate the first few values of P_t we see that

$$P_o = 45 + 10(-1)^0 = 45 + 10 = 55$$

$$P_1 = 45 + 10(-1)^1 = 45 - 10 = 35$$

$$P_2 = 45 + 10(-1)^2 = 45 + 10 = 55$$

$$P_3 = 45 + 10(-1)^3 = 45 - 10 = 35$$

$$P_4 = 45 + 10(-1)^4 = 45 + 10 = 55 \text{ etc.}$$

Price therefore continually fluctuates between 35 and 55.

In this example, $\left|\frac{a}{b}\right| = \left|\frac{2}{-2}\right| = |-1| = 1$

Therefore, as $t \to \infty, P_t$ neither converges on its equilibrium level nor explodes until the market collapses. This fluctuation between two price levels from year to year is sometimes observed in certain agricultural markets.

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Assignment on mAthemAticAl economics (30 %)

General instructions: Attempt all the questions and show all the necessary steps clearly.

1. Differentiate the following concepts(**1.5 pts each**)
   a. Mathematical vs non-mathematical
   b. Static analysis vs comparative analysis
   c. Static analysis vs dynamic analysis
   d. Unconstrained vs constrained optimization
2. Suppose the total cost function for the production of x units of a product is given by:$TC=2x^2-6x+20$
   a. Find the instantaneous rate of change of total cost with respect to the number of unit produced(**2pts**)
   b. Find the level of production at which this rate of change equals zero (**2 pts**).
   c. Find the level of output that minimizes AC (**2 pts**)
3. Find the Mclaurin series for the following function

   $f(x) = 20 + 16x + 72x^2 + 3x^3$ ***(2 pts)***
4. Given the function $Z = f(x, y) = 3x^2 -2x-3xy-7y + 5y^2$
   a. Find the critical values (**1 pt**)
   b. Optimize the function by using the Hessian determinant and state sign definiteness (**2 pts**).
   c. Find the maximum or minimum value (**1 pt**).

5. If $Q_d=ae^{-2p}$, calculate the price elasticity of demand when p=2 (**2 pts**)
6. Find the maxima or minima of the function, $z = 10x^2 + 12y^2 - 2xy$
   subject to 48 = 2x + 4y (**4 pts**)
7. If Q_t^d = 180 - $0.9P_t$ and Q_t^s = -24 + $0.8P_{t-1}$, say whether or not the long-run equilibrium price is stable and then use the difference equation method to calculate price in the thirtieth time period after a sudden one-off increase in quantity to 117 (**3 pts**).
8. There is initially an equilibrium in the macroeconomic model

$$Y_t = C_I + I_t$$
$$C_t = 2{,}500 + 0.9\ Y_{t-1}$$

with the level of I_t set at I_t=100. Investment is then increased to 1,500 where it remains for future time periods. Calculate what the level of Y_t will be in the fourth time period after this investment increase (**3 pts**).

GOOD LUCK!!!