Chapter 3

Canonical forms

A field $\mathbb{F}$ is said to be algebraically closed if every polynomial o degree greater than zero with coefficients in $\mathbb{F}$ has zero in $\mathbb{F}$.

3.1 Elementary row and column operations on matrices

Let K be a field and $\mathbb{F}[x]$ is the polynomial domain. We wish to study matrices over $\mathbb{F}[x]$, consider elementary operations in such matrices. That is The matrix operations of

1. *Interchanging two rows or columns, $(R_i \leftrightarrow R_j)$.*
2. *Adding a multiple of one row or column to another, $(R_i \leftrightarrow R_i + kR_j, k \neq 0)$.*
3. *Multiplying any row or column by a nonzero element. $(R_i \leftrightarrow kR_j)$.*

Each of the operations is called elementary operations.

Definition 3.1.0.1. *A matrix obtained from the identity matrix I by applying an elementary operation is called* ***elementary matrix****.*

Note: - Every elementary matrix is non-singular and its inverse is also elementary matrix.

3.2 Equivalence of matrices of polynomial

Definition 3.2.0.2. *:- Let A and B be matrices over $F(x)$. If B is obtained from A by performing any succession of $F(x)$ elementary operations, A is said to $F(x)$ equivalent to B. Notation: - $A \equiv B$, B is obtained by $\mathbb{F}(x)$ elementary operations of A,and $A \equiv B$ iff $B = PAQ$ where P, Q are products $\mathbb{F}(x)$ elementary matrix.*

Example 3.2.0.2. *Show that*

$$A \overset{\mathbb{F}[x]}{\equiv} B;$$

where

a. $A(x) = \begin{pmatrix} x & x+1 \\ x^2 - x & x^2 - x \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

b. $A(x) = \begin{pmatrix} x^2 & x+1 \\ x-1 & x^2 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ 0 & x^4 - x^2 + 1 \end{pmatrix}.$

3.2.1 Determinantal divisors and related invariants

The K^{th} order minor

Definition 3.2.1.1. *Let A be an $n \times n$ matrix over $\mathbb{F}(x)$, $A = (a_{ij})$, if A_{ij} is a matrix obtained by deleting the i^{th} and j^{th} column of A. The scalar $M_{ij} = det(A_{ij})$ is defined as the $(i, j)^{th}$ minor of A. The $k \times k$ sub-matrix of A is formed by deleting $n - k$ rows and $n - k$ columns of A (that is obtained when the first k rows and columns of A are retained) is called the* ***principal sub matrix*** *and the determinant of principal sub matrix of A is called a* ***principal minor*** *of A. The* ***leading principal minor*** *of A of order k is the minor of order k obtained by deleting the last $n - k$ rows and columns.*

Example 3.2.1.1. $A = \begin{pmatrix} 1 & 2x & 3 \\ 0 & 4 & 5x \\ x & 2 & 1 \end{pmatrix}$. *Find the 1^{st}, 2^{nd} and 3^{rd} order minors of A.*

Solution:

rows-columns	*1,2 (delete 3^{rd} column)*	*1,3 (delete 2^{nd} column)*	*2,3 (delete 1^{st} column)*
1,2 (delete 3^{nd} row)	$\left\lvert\begin{pmatrix} 1 & 2x \\ 0 & 4 \end{pmatrix}\right\rvert = 4$	$\left\lvert\begin{pmatrix} 1 & 3 \\ 0 & 5x \end{pmatrix}\right\rvert = 5x$	$\left\lvert\begin{pmatrix} 2x & 3 \\ 4 & 5x \end{pmatrix}\right\rvert = 10x^2 - 12$
1,3 (delete 2^{nd} row)	$\left\lvert\begin{pmatrix} 1 & 2x \\ x & 2 \end{pmatrix}\right\rvert = 2 - 2x^2$	$\left\lvert\begin{pmatrix} 1 & 3 \\ x & 1 \end{pmatrix}\right\rvert = 1 - 3x$	$\left\lvert\begin{pmatrix} 2x & 3 \\ 2 & 1 \end{pmatrix}\right\rvert = 2x - 6$
2,3 (delete 1^{st} row)	$\left\lvert\begin{pmatrix} 0 & 4 \\ x & 2 \end{pmatrix}\right\rvert = -4x$	$\left\lvert\begin{pmatrix} 0 & 5x \\ x & 1 \end{pmatrix}\right\rvert = -5x^2$	$\left\lvert\begin{pmatrix} 1 & 3 \\ x & 1 \end{pmatrix}\right\rvert = 1 - 3x$

