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Arba Minch University

Numerical Method(Math-2073/53)
Lecture note: Chapter-VI

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

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Contents

Chapter 1

Numerical Solution of Differential Equations

Definition 1.1

A differential equation *is a mathematical equation that relates some function with its derivatives. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology.*

In pure mathematics, differential equations are studied from several different perspectives, mostly concerned with their solutions the set of functions that satisfy the equation. Only the simplest differential equations are solvable by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form.

If a self- contained formula for the solution is not available, the solution may be numerically approximated using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

Differential equations can be divided into several types. Apart from describing the properties of the equation itself, these classes of differential equations can help inform the choice of approach to a solution. Commonly used distinctions include whether the equation is: Ordinary/Partial, Linear/Non-linear, and Homogeneous/In-homogeneous. This list is far from exhaustive; there are many other properties and sub-classes of differential equations which can be very useful in specific contexts.

Ordinary differential equations

Definition 1.2

An ordinary differential equation (ODE) *is an equation containing a function of one independent variable and its derivatives. The term "ordinary" is used in contrast with the term partial differential equation which may be with respect to more than one independent variable.*

Linear differential equations, which have solutions that can be added and multiplied by coefficients, are well-defined and understood, and exact closed-form solutions are obtained.

By contrast, ODE s that lack additive solutions are nonlinear, and solving them is far more intricate, as one can rarely represent them by elementary functions in closed form: Instead, exact and analytic solutions of ODE s are in series or integral form. Graphical and numerical methods, applied by hand or by computer, may approximate solutions of ODE s and perhaps yield useful information, often sufficing in the absence of exact, analytic solutions.

Partial differential equations

Definition 1.3:

is a differential equation that contains unknown multivariable functions and their partial derivatives. (This is in contrast to ordinary differential equations, which deal with functions of a single variable and their derivatives.) PDEs are used to formulate problems involving functions of several variables, and are either solved in closed form, or used to create a relevant computer model.

PDEs can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics. These seemingly distinct physical phenomena can be formalised similarly in terms of PDEs. Just as ordinary differential equations often model one-dimensional dynamical systems, partial differential equations often model multidimensional systems. PDEs find their generalization in stochastic partial differential equations.

Examples

In the first group of examples, let u be an unknown function of x, and let $c\&\omega$ be known constants. Note both ordinary and partial differential equations are broadly classified as linear and nonlinear.

1. In-homogeneous first-order linear constant coefficient ordinary differential equation:
$$\frac{du}{dx} = cu + x^2.$$
2. Homogeneous second-order linear ordinary differential equation:
$$\frac{d^2u}{dx^2} - x\frac{du}{dx} + u = 0.$$
3. Homogeneous second-order linear constant coefficient ordinary differential equation describing the harmonic oscillator:
$$\frac{d^2u}{dx^2} + \omega^2 u = 0.$$
4. In-homogeneous first-order nonlinear ordinary differential equation:
$$\frac{du}{dx} = u^2 + 4.$$
5. Second-order nonlinear (due to sine function) ordinary differential equation describing the motion of a pendulum of length L:
$$L\frac{d^2u}{dx^2} + g\sin u = 0.$$
In the next group of examples, the unknown function u depends on two variables x and t or x and y.

6. Homogeneous first-order linear partial differential equation:

$$\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = 0.$$

7. Homogeneous second-order linear constant coefficient partial differential equation of elliptic type, the Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}.$$

1.1 Euler Method

Why numerical solutions? For many of the differential equations we need to solve in the real world, there is no "nice" algebraic solution. That is, we can't solve it using the techniques we have met in this chapter (separation of variables, integrable combinations, or using an integrating factor), or other similar means.

As a result, we need to resort to using numerical methods for solving such DEs. The concept is similar to the numerical approaches we saw in an earlier integration chapter (Trapezoidal Rule, Simpson's Rule and Riemann Sums).

Even if we can solve some differential equations algebraically, the solutions may be quite complicated and so are not very useful. In such cases, a numerical approach gives us a good approximate solution.

