Chapter 1

Revision on Set Theory

1.1 Sets and set operations,ordered pairs

Introduction

Set theory is a branch of mathematical logic that studies sets, which informally are collections of objects. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. The language of set theory can be used to define nearly all mathematical objects.

Definition 1.1.1. *A **set** is a (unordered) well-defined collection of objects.* [a] *To indicate that we are considering a set, the objects (or the description) are put inside a pair of set braces, $\{\}$.) These objects are sometimes called elements or members of the set and can be anything: numbers, people, letters of the alphabet, and so on.*

[a] Well defined: (if there is a way to determine that an object belongs to the set or not.

Georg Cantor, one of the founders of set theory, gave the following definition of a set at the beginning of his Beiträge zur Begründung der transfiniten Mengenlehre:
"A set is a gathering together into a whole of definite, distinct objects of our perception or of our thought—which are called **elements of the set**. "

Terminology

For a set A having an element x, the following are all used synonymously:

- x is a member of A
- x is contained in A
- x is included in A
- x is an element of the set A
- A contains x
- A includes x

Notation We specify a set by specifying its members. The curly brace notation is used for this purpose. $\{1, 2, 3\}$ is the set containing 1, 2, 3 as members.

The curly brace notation can be extended to specify a set by specifying a rule for set membership. (“—” means “such that”.)
There are four ways of representing sets

Representation of sets

1. **Roster method (Tabular form)** In this method a set is represented by listing all its elements, separating these by commas and enclosing these in curly bracket. If V be the set of vowels of English alphabet, it can be written in Roster form as :

$$V = \{a, e, i, o, u\}.$$

   e.g., If A be the set of natural numbers less than 7. then $A = \{1, 2, 3, 4, 5, 6\}$, is in the Roster form.

2. **Set-builder form** In this form elements of the set are not listed but these are represented by some common property.
   e.g., Let V be the set of vowels of English alphabet then V can be written in the set builder form as: $V = \{x : x \textit{ is a vowel of English alphabet}\}$.
   (a) $A = \{-3, -2, -1, 0, 1, 2, 3\}$
   (b) $B = \{3, 6, 9, 12\}$
   Solution :
   (a) $A = \{x : x \in \mathbb{Z} \textit{ and } 3 \leq x \leq 3\}$
   (b) $B = \{x : x = 3n \textit{ and } n \in \mathbb{N}, n \leq\}$.

3. **Interval Notation** Used to describe subsets of sets upon which an order is defined, e.g., numbers.
   $[a, b] = \{x|a \leq x \leq b\}$,
   $[a, b) = \{x|a \leq x < b\}$,
   $(a, b] = \{x|a < x \leq b\}$, $(a, b) = \{x|a < x < b\}$ closed interval $[a, b]$ open interval (a, b) half-open intervals $[a, b)$ and $(a, b]$.

4. **Venn diagram** British mathematician John Venn (1834-1883 AD) introduced the concept of diagrams to represent sets. According to him universal set is represented by the interior of a rectangle and other sets are represented by interior of circles. It is a drawing in which geometric figures. It is a way of depicting the relationship between sets.One use of Venn diagrams is to illustrate the effects of set operations.

Example 1.1.1. *Write the following in Roster form.*
(a) $C = \{x : x \in \mathbb{N} \textit{ and } 50 \leq x \leq 60\}$
(b) $D = \{x : x \in \mathbb{R} \textit{ and } x^2 - 5x + 6 = 0\}$.
Solution :
(a) $C = \{50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60\}$
(b) $x^2 - 5x + 6 = 0 \Rightarrow (x - 3)(x - 2) = 0$.
$\Rightarrow x = 3, 2 \textit{ and } D = \{2, 3\}$.

Sets are conventionally denoted with capital letters.
There are two ways of describing, or specifying the members of, a set.

1. *One way is by intensional definition, using a rule or semantic description:*
   A is the set whose members are the first four positive integers. B is the set of colors of the Ethiopian flag.

2. *The second way is by extension – that is, listing each member of the set.*

An extensional definition is denoted by enclosing the list of members in curly brackets:

$$C = \{4, 2, 1, 3\}$$

$$D = \{green, yellow, red\}.$$

Characteristics of sets
A set is uniquely identified by its members.

$$\left\{x \middle| x \text{ is an even prime}\right\}$$
$$\left\{x \middle| x \text{ is a positive square root of } 4\right\}$$
$$\{2\}$$

1.2 Classification of Sets

Equal and Equivalent Sets *Moreover, the sets A, B are said to be equal if and only if every element of A is also an element of B, and every element of B is an element of A and **equivalent** if they have the equal number elements.*
Repetition of members is inconsequential in specifying a set.
The expressions

$$\begin{aligned}&\{1, 2, 3\}\\ &= \left\{1, 1, 1, 1, 2, 3\right\}\\ &= \left\{x \middle| x \text{ is an even prime or } x \text{ is a positive square root of } 4 \text{ or } x = 1 \text{ or } x = 2 \text{ or } x = 3\right\}\end{aligned}$$

all specify the same set.
Sets are unordered. The expressions

$$\{1, 2, 3\} = \{3, 2, 1\} = \{2, 1, 3\}$$

all specify the same set.
Sets can have other sets as members.

Example 1.2.1. *, The set*

$$\left\{\{1, 2\}, \{2, 3\}, \{1, Newton\}\right\}.$$

Finite and infinite sets *A is said to be an infinite set and B is said to be is finite set. A set is said to be finite if its elements can be counted and it is said to be infinite if it is not possible to count upto its last element.*

empty set

The set with no members is the ***empty or null set****. The expressions*

$$\{\}$$
$$\varnothing$$
$$\Big\{x : x \neq x\Big\}$$

all specify the empty set.
A set with exactly one member is called a ***singleton****. A set with exactly two members is called a* ***doubleton****.*

Thus $\{1\}$ is a singleton and $\{1, 2\}$ is a doubleton.

Subsets

Definition 1.2.1. *If every element of set A is also in B, then A is said to be a subset of B, written $A \subset B$ (pronounced A is contained in B).*
Equivalently, one can write $B \supseteq A$, read as B is a superset of A, B includes A, or B contains A.
The relationship between sets established by $\subset$ is called ***inclusion*** *or* ***containment****, and is given also for equal sets, that is, equality of sets is the same as mutual containment in each other: $A \subset B$ and $B \subset A$ is equivalent to $A = B$.*
If A is a subset of, but not equal to, B, then A is called a ***proper subset*** *of B, written $A \subsetneq B$, or simply $A \subset B$ (A is a proper subset of B), or $B \supsetneq A$ (B is a proper superset of A, $B \supset A$).*
The expressions $A \supset B$ and $B \supset A$ are used differently by different authors; some authors use them to mean the same as $A \subseteq B$ (respectively $B \supseteq A$), whereas others use them to mean the same as $A \subsetneq B$ (respectively $B \supsetneq A$).

Disjoint Sets *: Two sets are said to be disjoint if they do not have any common element. For example,sets $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$ are disjoint sets.*

Example 1.2.2. *Given that $A = \{2, 4\}$ and $B = \{x : x \text{ is a solution of } x^2 + 6x + 8 = 0\}$. Are A and B disjoint sets ?*
Solution :
If we solve $x^2 + 6x + 8 = 0$, we get $x = -4, -2$.

$$\therefore B = \{-4, -2\} \text{ and } A = \{2, 4\}.$$

Clearly ,A and B are disjoint sets as they do not have any common element.