The 2^{nd} order principal minors are

$$\left\lvert\begin{pmatrix} 1 & 2x \\ 0 & 4 \end{pmatrix}\right\rvert = 4, \ \left\lvert\begin{pmatrix} 1 & 3 \\ x & 1 \end{pmatrix}\right\rvert = 1 - 3x, \ \left\lvert\begin{pmatrix} 1 & 3 \\ x & 1 \end{pmatrix}\right\rvert = 1 - 3x.$$

And the 2^{nd} order leading principal minor is $\left\lvert\begin{pmatrix} 1 & 2x \\ 0 & 4 \end{pmatrix}\right\rvert = 4.$

definiteness and principal minors

Theorem 3.2.1.1. *Let A be a symmetric $n \times n$ matrix. Then we have:*

- *A is positive definite $\Delta D_k > 0$ for all principal minors.*
- *A is negative definite $\Leftrightarrow (-1)^k Dk > 0$ for all leading principal minors*
- *A is positive semidefinite $\Leftrightarrow \Delta_k \geq 0$ for all principal minors Δ_k.*
- *A is negative semidefinite $(-1)^k \Delta_k \geq 0$ for all principal minors.*

Example 3.2.1.2. *Determine the definiteness of the symmetric matrix* $A = \begin{pmatrix} 1 & 4 & 6 \\ 4 & 2 & 1 \\ 6 & 1 & 6 \end{pmatrix}$.

Solution:

$$D_1 = 1, D_2 = \left\lvert\begin{pmatrix} 1 & 4 \\ 4 & 2 \end{pmatrix}\right\rvert = -14, \ D_3 = \left\lvert\begin{pmatrix} 1 & 4 & 6 \\ 4 & 2 & 1 \\ 6 & 1 & 6 \end{pmatrix}\right\rvert = -109.$$

Let us compare with the criteria in the theorem: A is indefinite.

3.3 Smith canonical forms and invariant factors

Let A and B be $m \times n$ matrix over $F[x]$.

Lemma 3.3.0.1. *Suppose $B = PAQ$ where P is a product of $\mathbb{F}[x]$ elementary matrices,then every k^{th} order minor of B is a linear combination over $\mathbb{F}[x]$ of k^{th} order minor of A.*

Theorem 3.3.0.2. *Let $A,\ B$ be $m \times n$ matrices over $\mathbb{F}[x]$, then every k^{th} order minor of B is a linear combination over $\mathbb{F}[x]$ of k^{th} order minor of A.*

Corollary 3.3.0.1. *Let $A,\ B$ be $m \times n$ matrices over $\mathbb{F}[x]$, then $A \overset{\mathbb{F}[x]}{\equiv}$ implies $rank(A) = rank(B)$.*

Corollary 3.3.0.2. *Let $A \overset{\mathbb{F}[x]}{\equiv}$, $d_k(x) = gcd\{all\ K^{th}\ order\ minors\ of\ A\}$, then*

$$d_k(x) = gcd\{all\ k^{th}\ order\ minors\ of\ B\}.$$

Lemma 3.3.0.2. *Let A be a non-zero $m \times n$ matrix over $\mathbb{F}[x]$, then A is $\mathbb{F}[x]$ equivalent to $m \times n$ matrix over $\mathbb{F}[x]$ of the form: $\begin{pmatrix} f_1(x) & 0 \\ 0 & A_1 \end{pmatrix}$, where $f_1(x)$ is a monic polynomial of minimal degree among all non-zero elements of all matrices $\mathbb{F}[x]$ equivalent to A.*