Initial Value Problem: Consider

$$dy/dt = f(t, y) \text{ with } y(t_0) = y_0$$

From the definition of the derivative

$$dy/dt = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h}$$

Instead of taking the limit, fix h, so

$$dy/dt \approx \frac{y(x+h) - y(x)}{h}$$

Substitute into the differential equation and with algebra write

$$y(x+h) \approx y(t) + hf(t, y)$$

Euler's Method for a fixed h is

$$y(t+h) = y(t) + hf(t, y)$$

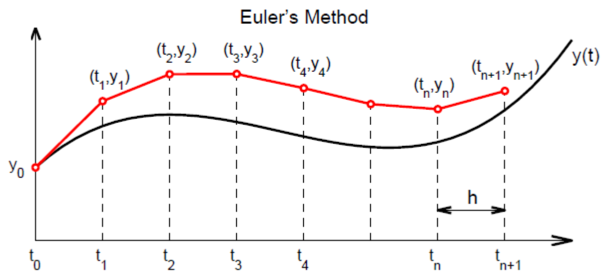
Geometrically, Euler's method looks at the slope of the tangent line

- The approximate solution follows the tangent line for a time step h.

- Repeat this process at each time step to obtain an approximation to the solution.

The ability of this method to track the solution accurately depends on the length of the time step, h, and the nature of the function $f(t, y)$. This technique is rarely used as it has very bad convergence properties to the actual solution.

Graph of Euler's Method



Euler's Method Formula: Euler's method is just a discrete dynamical system for approximating the solution of a continuous model

- Let $t_{n+1} = t_n + h$
- Define $y_n = y(t_n)$
- The initial condition gives $y(t_0) = y_0$
- **Euler's Method** is the discrete dynamical system

$$y_{n+1} = y_n + hf(t_n, y_n)$$

- Euler's Method only needs the initial condition to start and the right hand side of the differential equation (the slope field), $f(t, y)$ to obtain the approximate solution

Definition 1.4: Algorithm (Euler's Method)

Consider the initial value problem

$$dy/dt = f(t, y), y(t_0) = y_0.$$

Let h be a fixed stepsize and define $t_n = t_0 + nh$. Also, let $y(t_n) = y_n$.
Euler's Method *for approximating the solution to the IVP satisfies the difference equation*

$$y_{n+1} = y_n + hf(t_n, y_n).$$

Example 1.1

Consider the model

$$dy/dt = y + t \text{ with } y(0) = 3$$

Find the solution to this initial value problem
Exact Solution:
With the initial condition the solution is

$$y(t) = 4e^t - t - 1$$

Numerical solution *Euler's formula with h = 0.25 is*

$$y_{n+1} = y_n + 0.25(y_n + t_n)$$

t_n	Euler solution y_n
$t_0 = 0$	$y_0 = 3$
$t_1 = 0.25$	$y_1 = y_0 + h(y_0 + t_0) = 3 + 0.25(3 + 0) = 3.75$
$t_2 = 0.5$	$y_2 = y_1 + h(y_1 + t_1) = 3.75 + 0.25(3.75 + 0.25) = 4.75$
$t_3 = 0.75$	$y_3 = y_2 + h(y_2 + t_2) = 4.75 + 0.25(4.75 + 0.5) = 6.0624$
$t_4 = 1$	$y_4 = y_3 + h(y_3 + t_3) = 6.0624 + 0.25(6.0624 + 0.75) = 7.7656$

Actual solution is $y(1) = 8.8731$, *so the Euler solution has a* 12.5% *error*
If h = 0.1, after 10 steps $y(1) \approx y_{10} = 8.3750$ *with* 5.6% *error*

Euler's formula with different h is

t_n	$h = 0.2$	$h = 0.1$	$h = 0.05$	$h = 0.025$	Actual
0.2	3.6	3.64	3.662	3.6736	3.6856
0.4	4.36	4.4564	4.5098	4.538	4.5673
0.6	5.312	5.4862	5.5834	5.6349	5.6885
0.8	6.4944	6.7744	6.9315	7.015	7.1022
1	7.9533	8.375	8.6132	8.7403	8.8731
2	21.7669	23.91	25.16	25.8383	26.5562
% Err	−18.0	−9.96	−5.26	−2.70	

We see the percent error at $t = 2$ *(compared to the actual solution) declining by about 1/2 as h is halved*

1.2 Improved Euler's Method

Algorithm (Improved Euler's Method (or Heun Formula)) Consider the initial value problem

$$dy/dt = f(t, y), y(t_0) = y_0$$

. Let h be a fixed stepsize. Define $t_n = t_0 + nh$ and the approximate solution $y(t_n) = y_n$.