Example 1.2.3. *If $A = \{x : x \text{ is a vowel of English alphabet}\}$ and $B = \{y : y \in \mathbb{N} \text{ and } y \leq 5\}$. Is (i) $A = B$ (ii) $A \approx B$?*
Solution :

$$A = \{a, e, i, o, u\}, \;\; B = \{1, 2, 3, 4, 5\}.$$

Each set is having five elements but elements are different.

$$\therefore A \neq B \text{ but } A \approx B.$$

Exercise 1.2.1. *Which of the following sets* $A = \{x : x \text{ is a point on a line}\}$, $B = \{y : y \in \mathbb{N} \text{ and } y \leq 50\}$ *are finite or infinite ?*

Example 1.2.4. *The expression* $\{1, 2\} \subseteq \{1, 2, 3\}$ *says that* $\{1, 2\}$ *is a subset also proper subet of* $\{1, 2, 3\}$.
The empty set is a subset of every set.
Every set is a subset of itself.

Cardinality of sets

Definition 1.2.2. *The cardinality of a set* S*, denoted* $|S|$*, is the number of members of* S*.*

Example 1.2.5. *If* $B = \{blue, white, red\}$*, then* $|B| = 3$.
There is a unique set with no members, called the empty set. The cardinality of the empty set is zero.

Example 1.2.6. *The set of all three-sided squares has zero members and thus is the empty set. Though it may seem trivial, the empty set, like the number zero, is important in mathematics. Indeed, the existence of this set is one of the fundamental concepts of axiomatic set theory*[1]*.*
Some sets have infinite cardinality. The set $\mathbb{N}$ *of natural numbers, for instance, is infinite. Some infinite cardinalities are greater than others.*
For instance, the set of real numbers has greater cardinality than the set of natural numbers.
However, it can be shown that the cardinality of (which is to say, the number of points on) a straight line is the same as the cardinality of any segment of that line, of the entire plane, and indeed of any finite-dimensional Euclidean space.

Example 1.2.7. *Let* $A = \{1, 2, 3, 4, 5\}$, $|A| = 5$.

Power sets

A power set of a set is the set of all its ***subsets****.* $\mathcal{P}$ *is used for the power set.*
Note that the empty set and the set itself are members of the power set.

$$\mathcal{P}\{1, 2, 3\} = \Big\{\varnothing, \{1\}, \{2\}, \{3\}, \{1,\ 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}.$$

If set S *is finite with* $|S| = n$*, then* $|\mathcal{P}(S)| = 2^n$.

Universal set

- The universal set is denoted by U*: the set of all objects under the consideration.*

[1] axiomatic set theory

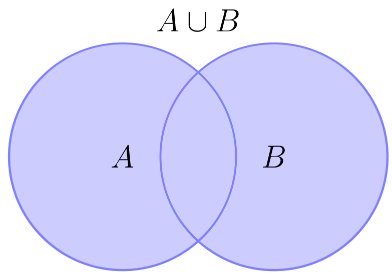


Figure 1.1 – The union of sets A and B

1.2.1 set operations

Sets can be combined in a number of different ways to produce another set. Here four basic operations are introduced and their properties are discussed.

Definition 1.2.3. *(Union): The union of sets A and B, denoted by $A \cup B$, is the set defined as*

$$A \cup B = \{x | x \in A \vee x \in B\}$$

Example 1.2.8. *: If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.*

Example 1.2.9. *: If $A = \{1, 2, 3\}$ and $B = \{1, 2, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.*

A more general form of the principle can be used to find the cardinality of any finite union of sets:

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n| &= (|A_1| + |A_2| + |A_3| + \ldots |A_n|) \\ &- (|A_1 \cap A_2| + |A_1 \cap A_3| + \ldots |A_{n-1} \cap A_n|) \\ &+ \ldots \\ &+ (-1)^{n-1} (|A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n|) . \end{aligned}$$

Basic properties of unions

- $A \cup B = B \cup A$.
- $A \cup (B \cup C) = (A \cup B) \cup C$.
- $A \subset (A \cup B)$.
- $A \cup A = A$.
- $A \cup U = U$.
- $A \cup \emptyset = A$.
- $A \subset B$ *if and only if* $A \cup B = B$.

Definition 1.2.4. *(Intersection): The intersection of sets A and B, denoted by $A \cap B$, is the set defined as*

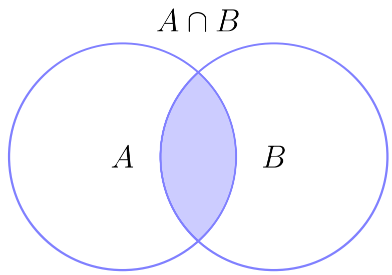


Figure 1.2 – The intersection of sets A and B

$$A \cap B = \{x|x \in A \wedge x \in B\}$$

Example 1.2.10. *: If $A = \{1, 2, 3\}$ and $B = \{1, 2, 4, 5\}$, then $A \cap B = \{1, 2\}$.*

Example 1.2.11. *: If $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then $A \cap B = \emptyset$.*

basic properties of intersections:

Let A and B be two sets. Then

1. $A \cap B = B \cap A$ *(Commutativity).*
2. $A \cap (B \cap C) = (A \cap B) \cap C$ *(Associativity).*
3. $A \cap B \subset A$ *and* $A \cap B \subset B$
4. $A \cap A = A$, $A \cap U = A$.
5. $A \cap \emptyset = \emptyset$.
6. $A \subset B$ *if and only if* $A \cap B = A$.

For sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|. \quad \textit{The principle of inclusion and exclusion}$$

Generally, The inclusion–exclusion principle is a counting technique that can be used to count the number of elements in a union of two sets, if the size of each set and the size of their intersection are known. It can be expressed symbolically as

- *If A and B are finite sets, then $n(A \cup B) = n(A) + n(B) - n(A \cap B.)$*
- *If $A \cap B = \emptyset$, then $n(A \cup B) = n(A) + n(B)$.*

Definition 1.2.5. *(Difference or Relative Complement): The difference of sets A from B , denoted by $A - B$, is the set defined as*

$$A - B = \{x|x \in A \wedge x \notin B\}$$

For sets A and B,

$$n(A - B) = n(A) - n(A \cap B).$$

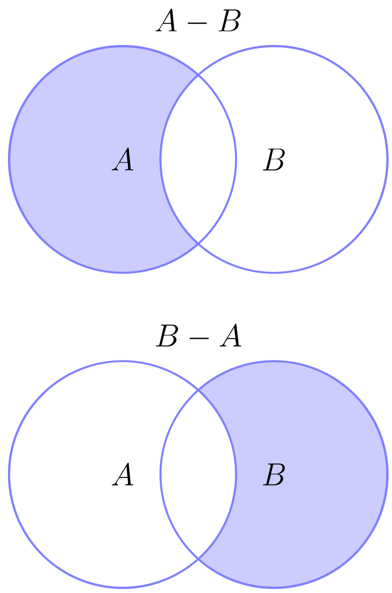


Figure 1.3 – The difference of sets A and B respectively

Example 1.2.12. *: If* $A = \{1, 2, 3\}$ *and* $B = \{1, 2, 4, 5\}$ *, then* $A - B = \{3\}$ *.*

Example 1.2.13. *: If* $A = \{1, 2, 3\}$ *and* $B = \{4, 5\}$ *, then* $A - B = \{1, 2, 3\}$ *.*
Note that in general $A - B \neq B - A$*.*

Definition 1.2.6. *(Absolute Complement): For a set* A*, the difference* $U - A$ *, where* U *is the universe, is called the complement of* A *and it is denoted by* A'*.*
Thus is the set of everything that is not in A*.*

Some basic properties of complements:

- $A\ B \neq B\ A$ *for* $A \neq B$.
- $A \cup A' = U$.
- $A \cap A' = \emptyset$.
- $(A')' = A$.
- $\emptyset \setminus A = \emptyset$.
- $A \setminus \emptyset = A$.
- $A \setminus A = \emptyset$.
- $A \setminus U = \emptyset$.
- $A \setminus A' = A$ *and* $A' \setminus A = A'$.
- $U' = \emptyset$ *and* $\emptyset' = U$.
- $A \setminus B = A \cap B'$.
- *if* $A \subset B$ *then* $A \setminus B = \emptyset$.