Theorem 3.3.0.3. *Let $A,\ B$ be equivalent matrices, $B,\ d_k(x) = gcd\{all\ k^{th}\ order\ minors\ of\ A\}$, then*

$$d_k = f_1(x).f_2(x)...f_k(x).$$

$$D = \begin{pmatrix} f_1(x) & 0 & 0 \\ 0 & f_2(x) & 0 \\ 0 & 0 & f_3(x) \end{pmatrix}.$$

$$D_1 = gcd\{f_1(x), f_2(x), f_3(x)\} = f_1(x),$$

$$D_2 = gcd\{f_1f_2, f_2f_3, f_1f_3\} = f_1f_2.$$

$$\vdots$$

Theorem 3.3.0.4. *Let $A,\ B$ be similar matrices. The polynomials $f_1(x),\ f_2(x),\ ..., f_k(x)$ are uniquely determined by the matrix A.*

Definition 3.3.0.2. *Let A and B be similar matrices. The polynomial $f_1(x),\ f_2(x),\ ..., f_k(x)$ are called the invariant factors of A.*
The matrix B is called the smith canonical form of A and denoted by $SCF(A)$.

Corollary 3.3.0.3. *Let $A_1,\ A_2$ be $m \times n$ matrices over $\mathbb{F}[x]$. $A_1 \overset{\mathbb{F}[x]}{\equiv} A_2$ iff $A_1,\ A_2$ have the same invariant factors.*

Example 3.3.0.3. *Find the invariant factors of $A = \begin{pmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 2 & 3 & x-1 \end{pmatrix}$ and its smith canonical form.*

Solution:
The 1^{st} order minors A are $0,\ 1,\ x,\ 2,\ 3,\ x-1$.

$$d_1(x) = f_1(x) = gcd\{0, 1, 2, 3, x, x-1\} = 1.$$

2^{nd} *order minors are*

rows-columns	*1,2 (delete 3^{rd} column)*	*1,3 (delete 2^{nd} column)*	*2,3 (delete 1^{st} column)*
1,2 (delete 2^{nd} row)	$\begin{vmatrix} x & 1 \\ 0 & x \end{vmatrix} = x^2$	$\begin{vmatrix} x & 0 \\ 0 & 1 \end{vmatrix} = x$	$\begin{vmatrix} 1 & 0 \\ x & 1 \end{vmatrix} = 1$
1,3 (delete 1^{st} row)	$\begin{vmatrix} x & 1 \\ 2 & 3 \end{vmatrix} = 3x - 2$	$\begin{vmatrix} x & 0 \\ 2 & x-1 \end{vmatrix} = x^2 - x$	$\begin{vmatrix} 1 & 0 \\ 3 & x-1 \end{vmatrix} = x - 1$
2,3 (delete 1^{st} row)	$\begin{vmatrix} 0 & x \\ 2 & 3 \end{vmatrix} = -2x$	$\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} = -5x^2$	$\begin{vmatrix} x & 1 \\ 3 & x-1 \end{vmatrix} = x^2 - x - 3$

The 3^{rd} order minor is $|A| = x^3 - x^2 - 5x + 2$,

$$d_3(x) = gcd\{x^3 - x^2 - 5x + 2\} = x^3 - x^2 - 5x + 2,\ f_3(x) = \frac{d_3(x)}{d_2(x)} = d_3(x) = x^3 - x^2 - 5x + 2.$$

The similarity invariants of A are $f_1(x) = 1,\ f_2(x) = 1,\ f_3(x) = x^3 - x^2 - 5x + 2$.
The smith canonical form of A is

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^3 - x^2 - 3x + 2 \end{pmatrix}.$$

Exercise 3.3.0.5. *Find the smith canonical form and invariant factors of the following matrices.*
a.

$$A = \begin{pmatrix} x & x \\ x^2 + x & x \end{pmatrix} \qquad b.B = \begin{pmatrix} x & x^2 \\ x^3 & x^5 \end{pmatrix} \qquad c.C = \begin{pmatrix} x & x+1 \\ x+2 & x+3 \end{pmatrix}$$

3.3.1 Similarity of matrices and invariant factors

Theorem 3.3.1.1. *Let A be a matrix over $\mathbb{F}[x]$ is a product of $F[x]$ elementary matrix iff $det(A) \neq 0$.*

Theorem 3.3.1.2. *Let A be a square matrix over $F[x]$. A has an inverse over $F[x]$ iff A is a product of $F[x]$ elementary matrix.*

Example 3.3.1.1. *Determine whether or not the following matrices over $F[x]$ are non-singular or not ($\mathbb{F} = \mathbb{R}$). $A = \begin{pmatrix} x & x+1 \\ x+2 & x+3 \end{pmatrix}$ b. $B = \begin{pmatrix} 1 & x \\ x & x^2+1 \end{pmatrix}$*
Solution: $det(A) = -2 \neq 0$. Hence A is invertible.