1. Approximate y by Euler's Method

$$ye_n = y_n + hf(t_n, y_n)$$

2. Improved Euler's Method is the difference formula

$$y_{n+1} = y_n + \frac{h}{2}(f(t_n, y_n) + f(t_n + h, ye_n))$$

or

$$\begin{aligned} k_1 &= hf(t_n, y_n) \\ k2 &= hf(t_n + h, y_n + k_1) \\ y_{n+1} &= y_n + \frac{k_1 + k_2}{2} \end{aligned}$$

Improved Euler's Method Formula: This technique is an easy extension of Euler's Method

- The Improved Euler's method uses an average of the Euler's method and an Euler???s method approximation to the function
- This technique requires two function evaluations, instead of one
- Simple two step algorithm for implementation
- Can show this converges as $O(h^2)$, which is significantly better than Euler's method

Example 1.2

Improved Euler's Method: Consider the initial value problem:

$$dy/dt = y + t \text{ with } y(0) = 3$$

- *Numerically solve this using Euler's Method and Improved Euler's Method using* $h = 0.1$
- *Compare these numerical solutions*

Solution: *Let* $y_0 = 3$, *the Euler's formula is*

$$y_{n+1} = y_n + h(y_n + t_n) = y_n + 0.1(y_n + t_n)$$

The Improved Euler's formula is

$$ye_n = y_n + h(y_n + t_n) = y_n + 0.1(y_n + t_n)$$

with

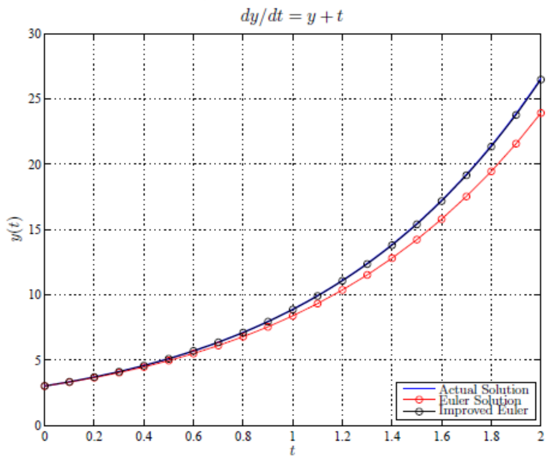
$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}((y_n + t_n) + (ye_n + t_n + h)) \\ y_{n+1} &= y_n + 0.05(y_n + ye_n + 2t_n + 0.1) \end{aligned}$$

Example

Solution: Below is a table of the numerical computations

t	Euler's Method	Improved Euler	Actual
0	$y_0 = 3$	$y_0 = 3$	$y(0) = 3$
0.1	$y_1 = 3.3$	$y_1 = 3.32$	$y(0.1) = 3.3207$
0.2	$y_2 = 3.64$	$y_2 = 3.6841$	$y(0.2) = 3.6856$
0.3	$y_3 = 4.024$	$y_3 = 4.0969$	$y(0.3) = 4.0994$
0.4	$y_4 = 4.4564$	$y_4 = 4.5636$	$y(0.4) = 4.5673$
0.5	$y_5 = 4.9420$	$y_5 = 5.0898$	$y(0.5) = 5.0949$
0.6	$y_6 = 5.4862$	$y_6 = 5.6817$	$y(0.6) = 5.6885$
0.7	$y_7 = 6.0949$	$y_7 = 6.3463$	$y(0.7) = 6.3550$
0.8	$y_8 = 6.7744$	$y_8 = 7.0912$	$y(0.8) = 7.1022$
0.9	$y_9 = 7.5318$	$y_9 = 7.9247$	$y(0.9) = 7.9384$
1	$y_{10} = 8.3750$	$y_{10} = 8.8563$	$y(1) = 8.8731$

Graph of Solution: Actual, Euler's and Improved Euler's



The Improved Euler's solution is very close to the actual solution

Comparison of the numerical simulations

- *It is very clear that the Improved Euler's method does a substantially better job of tracking the actual solution*
- *The Improved Euler's method requires only one additional function, $f(t, y)$, evaluation for this improved accuracy At $t = 1$, the Euler's method has a -5.6% error from the actual solution*
- *At $t = 1$, the Improved Euler's method has a -0.19% error from the actual solution*