Example 1.2.14. *Two programs were broadcast on television at the same time; one was the Big Game and the other was Ice Stars. The Nelson Ratings Company uses boxes attached to television sets to determine what shows are actually being watched. In its survey of* 1000 *homes at the midpoint of the broadcasts, their equipment showed that* 153 *households were watching both shows,* 736 *were watching the Big Game and* 55 *households were not watching either. How many households were watching only Ice Stars? What percentage of the households were not watching either broadcast?*

Solution

Let B *the set of Big Game and* I *Ice Stars.*

$|B \cap I| = 153$ *households were watching both broadcasts.*

$n(B) = 736$,*were watching the Big Game.*

Since we already have 153 *in that part of* B *that is in common with* I, *the remaining part of* B *will have* $736 - 153 = 583$.

This tells us that 583 *households were watching only the Big Game.*

We are also told that 55 *households were watching neither program,*

$$\Rightarrow n((\overline{I \cup B})) = 55.$$

Finally, we know that the total of everything should be 1000, $n(U) = 1000$.

Since only one area does not yet contain a number it must be the missing amount to add up to 1000.

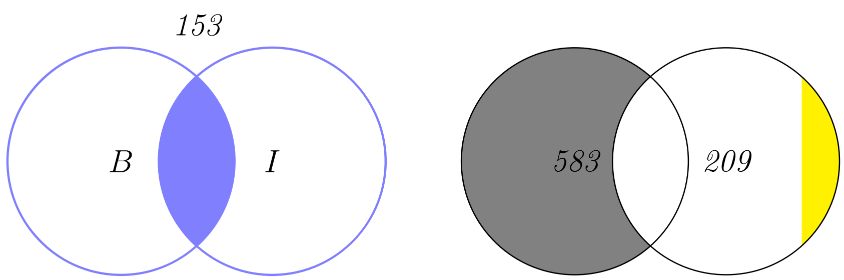
We add the three numbers that we have, $583 + 153 + 55 = 791$, *and subtract that total from 1000,* $1000 - 791 = 209$, *to get the number that were watching only Ice Stars.*

Now, we have the information needed to answer any questions about the survey results.

In particular, we were asked how many households were watching only Ice Stars, we found this

number to be 209.
We were also asked what percentage of the households were watching only the Big Game. The number watching only the game was found to be 583, so we compute the percentage,

$$\frac{583}{1000} * 100\% = 58.3\%.$$



Exercise 1.2.2. *In a recent survey people were asked if the took a vacation in the summer, winter, or spring in the past year. The results were 73 took a vacation in the summer, 51 took a vacation in the winter, 27 took a vacation in the spring, and 2 had taken no vacation. Also, 10 had taken vacations at all three times, 33 had taken both a summer and a winter vacation, 18 had taken only a winter vacation, and 5 had taken both a summer and spring but not a winter vacation.*

1. *How many people had been surveyed?*
2. *How many people had taken vacations at exactly two times of the year?*
3. *How many people had taken vacations during at most one time of the year?*
4. *What percentage had taken vacations during both summer and winter but not spring?*

Proposition 1.2.1. *: Let A, B, C be sets. Then*

1. $A \cup A = A$, $A \cap A = A$, *and* $A \setminus A = A$;
2. $A \cup \emptyset = A$ *and* $A \cap \emptyset = \emptyset$;
3. $A \cup (B \cup C) = (A \cup B) \cup C$ *and* $A \cap (B \cap C) = (A \cap B) \cap C$;
4. $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$;
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof
We will prove (1) and (3) and leave the remaining results to be proven in the exercises.
(1) Observe that

$$\begin{aligned} A \cup A &= \{x : x \in A \text{ or } x \in A\} \\ &= \{x : x \in A\} \\ &= A. \end{aligned}$$

and

$$\begin{aligned} A \cap A &= \{x : x \in A \text{ and } x \in A\} \\ &= \{x : x \in A\} \\ &= A. \end{aligned}$$

Also, $A \setminus A = A \cap A' = \emptyset$
(3) For sets $A, B,$ *and* C*,*

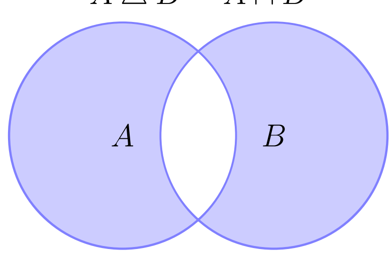
$$\begin{aligned} A \cup (B \cup C) &= A \cup \{x : x \in B or x \in C\} \\ &= \{x : x \in A \text{ or } x \in B \text{ or } x \in C\} \\ &= \{x : x \in A \text{ or } x \in B\} \cup C \\ &= (A \cup B) \cup C \end{aligned}$$

A similar argument proves that $A \cap (B \cap C) = (A \cap B) \cap C$.
The fourth set operation is the ***Cartesian product*** *We first define an ordered pair and Cartesian product of two sets using it. Then the Cartesian product of multiple sets is defined using the concept of* $n-$*tuple.*

Definition 1.2.7. *An extension of the complement is the* ***symmetric difference****, defined for sets* A*,* B *as*

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

$A \triangle B = \overline{A \cap B}$



Example 1.2.15. *, the symmetric difference of* $\{7, 8, 9, 10\}$ *and* $\{9, 10, 11, 12\}$ *is the set* $\{7, 8, 11, 12\}$.

Some Properties of Union and Intersection *Based on the preceding definitions, we can derive some useful properties for the operations on sets. The proofs of these properties are left as* *an exercise to the reader. The union and intersection operations are commutative. That is, for sets* A, B

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Furthermore, they are associative. That is, for sets A, B, C

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Furthermore, union distributes over intersection and intersection distributes over union. That is, for sets A, B, C

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Theorem: De Morgan's laws

Two important propositions for sets are De Morgan's laws. They state that, for sets A, B, C

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$$
$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

When A *is a universe to which* B *and* C *belong, De Morgan's laws can be stated more simply as,*

$$(B \cup C)^C = B^C \cap C^C$$
$$(B \cap C)^C = B^C \cup C^C$$

In words:
"The complement of a union is the intersection of the complements." "The complement of an intersection is the union of the complements."

Proof

Families of sets

A set of sets is usually referred to as a family or collection of sets. Often, families of sets are written with either a script or Fraktur font to easily distinguish them from other sets. For a family of sets $\mathfrak{A}$ *, define the union and intersection of the family by,*

$$\bigcup \mathfrak{A} = \bigcup_{A \in \mathfrak{A}} A = \{x : x \text{ is in some } A \in \mathfrak{A}\}$$
$$\bigcap \mathfrak{A} = \bigcap_{A \in \mathfrak{A}} A = \{x : x \text{ is in all } A \in \mathfrak{A}\}$$

For a family of sets, we say that it is pairwise disjoint if any two distinct sets we choose from the family are disjoint.