Theorem 3.3.1.3. *Let A be an nxn matrix over $\mathbb{F}[x]$, $X_A(x)$ = the characterstics of A. Then $X_A(x)$ is the product of the invariant factors of $(xI - A)$. (i.e.If $f_1(x), f_2(x), ..., f_n(x)$ are invariant factors of $(xI - A)$, $X_A(x) = f_1(x).f_2(x)....f_n(x)$.*
***Proof**:*

Definition 3.3.1.1. *Let A be an nxn matrix over the field $\mathbb{F}$, then the invariant factors of $(xI - A)$ is called **similarity invariant** of A.*

Theorem 3.3.1.4. *Let A be an nxn matrix over $\mathbb{F}[x]$, $M_A(x)$ = the minimal polynomial of A, then $M_A(x) = f_n(x)$,where $f_n(x)$ if the similarity invariant of a of highest degree of A.*

3.4 The rational canonical forms

Consider the monic polynomial of degree 'n'.

$$f(x) = x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = x^n - (-a_1x^{n-1} - ... - a_{n-1}x - a_n)$$

associate with $f(x)$ is the $n \times n$ matrix. The companion matrix is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{pmatrix}.$$

It is denoted by $C(f)$.
Note:-
If n=1,n=degree of polynomial $f(x) = x - a_0$ the companion matrix $C(f) = (a_0)$.
If n=2 and $f(x) = x^2 - (a_1x + a_2)$, then $C(f) = \begin{pmatrix} 0 & 1 \\ a_2 & a_1 \end{pmatrix}$.
If n=4, $f(x) = x^4 - (3x^3 + 2x^2 - 5x + 2)$, then companion matrix

$$C(f) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & -5 & 2 & 3 \end{pmatrix}.$$

Theorem 3.4.0.5. *Let $f(x) = x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$, then the characteristic and minimal polynomial of the $C(f)$ are both equal to $f(x)$.*

Theorem 3.4.0.6. *Let $f(x) = x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$ be monic polynomial of degree n. Then*

$$[xI - C(f)] \overset{\mathbb{F}[x]}{\equiv} diag(1, 1, ..., f(x)).$$

Theorem 3.4.0.7. *Let $f_1(x).f_2(x).....f_r(x)$ be non constant monic polynomials over $\mathbb{F}$ such that $f_i|f_(i+1)$ for $i = 1 : r - 1$, the campanion matrix $Ci = C(f_i), i = 1, 2, ..., r$. Then the matrix*

$$B = diag(C_1, C_2, ..., C_r)$$

has

$$f_1(x).f_2(x).....f_r(x)$$

its non-trivial similarity invariants.

Theorem 3.4.0.8. *every non-singular matrix A over a field $\mathbb{F}$ is similar to a diagonal block matrix, where each diagonal block is the companion matrix of the non-trivial similarity invariant of A.*

Definition 3.4.0.2. *A diagonal block matrix $diag(C_1, C_2, ..., C_r)$ of the above theorem is called **rational canonical form** for the matrix similar to A.*

Example 3.4.0.2. *Let* $A = \begin{pmatrix} 6 & 2 & -2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$. *Find the rational canonical form of matrices which are similar to A.*
Solution:
similarity invariants are obtained from the matrix similar to A which is $(xI - A)$.

$$(xI - A) = x\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 6 & 2 & -2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} x-6 & -2 & 2 \\ 2 & x-2 & -2 \\ -2 & -2 & x-2 \end{pmatrix}.$$