1.3 The Runge-Kutta method

Although Euler's method is easy to implement, this method is not so efficient in the sense that to get a better approximation, one need a very small step size. One way to get a better accuracy is to include the higher order terms in the Taylor expansion in the formula. But the higher order terms involve higher derivatives of y. The Runge-Kutta methods attempt to obtain greater accuracy and at the same time avoid the need for higher derivatives, by evaluating the function $f(x, y)$ at selected points on each subintervals. A general Runge Kutta algorithm is given as

$$y_{n+1} = y_n + h\phi(x_n, y_n, h) \tag{1.1}$$

The function ϕ is termed as *increment function*. The m^{th} order Runge-Kutta method gives accuracy of order $O(h^m)$. The function ϕ is chosen in such a way that when expanded the right hand side of (1.1) matches with the Taylor series up to desired order. This means that for a second order Runge-Kutta method the right side of (1.1) matches up to second order terms of Taylor series.

1.3.1 Second Order Runge-Kutta

The Second order Runge Kutta methods are known as **RK2** methods. For the derivation of second order Runge Kutta methods, it is assumed that ϕ is the weighted average of two functional evaluations at suitable points in the interval $[x_n, x_{n+1}]$, i.e., $\phi(x_n, y_n, h) = w_1k_1 + w_2k_2$. Thus, we have:

$$y_{n+1} = y_n + [w_1k_1 + w_2k_2] \tag{1.2}$$

where

$$k_1 = hf(x_n, y_n),\ k_2 = hf(x_n + \alpha h, y_n + \beta k_1) \tag{1.3}$$

Here w_1, w_2, α and β are constants to be determined so that equation (1.2) agrees with the Taylor algorithm of a possible higher order.

Now, let's write down the Taylor series expansion of y in the neighborhood of x_n correct to the h^2 term i.e

$$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2}f'(x_n, y(x_n)) + O(h^3) \tag{1.4}$$

Then, using chain rule for the derivative $f'(x_n, y(x_n))$ we get

$$f'(x_n, y(x_n)) = \frac{\partial f(x_n, y(x_n))}{\partial x} + f(x_n, y(x_n))\frac{\partial f(x_n, y(x_n))}{\partial y},$$

Thus we have

$$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2}\left[\frac{\partial f(x_n, y(x_n))}{\partial x} + f(x_n, y(x_n))\frac{\partial f(x_n, y(x_n))}{\partial y}\right] + O(h^3) \tag{1.5}$$

In addition, equation (1.2) and (1.3) can be rewritten as:

$$\begin{aligned} y_{n+1} &= y_n + w_1hf(x_n, y_n) + w_2hf(x_n + \alpha h, y_n + \beta hf(x_n, y_n)) \\ &= y_n + w_1hf(x_n, y_n) + w_2h\left[f(x_n, y_n) + \alpha h\frac{\partial f(x_n, y_n)}{\partial x} + \beta hf(x_n, y_n)\frac{f(x_n, y_n)}{\partial y} + O(h^2)\right] \\ &= y_n + h(w_1 + w_2)f(x_n, y_n) + h^2\left[w_2\alpha\frac{\partial f(x_n, y_n)}{\partial x} + w_2\beta f(x_n, y_n)\frac{\partial f(x_n, y_n)}{\partial y}\right] + O(h^3) \end{aligned}$$

Therefore,

$$y_{n+1} = y_n + h(w_2 + w_2)f(x_n, y_n) + h^2\left[w_2\alpha\frac{\partial f(x_n, y_n)}{\partial x} + w_2\beta f(x_n, y_n)\frac{\partial f(x_n, y_n)}{\partial y}\right] + O(h^3). \tag{1.6}$$

Assuming $y(x_n) \approx y_n$ and comparing equations (1.5) and (1.6) yields

$$w_1 + w_2 = 1, \qquad w_2\alpha = \frac{1}{2} \qquad \text{and} \qquad w_2\beta = \frac{1}{2}. \tag{1.7}$$

Observe that four unknowns are to be evaluated from three equations. Accordingly many solutions are possible for (1.7). Two examples of second-order Runge-Kutta methods of the form (1.2) and (1.3) are the modified Euler method and the improved Euler method.