1.2.2 Ordered Pairs

Definition 1.2.8. *(ordered pair): An* ***ordered pair*** *is a pair of objects with an order associated with them.*
If objects are represented by x *and* y*, then we write the ordered pair as* $< x, y >$*.*
Two ordered pairs $< a, b >$ *and* $< c, d >$ *are equal if and only if* $a = c$ *and* $b = d$*.*

Example 1.2.16. *The ordered pair* $< 1, 2 >$ *is not equal to the ordered pair* $< 2, 1 >$*.*

Definition 1.2.9. *(Cartesian product): A new set can be constructed by associating every element of one set with every element of another set.*
The set of all ordered pairs (a, b)*, where* a *is an element of* A *and* b *is an element of* B*, is called the* ***Cartesian product*** *of* A *and* B *and is denoted by* $A \times B$*. The concept of Cartesian product can be extended to that of more than two sets. First we are going to define the concept of ordered* $n-$*tuple.*

Example 1.2.17.

$$\{1, 2\} \times \{red, white, green\} = \{(1, red), (1, white), (1, green), (2, red), (2, white), (2, green)\}.$$

$$\{1, 2\} \times \{1, 2\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}.$$

$$\{a, b, c\} \times \{d, e, f\} = \{(a, d), (a, e), (a, f), (b, d), (b, e), (b, f), (c, d), (c, e), (c, f)\}.$$

Exercise 1.2.3. *Let* $A = \{a, b, c\}, \ B = \{d, e\}, \ C = \{a, d\}$*. Find*
(i) $A \times B$ *(ii)* $B \times A$ *(iii)* $A \times (B \cup C)$
(iv) $(A \cap C) \times B$ *(v)* $(A \cap B) \times C$ *(vi)* $A \times (B \setminus C)$ *.*

Definition 1.2.10. *(ordered* $n-$*tuple): An ordered* $n-$*tuple is a set of* n *objects with an order associated with them (rigorous definition to be filled in). If* n *objects are represented by* $x_1, x_2, ..., x_n$*, then we write the ordered* $n-$*tuple as* $(x_1, x_2, ..., x_n)$ *.*

Example 1.2.18. *The ordered* $3-$*tuple* $(1, 2, 3)$ *is not equal to the ordered* $n-$*tuple* $(2, 3, 1)$*.*

Definition 1.2.11. *(Cartesian product): Let $A_1, ..., A_n$ be n sets. Then the set of all ordered $n-$tuples $(x_1, ..., x_n)$, where $x_i \in A_i$ for all i, $1 \leq i \leq n$, is called the* ***Cartesian product*** *of $A_1, ..., A_n$, and is denoted by $A_1 \times ... \times A_n$.*

Basic Properties Cartisian Product

Let A, B and C be sets. Then

- $A \times \emptyset = \emptyset$.
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
- $(A \cup B) \times C = (A \times C) \cup (B \times C)$.
- *If A, or $B = \emptyset$ or $A = B = \emptyset$, then $A \times B = B \times A = \emptyset$.*

If A and B are finite sets; then the ***cardinality*** *of the Cartesian product is the product of the* ***cardinalities****:*

$$|A \times B| = |B \times A| = |A| \times |B|.$$

1. *In a class of 40 students, 15 like to play cricket and football and 20 like to play cricket. How many like to play football only but not cricket?*
   Solution: Let C = Students who like cricket

   F = Students who like football

   $C \cap F$ = Students who like cricket and football both

   $C - F$ = Students who like cricket only

   $F - C$ = Students who like football only.

   Given:
   $n(C) = 20,\ n(C \cap F) = 15,\ n(C \cup F) = 40$

   Required:
   $n(F) =?$

$$\Rightarrow n(C \cup F) = n(C) + n(F) - n(C \cap F)$$

$$\Rightarrow 40 = 20 + n(F) - 15 \Rightarrow 40 = 5 + n(F) \Rightarrow 40–5 = n(F)$$

$$\Rightarrow n(F) = 35$$

$$\Rightarrow n(F - C) = n(F) - n(C \cap F) = 35 - 15 = 20.$$

Therefore, Number of students who like football only but not cricket $= 20$.

2. *There is a group of 80 persons who can drive scooter or car or both. Out of these, 35 can drive scooter and 60 can drive car. Find how many can drive both scooter and car? How many can drive scooter only? How many can drive car only?*
   Solution:
   Let $S = \{$ *Persons who drive scooter* $\}$

   $C = \{$ *Persons who drive car*$\}$

   Given, $n(S \cup C) = 80,\ n(S) = 35\ n(C) = 60.$

   Therefore,

$$n(S \cup C) = n(S) + n(C) - n(S \cap C)$$
$$\Rightarrow 80 = 35 + 60 - n(S \cap C)$$
$$\Rightarrow 80 = 95 - n(S \cap C)$$
$$\textit{Therefore, } n(S \cap C) = 95\text{–}80 = 15$$

   Therefore, 15 persons drive both scooter and car.

   Therefore, the number of persons who drive a scooter only $= n(S) - n(S \cap C) = 35 - 15 = 20.$

   Also, the number of persons who drive car only $= n(C) - n(S \cap C) = 60 - 15 = 45$

3. *It was found that out of 45 girls, 10 joined singing but not dancing and 24 joined singing. How many joined dancing but not singing? How many joined both?*

   Solution:
   Let $S = \{$*Girls who joined singing*$\}$

   $D = \{$*Girls who joined dancing*$\}$.

   Number of girls who joined dancing but not singing = Total number of girls - Number of girls who joined singing

$$45 - 24 = 21.$$
$$\textit{Now, } n(S - D) = 10,\ n(S) = 24$$
$$\textit{Therefore, } n(S - D) = n(S) - n(S \cap D)$$
$$\Rightarrow n(S \cap D) = n(S) - n(S - D)$$
$$= 24 - 10$$
$$= 14$$

   Therefore, number of girls who joined both singing and dancing is 14.

Set identities *There are a number of set identities that the set operations of union, intersection, and set difference satisfy. They are very useful in calculations with sets. Below we give a table of such set identities, where U is a universal set and A, B, and C are subsets of U.*

Commutative Laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative Laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive Laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotent Laws	$A \cap A = A$	$A \cap A = A$
Absorption Laws	$A \cap (A \cup B) = A$	$A \cup (A \cap B) = A$
Identity Laws	$A \cup \emptyset = A$	$A \cup U = A$
Universal Bound Laws	$A \cup U = U$	$A \cap \emptyset = \emptyset$
DeMorgan's Laws	$(A \cap B)^c = A^c \cup B^c$	$(A \cap B)^c = A^c \cup B^c$
Complement Laws	$U^c = \emptyset$	$A \cap A^c = \emptyset$
Complents of U and $\emptyset$	$U^c = \emptyset$	$\emptyset^c = U$
Double Complement Law	$(A^c)^c = A$	$\emptyset^c = U$
Set Difference Law	$A - B = A \cap B^c$	

Proof

Set theory is seen as the foundation from which virtually all of mathematics can be derived. For example, structures in abstract algebra, such as groups, fields and rings, are sets closed under one or more operations.

*One of the main applications of naive set theory is **constructing relations**.*

Summary

- $\cup$ *is Union: is in either set or both sets.*
- $\cap$ *is Intersection: only in both sets.*
- $-$ *is Difference: in one set but not the other.*
- A^c *is the Complement of A: everything that is not in A.*
- $\triangle$ *is symmetric difference: is in each set but not both.*
- *Empty Set: the set with no elements. Shown by* $\{\}$.
- *Universal Set: all things we are interested in.*

1.3 Relations and functions

1.3.1 Relation

Definition 1.3.1. *A **relation***

1. *is an association between objects.*

2. *from a set A to B is a set of ordered pairs* $\{(a, b) : a \in A\ b \in B\}$. *We write aRb, say that a is related to b by R and* $R \subseteq A \times B$.

If A and B are two sets then a relation R from A to B is a sub set of $A \times B$. *If*

1. $R = \emptyset$, *R is called a void relation.*
2. $R = A \times B$, *R is called a universal relation.*
3. *If R is a relation defined from A to A, it is called a relation defined on A.*
4. $R = \{(a, a)\ \forall a \in A\}$, *is called the identity relation.*

Definition 1.3.2. ***Domain:*** *Let* $R \subset A \times B$ *be a relation. The* ***domain*** *of R is the set*

$$dom(R) = \{x \in A : \exists y(y \in B \wedge (x, y) \in R)\},$$

of the given relation, and the ***range*** *of R is the set*

$$ran(R) = \{y \in B : \exists x(x \in A \wedge (x, y) \in R)\}.$$

Example 1.3.1. *If* $R = \{(1, 3), (2, 4), (2, 5)\}$, *then* $dom(R) = \{1, 2\}$ *and* $ran(R) = \{3, 4, 5\}$.