Now,

$$d_1(x) = \text{the } gcd\{\text{of } 1^{st} \text{ order minor of } (xI - A)\}$$
$$= gcd\{x-6,\ -2,\ 2,\ x-2\} = 1 = f_1(x),$$

$$d_2(x) = gcd\{\text{of all } 2^{nd} \text{ order minors of } (xI-A)\} = gcd\{x^2-4x,\ 2x-8,\ -2x+8,\ x^2-8x+16\} = x-4.$$

$$f_2(x) = \frac{d_2(x)}{d_1(x)} = \frac{(x-4)}{1} = x-4,$$

$$d_3(x) = det(xI - A) = x^3 - 10x^2 + 32x + 32 = (x-4)^4(x-2),$$

$$f_3(x) = \frac{d_3(x)}{d_2(x)} = \frac{(x-4)^4(x-2)}{(x-4)} = (x-4)(x-2) = x^2 - 6x + 8.$$

$$f_1(x),\ f_2(x),\ f_3(x)$$

are similarity invariants of a matrix similar to A. But they are non trivial.
The companion matrix for $f_2(x)$ *is* $C(f_2) = (4)$, *and the companion matrix for* $f_3(x)$ *is* $C(f_3) = \begin{pmatrix} 0 & 1 \\ -8 & 6 \end{pmatrix}$.
Therefore, the rational canonical form of the matrices similar to A is

$$C(f) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -8 & 6 \end{pmatrix}.$$

The minimal polynomial $\mathcal{M}_A(x)$ *of A is* $\mathcal{M}_A(x) = x^2 - 6x + 8$.

3.4.1 Elementary divisors

Let $\mathcal{X}_A(x)$ *and* $f_1(x).f_2(x).....f_r(x)$ *be the characteristic polynomial and the similarity invariants of an* $n \times n$ *matrix A over a field* $\mathbb{F}$.
Suppose that the $\mathcal{X}_A(x) = p_1^{e_1}p_2^{e_2}...p_t^{e_t}$, *where* $p_1,\ p_2,\ ..., p_t$ *are distinct monic polynomials which are irreducible over* $\mathbb{F}$ *and each* e_i *is positive integer and the non-trivial similarity invariant of A is given by*

$$f_i = p_1^{e_{i1}}p_2^{e_{i2}}...p_t^{e_{it}}.$$

Since $f_i | f_{i+1},\ e_{i+1,j} \geq e_{ij}$.

Definition 3.4.1.1. *The polynomial* p_j *which appears in similarity invariants of A with non-zero exponents* e_{ij} *are called the elementary divisors of A over the field* $\mathbb{F}$. *Remark:*
$p_j(x)$ *is a monic polynomial.* $p_j(x)$ *is a power of irreducible polynomial (one cannot be factored).*
The characteristic polynomial of A is a product elementary divisors.
Hence the minimal polynomial is the least common multiple of all elementary divisors.

Example 3.4.1.1. *Let $f_1(x) = (x-1)(x+1)$, $f_2(x) = (x^2-1)(x+1)(x^2+2)$, $f_3(x) = (x-1)^2(x-1)^2(x^2+2)$ be similarity invariants of A, then the elementary divisors of A are*

$$(x-1)(x+1),\ (x-1)^2,\ x+1,\ x^2+2,\ (x-1)^2,\ (x+1)^2,\ x^2+2.$$

Example 3.4.1.2. *Suppose $x,\ x,\ x^2,\ x+1,\ (x-1)^2,\ (x-1)^2,\ x-1,\ (x-1)^3$ are the elements of the elementary divisors of A, then the non trivial similarity invariants of are*

$$f_1(x) = x,\ f_2(x) = x(x-1)(x+1),\ f_3(x) = (x)^2(x-1)^3(x-1)^2).$$

Theorem 3.4.1.1. *Let A and B be an $n \times n$ matrix over a field $\mathbb{F}$. $A \sim B$ if and only if A and B have the same elementary divisor.*

3.5 The normal and Jordan canonical forms

3.5.1 Normal canonical forms

Let $f(x) = x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$. We can write $f(x) = p_1^{e_1}p_2^{e_2}...p_t^{e_t}$,where p_j are distinct monic, irreducible polynomials over $\mathbb{F}[x]$ and $e_i^{'}s$ are positive integers .