(a) **The modified Euler method** In this case we take $\beta = \frac{1}{2}$ obtain

$$y_{n+1} = y_n + hf\left(x_n + \frac{1}{2}h, y_n + \frac{h}{2}f(x_n, y_n)\right).$$

(b) **The improved Euler method, usually called RK2** This is arrived at by choosing $\beta = 1$ which gives

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + h, y_n + k_1), \\ y_{n+1} &= y_n + \frac{1}{2}(k_1 + k_2). \end{aligned}$$

1.3.2 Fourth Order Runge-Kutta

A similar but more complicated analysis is used to construct Runge-Kutta methods of higher order. One of the most frequently used methods of the Runge-Kutta family is often known as the classical **fourth-order** method, **RK4**, given by:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{1.8}$$

where

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1) \\ k_3 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2) \\ k_4 &= hf(x_n + h, y_n + k_3). \end{aligned}$$

Example 1.3

Consider the initial value problem

$$y' = y, \quad y(0) = 1.$$

Approximate y(0.05) with a step-size $h = 0.01$ using RK2 and RK4.
Solution: *Here $f(x, y) = y, x_0 = 0, \quad y_0 = 1, \quad$ and $\quad h = 0.01$. First let's use RK2 for $n = 0, 1, \cdots, 5$. A $x = 0.01$ or when $n = 0$:*

$$\begin{aligned} k_1 &= hf(x_0, y_0) = hy_0 = 0.010000 \\ k_2 &= hf(x_0 + h, y_0 + k_1) = h(y_0 + k_1) = 0.01(1 + 0.01) = 0.010100 \\ y_1 &= y_0 + \frac{1}{2}(k_1 + k_2) = 1.0 + 0.5(0.010000 + 0.010100) = 1.010050 \end{aligned}$$

Example

At $x = 0.02$ *or when* $n = 1$*:*

$$k_1 = hf(x_1, y_1) = hy_1 = 0.01(1.010050) = 0.010100$$
$$k_2 = hf(x_1 + h, y_1 + k_1) = h(y_1 + k_1) = 0.01(1.010050 + 0.010100) = 0.010202$$
$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2) = 1.010050 + 0.5(0.010100 + 0.010202) = 1.020201$$

Repeating this until $x = 0.05$ *we get the following*

x_i	k_1	k_2	y_i
0.0	–	–	1.000000
0.1	0.010000	0.010100	1.010050
0.2	0.010100	0.010202	1.020201
0.3	0.010202	0.010304	1.030454
0.4	0.010305	0.010408	1.040810
0.5	0.010408	0.010512	1.051270

Therefore, $y(0.05) = 1.051270$
Now, let us use RK4 for $n = 0, 1, \cdots 5$.
A $x = 0.01$ *or when* $n = 0$*:*

$$k_1 = hf(x_0, y_0) = hy_0 = 0.010000$$
$$k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) = h(y_0 + \frac{1}{2}k_1) = 0.01(1 + 0.005) = 0.010050$$
$$k_3 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) = h(y_0 + \frac{1}{2}k_2) = 0.01(1 + 0.005025) = 0.010050$$
$$k_4 = hf(x_0 + h, y_0 + k_3) = h(y_0 + k_3) = 0.01(1 + 0.010050) = 0.010101$$
$$y_1 = y_0 + \frac{1}{2}(k_1 + 2k_2 + 2k_3 + k_4) = 1.0 + \frac{1}{6}(0.010000 + 2 \times 0.010050 + \times 0.010050 + 0.010101)$$
$$= 1.010050$$

At $x = 0.02$ *or when* $n = 1$*:*

$$k_1 = hf(x_1, y_1) = hy_1 = 0.01(1.010050) = 0.010101$$
$$k_2 = hf(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1) = h(y_1 + \frac{1}{2}k_1) = 0.01(1.010050 + 0.005050) = 0.010151$$
$$k_3 = hf(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2) = h(y_1 + \frac{1}{2}k_2) = 0.01(1.010050 + 0.5 \times 0.010151) = 0.010151$$
$$k_4 = hf(x_1 + \frac{1}{2}h, y_1 + k_3) = h(y_1 + k_3) = 0.01(1.010050 + 0.010151) = 0.010202$$
$$y_2 = y_1 + \frac{1}{2}(k_1 + 2k_2 + 2k_3 + k_4) = 1.010050 + \frac{1}{6}(0.010101 + 2 \times 0.010151 + 2 \times 0.010151 + 0.010202)$$
$$= 1.020201.$$