Example 1.3.2. *Let* $S = \{(x, y) : |x| = y \wedge x, y \in R\}$. *Notice that both* $(2, 2)$ *and* $(-2, 2)$ *are elements of S. Furthermore,* $dom(S) = R$ *and* $range(S) = [0, \infty)$.

Example 1.3.3. *Given that* $A = \{2, 4, 5, 6, 7\},\ B = \{2, 3\}$. *R is a relation from A to B defined by*

$$R = \{(a, b) : a \in A,\ b \in B \text{ and a is divisible by } b\}$$

find (i) R in the roster form (ii) Domain of R (iii) Range of R.
(iv) Repersent R diagramatically.
Solution :

$$\text{(i) } R = \{(2, 2), (4, 2), (6, 2), (6, 3)\}.$$
$$\text{(ii) Domain of } R = \{2, 4, 6\}.$$
$$\text{(iii) Range of } R = \{2, 3\}.$$

Exercise 1.3.1. *If R is a relation 'is greater than' from A to B, where* $A = \{1, 2, 3, 4, 5\}$ *and* $B = \{1, 2, 6\}$. *Find (i) R in the roster form. (ii) Domain of R (iii) Range of R.*

Example 1.3.4. *State the domain and range using the relation:* $\{(-1, 2), (0, 4), (0, -3), (1, -3)\}$.
(Hint: To list the domain and range you list them from least to greatest in set notation.)
Solution:
Domain $\{-1, 0, 1\}$ *; Range* $\{-3, 2, 4\}$

Definition 1.3.3. *A binary relation R on sets X and Y is a definite relation between elements of X and elements of Y . We write xRy if* $x \in X$ *and* $y \in Y$ *are related. One can also define relations on more than two sets, but we shall consider only binary relations and refer to them simply as relations. If* $X = Y$, *then we call R a relation on X.*

Definition 1.3.4. *A set R is an **(n-ary) relation** if there exist sets $A_0, A_1, \cdots, A_{n-1}$ such that*

$$R \subset A_0 \times A_1 \times \cdots \times A_{n-1}.$$

In particular, R is a unary relation if $n = 1$ and a binary relation if $n = 2$. If $R \subset A \times A$ for some set A, then R is a relation on A and we write (A, R).

Example 1.3.5. *The less-than relation on $\mathbb{Z}$ is defined as*

$$L = \{(a, b) : a, b \in \mathbb{Z} \wedge a < b\}.$$

Another approach is to use membership in the set of positive integers as our condition. That is,

$$L = \{(a, b : a, b \in \mathbb{Z} \wedge b - a \in \mathbb{Z}^+\}.$$

Hence, $(4, 7) \in L$ because $7 - 4 \in \mathbb{Z}^+$.
Representation of Relations using Graph
A relation can be represented using a directed graph.
The number of vertices in the graph is equal to the number of elements in the set from which the relation has been defined.
For each ordered pair (x, y) in the relation R, there will be a directed edge from the vertex 'x' to vertex 'y'.
If there is an ordered pair (x, x), there will be self- loop on vertex 'x'.

Types of Relation *When we are looking at relations, we can observe some special properties different relations can have.*

1. *The **Full Relation** between sets X and Y is the set*

$$X \times Y.$$

2. *The **Inverse Relation** R' of a relation R is defined as*

$$R' = \{(b, a) | (a, b) \in R\}.$$

3. *A relation R on set A is called **Reflexive** if $\forall \in A$ is related to a (aRa holds).*

4. *A relation R on set A is called **Irreflexive** if no $a \in A$ is related to a (aRa does not hold). The relation of equality, " $=$ " is reflexive. Observe that for, say, all numbers a (the domain is R): $a = a$ so " $=$ " is reflexive.*
   In a reflexive relation, we have arrows for all values in the domain pointing back to themselves:

   Note that $\leq$ is also reflexive ($a \leq a$ for any $a \in R$). On the other hand, the relation $<$ is not ($a < a$ is false for any $a \in R$).

5. *Symmetric*
   A relation is symmetric if, we observe that for all values of a and b: aRb implies bRa. The relation of equality again is symmetric. If $x = y$, we can also write that $y = x$ also. In a symmetric relation, for each arrow we have also an opposite arrow, i.e. there is either no arrow between x and y, or an arrow points from x to y and an arrow back from y to x:

   Neither $\leq$ nor $<$ is symmetric ($2 \leq 3$ and $2 < 3$ but neither $3 \leq 2$ nor $3 < 2$ is true).

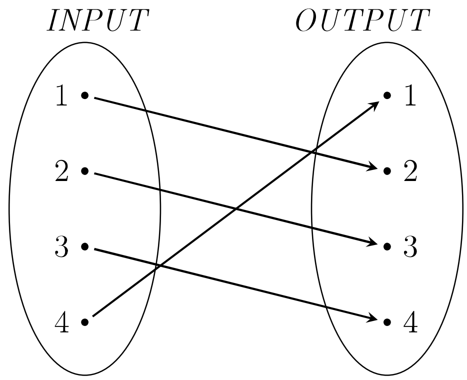
6. *Antisymmetric A relation is antisymmetric if we observe that for all values a and b: aRb and bRa implies that $a = b$.*
   Notice that antisymmetric is not the same as "not symmetric."
   Take the relation greater than or equal to, " $\geq$ " If $x \geq y$, and $y \geq x$, then y must be equal to x. a relation is anti-symmetric if and only if $a \in A$, $(a, a) \in R$.

7. *Transitive*
   A relation is transitive if for all values a, b, c: aRb and bRc implies aRc.
   The relation greater-than " $>$ " is transitive. If $x > y$, and $y > z$, then it is true that $x > z$. This becomes clearer when we write down what is happening into words. x is greater than y and y is greater than z. So x is greater than both y and z.
   The relation is-not-equal " $\neq$ " is not transitive. If $x \neq y$ and $y \neq z$ then we might have $x = z$ or $x \neq z$.

8. ***Trichotomy***
   A relation satisfies trichotomy if we observe that for all values a and b it holds true that: aRb or bRa The relation is-greater-or-equal satisfies since, given 2 real numbers a and b, it is true that whether $a \geq b$ or $b \geq a$ (both if $a = b$).

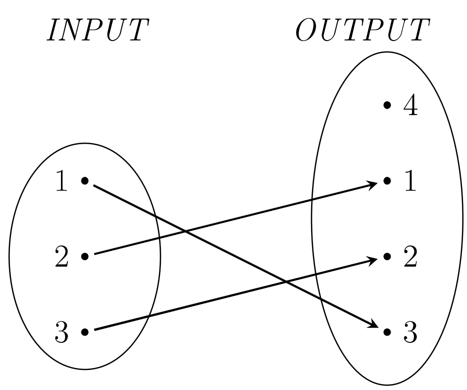
Example 1.3.6. *[]*

- *If $R = \{(1, 2), (2, 3)\}$, then R' will be $\{(2, 1), (3, 2)\}$.*
- *The relation $R = \{(a, a), (b, b)\}$ on set $X = \{a, b\}$ is reflexive.*
- *The relation $R = \{(a, b), (b, a)\}$ on set $X = \{a, b\}$ is irreflexive.*
- *The relation $R = \{(1, 2), (2, 1), (3, 2), (2, 3)\}$ on set $A = \{1, 2, 3\}$ is symmetric.*
- *The relation $R = \{(x, y) \to N | x \leq y\}$ is anti-symmetric since $x \leq y$ and $y \leq x$ implies $x = y$.*
- *The relation $R = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$ on set $A = \{1, 2, 3, 4\}$ is transitive.*



1.3.2 Function

This relation f from set A to B where every element of A has a unique image in B is defined as a function from A to B. So we observe that in a function no two ordered pairs have the same first element.