Lemma 3.5.1.1. *let $C = C(f)$.Then $C \sim diag(C_1, C_2, ..., C_t)$ where $C_i = f(p_i^{e_i})$.*

Lemma 3.5.1.2. *Let A be $n \times n$ matrix over a field $\mathbb{F}$, $g_1, g_2, ..., g_n$ be the elementary divisors of A, $C_i = f(g_i)$ for $i = 1, 2, ..., r$. Then $A \sim daiag(C_1, C_2, ..., C_r)$.*

Definition 3.5.1.1. *Let $C_1, C_2, ..., C_r$ be the matrix $diag(C_1, C_2, ..., C_r)$ is called* ***normal canonical form*** *for the matrix similar to A.*

Example 3.5.1.1. *Find the normal canonical form of a matrix similar to*

$$A = \begin{pmatrix} 6 & 2 & 2 \\ -2 & 2 & 0 \\ 0 & 0 & 2. \end{pmatrix}.$$

Solution:

Step1:- find elementary divisor of $(xI - A) = \begin{pmatrix} x-6 & -2 & -2 \\ 2 & x-2 & 0 \\ 0 & 0 & x-2 \end{pmatrix}$.

The 1^{st} order minors of $(xI - A)$ are $x-6,\ -2,\ -2,\ -2,\ 2,\ 0,\ x-2$.

$$d_1(x) = gcd\{0, -2, 2, x-6, x-2\} = 1 = f_1(x).$$

The 2^{nd} order minors of $(xI - A)$ are

$$x^2 - 8x + 16, x^2 - 4x + 4, -(x-2), 4, 2(x-2), x^2 - 8x + 10, x - 4$$

$$d_2(x) = gcd x^2 - 8x + 16, x^2 - 4x + 4, -(x-2), 4, 2(x-2), x^2 - 8x + 10, x - 4 = 1,$$

$$f_2(x) = \frac{d_2(x)}{d_1(x)} = 1.$$

The 3^{rd} order minor of $(xI - A)$ is $(x-2)(x-4)^2 = d_3(x)$.

$$f_3(x) = \frac{d_3(x)}{d_2(x)} = (x-2)(x-4)^2.$$

The elementary divisors of A are $(x-2)$ and $(x-4)^2$.
Step 2:- compute the companion matrix for the elementary divisors of $(xI - A)$. Let $C_1 = C(x-2) = (2)$, $C_2 = C((x-4)^2) = C(x^2 - 8x + 16) = \begin{pmatrix} 0 & 1 - 16 & 8 \end{pmatrix}$.
Step 3:-

$$A \sim diag(C_1, C_2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -16 & 8 \end{pmatrix}$$

which is the normal canonical form of any matrix similar to A.

3.5.2 Jordan Canonical Form

Suppose the elementary divisors $p_j^{e_{ij}}$ of an $n \times n$ matrix A are of the form $(x - \lambda_j)^{e_{ij}}$. We define the Jordan block corresponding to the elementary divisor $p_j^{e_{ij}} = (x - \lambda_j)^{e_{ij}}$ to be the $e_{ij} \times e_{ij}$ of matrix J_j, given by

$$J_j = \begin{pmatrix} \lambda_{j1} & 1 & 0 & \dots & 0 \\ 0 & \lambda_{j2} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{j2} \end{pmatrix}.$$

Theorem 3.5.2.1. *Let $\mathbb{F}$ be algebraically closed field. If the $n \times n$ matrix A over $\mathbb{F}$ has r elementary divisors with associated Jordan $J_1, J_2, \cdots, J_r$, then $A \sim diag(J_1, J_2, ..., J_r)$.*

Definition 3.5.2.1. *If $A \sim diag(J_1, J_2, ..., J_r)$ is called the **Jordan canonical form** of A.*

Example 3.5.2.1. *Let $A = \begin{pmatrix} 6 & 2 & 2 \\ -2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Find the jordan canonical form of the matrix similar to A.*
Solution: From previous example we have seen $(x-2)$ and $(x-4)^2$ are elementary divisors of A.
Let J_1 be the jordan canonical form of $(x-2)$, $J_1 = (2)$.
Let J_2 be the jordan canonical form of

$$(x-4)^2, \ J_2 = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}.$$

Hence the Jordan canonical form of the matrices similar to A is given by

$$diag(J_1, J_2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$