Repeating the above procedure until $x = 0.05$ we get the following

x_i	k_1	k_2	k_3	k_4	y_i
0.0	–	–	–	–	1.000000
0.1	0.010000	0.010050	0.010050	0.010101	1.010050
0.2	0.010101	0.010151	0.010151	0.010202	1.020201
0.3	0.010202	0.010253	0.010253	0.010305	1.030455
0.4	0.010305	0.010356	0.010356	0.010408	1.040811
0.5	0.010408	0.010460	0.010460	0.010513	1.051271

Therefore, $y(0.05) = 1.051271$

Example 1.4

$$\begin{cases} y' = y - t^2 + 1 \\ y(0) = 0.5 \end{cases}$$

The exact solution for this problem is $y = t^2 + 2t + 1 - \frac{1}{2}e^t$, *and we are interested in the value of* y *for* $0 \leq t \leq 2$.

1. ***We first solve this problem using RK4 with*** $h = 0.5$. *From* $t = 0$ *to* $t = 2$ *with step size* $h = 0.5$, *it takes* 4 *steps:*$t_0 = 0,\ t_1 = 0.5,\ t_2 = 1,\ t_3 = 1.5,\ t_4 = 2.$

Step 0 $t_0 = 0, y_0 = 0.5.$

Step 1 $t_0 = 0.5$

$$\begin{aligned} k_1 &= hf(t_0, y_0) = 0.5f(0, 0.5) = 0.75 \\ k_2 &= hf(t_0 + h/2, y_0 + k_1/2) = 0.5f(0.25, 0.875) = 0 : 90625 \\ k_3 &= hf(t_0 + h/2, y_0 + k_2/2) = 0.5f(0.25, 0.953125) = 0.9453125 \\ k_4 &= hf(t_0 + h, y_0 + k_3) = 0.5f(0.5, 1.4453125) = 1.09765625 \\ y_1 &= y_0 + (k_1 + 2k_2 + 2k_3 + k_4) = 6 = 1.425130208333333 \end{aligned}$$

Step 2 $t_2 = 1$

$$\begin{aligned} k_1 &= hf(t_1, y_11) = 0.5f(0.5, 1.425130208333333) = 1.087565104166667 \\ k_2 &= hf(t_1 + h/2, y_1 + k_1/2) = 0.5f(0.75, 1.968912760416667) = 1.203206380208333 \\ k_3 &= hf(t_1 + h/2.y_1 + k_2/2) = 0.5f(0.75, 2.0267333984375) = 1.23211669921875 \\ k_4 &= hf(t_1 + h, y_1 + k_3) = 0.5f(1, 2.657246907552083) = 1.328623453776042 \\ y_2 &= y_1 + (k_1 + 2k_2 + 2k_3 + k_4)/6 = 2.639602661132812 \end{aligned}$$

Step 3 $t_3 = 1.5$

$$\begin{aligned} k_1 &= hf(t_2, y_2) = 0.5f(1, 2.639602661132812) = 1.319801330566406 \\ k_2 &= hf(t_2 + h/2, y_2 + k_1/2) = 0.5f(1.25, 3.299503326416016) = 1.368501663208008 \\ K_3 &= hf(t_2 + h/2, y_2 + k_2/2) = 0.5f(1.25, 3.323853492736816) = 1.380676746368408 \\ K_4 &= hf(t_2 + h, y_2 + K_3) = 0.5f(1.5, 4.020279407501221) = 1.385139703750610 \\ y_3 &= y_2 + (k_1 + 2k_2 + 2k_3 + k_4)/6 = 4.006818970044454 \end{aligned}$$

Step 4 $t_4 = 2$

$$\begin{aligned} k_1 &= hf(t_3, y_3) = 0.5f(1.5, 4.006818970044454) = 1.378409485022227 \\ k_2 &= hf(t_3 + h/2, y_3 + k_1/2) = 0.5f(1.75, 4.696023712555567) = 1.316761856277783 \\ K_3 &= hf(t_3 + h/2, y_3 + k_2/2) = 0..5f(1.75, 4.665199898183346) = 1.301349949091673 \\ K_4 &= hf(t_3 +, y_3 + K_3) = 0.5f(2, 5.308168919136127) = 1.154084459568063 \\ y_4 &= y_3 + (k_1 + 2k_2 + 2k_3 + k_4)/6 = 5.301605229265987 \end{aligned}$$