This leads us to the following definition.

Function

Definition 1.3.5. *A* ***function*** *or mapping (Defined as $f : X \to Y$) is a special type of relation in which each input has exactly one output. It is a relationship from elements of one set X to elements of another set Y (X and Y are non-empty sets). X is called* ***Domain*** *and Y is called* ***Codomain*** *of function* ***'f'****.*
Function 'f' is a relation on X and Y such that for each $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in R$. 'x' is called ***pre-image*** *and 'y' is called* ***image*** *of function f.*
A function relates an input to an output.

Example 1.3.7. *Tree grows 20 cm every year, so the height of the tree is related to its age using the function*

$$h : h(age) = age \times 20$$

So, if the age is 10 years, the height is:

$$h(10) = 10 \times 20 = 200cm.$$

Special rule *It must work for every possible input value.*
And it has only one relationship for each input value
Functions can be represented in several different ways; ordered pairs, table of values, mapping diagrams, graphs and in function notation.

So from the above example and the definition we observe that in a function no two ordered pairs have the same first element.
We also see that $\exists$ an element $\in B$. Thus here:
(i) the set B will be termed as co-domain and
(ii) the set $\{1, 2, 3\}$ is called the range.
And we can conclude that range is a subset of co-domain. Symbolically, this function can be written as

$$f : A \to B \text{ or } A \underset{\to}{f} B$$

Example 1.3.8. *Which of the following relations are functions from A to B.*
Write their domain and range. If it is not a function give reason ?

(a). $\{(1, -2), (3, 7), (4, -6), (8, 1)\}$, $A = \{1, 3, 4, 8\}$, $B = \{-2, 7, -6, 1, 2\}$.

(b). $\{(1, 0), (1, -1), (2, 3), (4, 10)\}$, $A = \{1, 2, 4\}$, $B = \{0, -1, 3, 10\}$.

(c). $\{(a, b), (b, c), (c, b), (d, c)\}$, $A = \{a, b, c, d, e\}$, $B = \{b, c\}$.

(d). $\{(2, 4), (3, 9), (4, 16), (5, 25), (6, 36)\}$, $A = \{2, 3, 4, 5, 6\}$, $B = \{4, 9, 16, 25, 36\}$.

(e). $\{(1, -1), (2, -2), (3, -3), (4, -4), (5, -5)\}$, $A = \{0, 1, 2, 3, 4, 5\}$, $B = \{-1, -2, -3, -4, -5\}$.

Solution :
(a) It is a function. *Domain*$= \{1, 3, 4, 8\}$, *Range* $= \{-2, 7, -6, 1\}$.

(b) It is not a function. *Because first two ordered pairs have same first elements.*
(c) It is not a function. *Domain*$= \{a, b, c, d\} \neq A$, *Range* $= \{b, c\}$.
(d) It is a function. *Domain* $= \{2, 3, 4, 5, 6\}$, *Range* $= \{4, 9, 16, 25, 36\}$.
(e) It is not a function . *Domain* $= \{1, 2, 3, 4, 5\} \neq A$, *Range* $= \{-1, -2, -3, -4, -5\}$.
Let us consider some functions which are only defined for a certain subset of the set of real numbers.

Example 1.3.9. *Find the domain of each of the following functions :*

$$a.\ y = \frac{1}{x} \quad b.\ y = \frac{1}{x-2} \quad c.\ y = \frac{1}{(x+2)(x-3)}$$

Solution

a. *The function* $y = \frac{1}{x}$ *can be described by the following set of ordered pairs.* $\{\cdots, (-2, \frac{-1}{2}), (-1, -1), (1, 1), ($
Here we can see that x *can take all real values except* 0 *because the corresponding image, i.e.,* $\frac{1}{0}$ *is not defined.*
$\therefore$ *Domain* $= \mathbb{R} - \{0\}$.

b. x *can take all real values except* 2 *because the corresponding image, i.e.,* $\frac{1}{(2-2)}$ *does not exist.*
$\therefore$ *Domain* $= \mathbb{R} - \{2\}$.

c) *Value of* y *does not exist for* $x = -2$ *and* $x = 3$.
$\therefore$ *Domain* $= \mathbb{R} - \{-2, 3\}$.

Classification of Functions

- ***Injective / One-to-one function***. *A function* $f : A \to B$ *is injective or one-to-one function if for every* $b \in B$, *there exists at most one* $a \in A$ *such that* $f(a) = b$.
  This means a function f *is injective if* $a_1 \neq a_2$ *implies* $f(a_1) \neq f(a_2)$.

- ***Surjective / Onto function*** *A function* $f : A \to B$ *is surjective (onto) if the image of* f *equals its range. Equivalently, for every* $b \in B$, *there exists some* $a \in A$ *such that* $f(a) = b$. *This means that for any* y *in* B, *there exists some* x *in* A *such that* $y = f(x)$.

- ***Bijective / One-to-one Correspondent***
  A function $f : A \to B$ *is bijective or one-to-one correspondent if and only if* f *is both injective and surjective.*

- ***Inverse of a Function*** *The inverse of a one-to-one corresponding function* $f : A \to B$, *is the function* $g : B \to A$, *holding the following property*

$$f(x) = y \Leftrightarrow g(y) = x$$

The function f *is called* ***invertible***, *if its inverse function* g *exists.*

Example 1.3.10. *Without using graph prove that the function $f : \mathbb{R} \to \mathbb{R}$ defiend by $f(x) = 4+3x$ is one-to-one.*
Solution : *For a function to be one-one function*

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2, \ \forall x_1, x_2 \in \mathbb{R}.$$

Now, $f(x_1) = f(x_2)$ gives $4 + 3x_1 = 4 + 3x_2$ or $x_1 = x_2$.
$\therefore f$ is a one-one function.
Pigeonhole Principle
Suppose X is a finite set with m elements, Y is a finite set with n elements, and $f : X \to Y$ is a function.

1) *If $m = n$; then f is injective iff i is surjective iff f is bijective.*

2) *If $m > n$; then f is not injective.*

3) *If $m < n$; then f is not surjective.*

Exercise 1.3.2. *Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 4x^3 - 5$ is a bijection.*
Solution : *Now $f(x_1) = f(x_2), \ \forall x_1, x_2 \in$ Domain.*
$\therefore 4x_1^3 - 5 = 4x_2^3 - 5$
$\Rightarrow x_1^3 = x_2^3$
$\Rightarrow x_1^3 - x_2^3 = 0 \Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0.$
$\Rightarrow \ x_1 = x_2$ *or* $x_1^2 + x_1x_2 + x_2^2 = 0$ *(rejected). It has no real value of x_1 and x_2 . $\therefore f$ is a one-one function.*
Again let $y = (x)$ where $y \in$ codomain, $x \in$ domain.
We have $y = 4x^3 - 5$ or $x = (\frac{y+5}{4})^{\frac{1}{3}}$
$\therefore$ For each $y \in$ codomain $\exists x \in$ domain such that $f(x) = y$.
Thus f is onto function.
$\therefore f$ is a bijection.

Exercise 1.3.3. *Prove that $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = x^2 + 3$ is neither one-one nor onto function.*

- ***Monotonic Function***
  Let $F : A \to B$ be a function then F is said to be monotonic on an interval (a, b) if it is either increasing or decreasing on that interval.
  For function to be increasing on an interval (a, b)

  $$x_1 < x_2 \Rightarrow F(x_1) < F(x_2) \ \forall x_1, \ x_2 \in (a, b)$$

  and for function to be decreasing on (a, b)

  $$x_1 < x_2 \Rightarrow F(x_1) > F(x_2) \ \forall x_1, \ x_2 \in (a, b)$$

  A function may not be monotonic on the whole domain but it can be on different intervals of the domain.

Example 1.3.11. *Consider the function* $F : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^2$.
Now, $\forall x_1, x_2 \in [0, \infty,$

$$x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$$

$\Rightarrow F$ *is a Monotonic Function on* $[0, \infty)$. *It is only increasing function on this interval).*
But $\forall x_1, x_2 \in (-\infty, 0)$,

$$x_1 < x_2 \Rightarrow F(x_1) > F(x_2)$$

$\Rightarrow F$ *is a Monotonic Function on* $(-\infty, 0]$.
(It is only a decreasing function on this interval) Therefore if we talk of the whole domain given function is not monotonic on $\mathbb{R}$ *but it is monotonic on* $(-\infty, 0)$ *and* $(0, \infty)$.

Even Function
A function is said to be an even function if for each x *of domain*

$$F(-x) = F(x)$$

Example 1.3.12. , *each of the following is an even function.*
(i) If $F(x) = x^2$, *then* $F(-x) = (-x)^2 = x^2 = F(x)$.
(ii) If $F(x) = cosx$, *then* $F(-x) = cos(-x) = cosx = F(x)$.
(iii) If $F(x) = |x|$, *then* $F(-x) = |-x| = x = F(x)$.

Odd Function
A function is said to be an odd function if for each x

$$f(-x) = -f(x)$$

Example 1.3.13. *If* $f(x) = x^3$, *then* $f(-x) = (-x)^3 = -x^3 = -f(x)$. *)*

cont.

- ***Greatest Integer Function (Step Function)***
  $f(x) = [x]$ which is the greatest integer less than or equal to x and $f(x)$ is called Greatest Integer Function or Step Function.
  Domain of the step function is the set of ***real numbers.***
  Range of the step function is the ***set of integers****.*

- ***Polynomial Function*** *Note : Functions of the type $f(x) = k$, where k is a constant is also called a constant function.*

- ***Rational Function*** *Function of the type $f(x) = \frac{h(x)}{g(x)}$; $g(x) \neq 0$ and $g(x)$, $h(x)$ are polynomial functions. are called rational functions.*

- ***Reciprocal Function***
  Functions of the type $y = \frac{1}{x}$, $x \neq 0$ is called a reciprocal function.

- *The function $f(x) = e^x$, where x is any real number is called an* ***Exponential Function.***

- ***Logarithmic Functions*** *Consider now the function*

$$y = e^x.$$

We write it equivalently as

$$x = loge^y$$

Thus, $y = loge^x$ is the inverse function of $y = e^x$.
The base of the logarithm is not written if it is e and so $loge^x$ is usually written as $logx$.
The corresponding laws of logarithms are

$$loga^{(mn)} = loga^m + loga^n$$

$$loga^{(m/n)} = loga^{-}loga^n$$

$$loga^{(}m^n) = nloga^m$$

$$logb^m = \frac{loga^m}{loga^b}$$

or

$$logb^m = loga^m.logb^a$$

Here $a, b > 0$, $a \neq 1$, $b \neq 1$.

cont.

- ***Composition of Functions***

  Two functions $f : A \to B$ and $g : B \to C$ can be composed to give a composition gof. This is a function from A to C defined by

$$(gof)(x) = g(f(x)).$$

Example 1.3.14. *If $f(x) = x + 1$ and $g(x) = x^2 + 2$, calculate fog and gof.*
Solution :

$$\begin{aligned} fog(x) &= f(g(x)) \\ &= f(x^2 + 2) \\ &= x^2 + 2 + 1 \\ &= x^2 + 3, \\ (gof)(x) = g(f(x)) &= g(x + 1) \\ &= (x + 1)2 + 2 \\ &= x + 1 + 2 \\ &= x + 3. \end{aligned}$$

Here , we see that $(fog) \neq gof$.
Some Facts about Composition
If f and g are one-to-one then the function (gof) is also one-to-one.
If f and g are onto then the function (gof) is also onto.
Composition always holds associative property but does not hold commutative property.

1.4 Equivalence relation

*A given binary relation '$\backsim$' on a set X is said to be an **equivalence relation** if and only if it is reflexive, symmetric and transitive (**R-S-T**). That is, for all $a, b, c \in X$:*

- *$a \backsim a$. (Reflexivity)*
- *If $a \backsim b$, then $b \backsim a$. (Symmetry)*
- *If $a \backsim b$ and $b \backsim c$, then $a \backsim c$. (Transitivity)*

*X together with the relation '$\backsim$', $(X, \backsim)$ is called a **setoid**.*
The equivalence class of a under $\backsim$, denoted $[a]$, is defined as $[a] = \{b \in X \mid a \sim b\}$

Example 1.4.1. *Let the set $\{a, b, c\}$ have the equivalence relation $\{(a, a),(b, b),(c, c),(b, c),(c, b)\}$. The following sets are equivalence classes of this relation: $[a] = \{a\}$, $[b] = [c] = \{b, c\}$.*

The set of all equivalence classes for this relation is $\{\{a\},\{b,c\}\}$.
This set is a partition of the set $\{a,b,c\}$.
The relation "$\geq$*" between real numbers is reflexive and transitive, but not symmetric.*

Example 1.4.2. $7 \geq 5$ *does not imply that* $5 \geq 7$.
The relation "has a common factor greater than 1 *with" between natural numbers greater than* 1*, is reflexive and symmetric, but not transitive.*

Example 1.4.3. *The natural numbers* 2 *and* 6 *have a common factor greater than* 1*, and* 6 *and* 3 *have a common factor greater than* 1*, but* 2 *and* 3 *do not have a common factor greater than* 1*.*
The empty relation R *on a non-empty set* X *(i.e.* aRb *is never true) is vacuously symmetric and transitive, but not reflexive. (If* X *is also empty then* R *is reflexive.)*
The relation "is approximately equal to" between real numbers, even if more precisely defined, is not an equivalence relation, because although reflexive and symmetric, it is not transitive, since multiple small changes can accumulate to become a big change.

Example 1.4.4. *Given the above information, determine which relations are reflexive, transitive, symmetric, or antisymmetric on the following - there may be more than one characteristic. (Answers follow.)* xRy *if*

1. $x = y$
2. $x < y$
3. $x^2 = y^2$
4. $x \leq y$

Answers

1. *Symmetric, Reflexive,and transitive*
2. *Transitive, Trichotomy*
3. *Symmetric, Reflexive, and transitive (*$x^2 = y^2$*is just a special case of equality, so all properties that apply to* $x = y$ *also apply to this case)*
4. *Reflexive, Transitive and Antisymmetric (and satisfying Trichotomy)*

Theorem 1.4.1. *Let* R *be an equivalence relation on a set* A*. Then the following are equivalent:*

(1) aRb

(2) $[a] = [b]$.

(3) $[a] \cap [b] \neq \emptyset$

Proof:

$1 \to 2$. Suppose $a, b \in A$ and aRb. We must show that $[a] = [b]$.
Suppose $x \in [a]$.
Then, by definition of $[a]$, aRx.
Since R is symmetric and aRb, bRa. Since R is transitive and we have both bRa and aRx, bRx.
Thus, $x \in [b]$.
Suppose $x \in [b]$.
Then bRx. Since aRb and R is transitive, aRx. Thus, $x \in [a]$.
We have now shown that $x \in [a]$ if and only if $x \in [b]$.
Thus, $[a] = [b]$.
$2 \to 3$. Suppose $a, b \in A$ and $[a] = [b]$.
Then $[a] \cap [b] = [a]$. Since R is reflexive, aRa; that is $a \in [a]$.
Thus $[a] = [a] \cap [b] \neq \emptyset$;
$3 \to 1$. Suppose $[a] \cap [b] \neq \emptyset$;.
Then there is an $x \in [a] \cap [b]$. By definition, aRx and bRx. Since R is symmetric, xRb and transitive, aRx and xRb, aRb.

Partition

Definition 1.4.1. *A partition of X is a set P of nonempty subsets of X, such that every element of X is an element of a single element of P. Each element of P is a cell of the partition. Moreover, the elements of P are pairwise disjoint and their union is X.*
Here, every set partitioned into equivalence classes of its elements and is the union of of those sets.

1.5 Order relation

Remark 1.5.1. *(partial order): A binary relation R on a non-empty set A is a partial order if and only if it is*

- *(1) reflexive,*
- *(2) antisymmetric, and*
- *(3) transitive.*

The ordered pair $< A, R >$ is called a poset (partially ordered set) when R is a partial order.

Example 1.5.1. *:*
• *The less-than-or-equal-to relation on the set of integers I is a partial order, and the set I with this relation is a poset.*
• *If $A = \{1, 2, 3\}$ then*

$$R = \{(1, 1), (2, 2), (3, 3), (2, 1), (3, 1), (3, 2)\}$$

is a partial order on A .
Do you recognize the order? It is the greater than or equal to relation " ³ " .
Note that the members in R are equivalent to the inequalities

$$1 \geq 1,\ 2 \geq 2,\ 3 \geq 3,\ 2 \geq 1,\ 3 \geq 1,\ 3 \geq 2.$$

• *(The Divide Ordering) Let D denote the relation "divides" on the set of natural numbers $\mathbb{N}$. For example, $1D7$, $2D7$, $3D9$, $7D21$ and so on. Show that D defines a partial order on the natural numbers.*
Solution:
Let notation "D" replaced by " | ".

- *Reflexive:* $n \mid n$, $\forall n \in \mathbb{N}$. *i.e.,* $(n, n) \in D$.
- *Antisymmtric* $(m|n) \wedge (n|m) \Rightarrow m = n$, $\forall m, n \in \mathbb{N}$.
- *Transitive:* $(m|n) \wedge (n|p) \Rightarrow m|p$, $\forall m, n, p \in \mathbb{N}$.

Exercise 1.5.1. *(Checking $\leq$)*

1. *Check to see if $\leq$ is a partial order on the real numbers.*
2. *(Ordered Sets) The power set of $A = \{a, b, c\}$ consists of the family of eight subsets:*

   $$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

   Show that set inclusion relation " $\subset$ " is a partial order on $\mathcal{P}(A)$.
3. *The subset relation on the power set of a set, say $\{1, 2\}$, is also a partial order, and the set $\{1, 2\}$ with the subset relation $(I, \leq)$ is a poset.*

Definition 1.5.1. • *(total/Linear order): A binary relation R on a set A is a total order if and only if it is*
(1) a partial order, and
(2) for any pair of elements a and b of A, $< a, b > \in R$ or $< b, a > \in R$.
*(Every element of the set are comparable. (**Dichotomy Law**))*
That is, every element is related with every an other element in the set.

Example 1.5.2. *The partial order " $\leq$ " on the real numbers is a total order, meaning that every two real numbers x and y are comparable;*
that is either $x \leq y$ or $y \leq x$.
On the other hand " $\subset$ " is not a total order on $\mathcal{P}(A)$ since there exists incomparable elements, such as the sets $\{1, 2\}$ and $\{3, 4\}$ where $\{1, 2\} \subset \{3, 4\}$ and $\{3, 4\} \subset \{1, 2\}$.

Chain and anti-chain

Note: a subset of a partially ordered set which is totally ordered by the inherited relation is called a ***chain*** *(It is a totally ordered subset of a poset S.*
An ant-chain *is a subset of a poset in which any distinct elements are incomparable.*
A certain chain C is maximal if there is no chain D such that $C \subset D$. ($|C| < |D|$. Also for anti-chain. ***e.g.,*** *Let us consider the poset $(\{1, 2, 3, 4, 5, 6, 9, 12, 18\}, |)$, recalling that $a|b$ is a is a divisor of b.*
The corresponding chains are $\{1, 2, 4, 12\}$, $\{1, 3, 6, 12\}$, $\{1, 3, 9, 18\}$, $\{1, 2, 6, 12\}$, $\{1, 3, 18\}$, $\{1, 5\}$.
Here we may note that $\{1, 2, 4, 12\}$ is the maximum chain. There are no chains of length greater than 4.
And $\{4, 6, 9, 5\}$ is our maximum antichain.

Proposition 1.5.1. *Equality is both an equivalence relation and a partial order. Equality is also the only relation on a set that is reflexive, symmetric and antisymmetric.*

A ***strict partial order*** *if it is irreflexive, transitive, and asymmetric.*

A ***partial equivalence relation*** *if it is transitive and symmetric. Transitive and symmetric imply reflexive if and only if for all $a \in X$, there exists $a, b \in X$ such that $a \sim b$.*

A reflexive and symmetric relation is a ***dependency relation****, if finite, and a* ***tolerance relation*** *if infinite.*
A preorder is reflexive and transitive.

A ***congruence relation*** *if it is an equivalence relation whose domain X is also the underlying set for an algebraic structure*[2]*, and which respects the additional structure.*

Any ***equivalence relation*** *if it is the negation of an apartness relation, though the converse statement only holds in classical mathematics (as opposed to constructive mathematics), since it is equivalent to the law of excluded middle.*

A ***serial relation*** *$\sim$ satisfies $\forall a, \exists b$ such that $a \sim b$.*
Evidently it is sufficient for a serial relation $\sim$ to be symmetric and transitive for it also to be reflexive.

Given our definitions, we can explore the connection between partial orders and strict orders. Based on what we know of ordinary $\leq$ and $<$, we expect the following result might be true. Which it is.

Proposition 1.5.2. *(Strict and Partial Orders)*
Let A be any set.

a) If $\leq$ is a partial order on A, then the relation $<$ defined by $x < y \leftrightarrow x \leq y \wedge x \neq y$. is a strict order on A.

b) If $<$ is a strict order on A, then the relation $\leq$ defined by $x \leq y \leftrightarrow x < y \vee x = y$ is a partial order on A.

[2]algebraic structure

Proof :
For part a), suppose $\leq$ *is a partial order with* $<$ *defined by* $x < y$ *iff* $x \leq y$ *and* $x \neq y$*.*
To show that $<$ *is a strict order, we must show that it is irreflexive and transitive.*
The relation $<$ *is clearly irreflexive: if* $x < y$*, then* $x \neq y$ *according to the definition.*
Now suppose $x < y$ *and* $y < z$*. Then by the transitivity of* $\leq$*,* $x \leq z$*.*
But $x \leq z$*; for if* $x = z$*, the antisymmetry of* $\leq$ *would mean, for instance, that* $x = y$*, which contradicts the supposition* $x < y$*.*
Thus $x < z$*, which establishes that* $<$ *is transitive.*
Part b is left as an exercise.

Example 1.5.3. *Discuss the meaning of the strict orders* $<$ *associated with the following partial orders:*

a) *if* $\leq$ *is the partial order of divisibility on the set of natural numbers.*

b) *if* $\leq$ *represents the subset ordering on a collection of sets.*

Solution
a) If $\leq$ *denotes the relation "is a divisor of," then* $<$ *indicates proper divisibility: here* $m < n$ *means* m *is a proper divisor of* n*.*
b) If $\leq$ *denotes "is a subset of," then ¡ indicates proper subset inclusion: here* $S < T$ *denotes the concrete relation* $S \subset T$*.*
Well-definedness under an equivalence relation
If $\sim$ *is an equivalence relation on* X*, and* $P(x)$ *is a property of elements of* X*, such that whenever* $x \sim y$*,* $P(x)$ *is true if* $P(y)$ *is true, then the property* P *is said to be well-defined or a class invariant under the relation* $\sim$*.*

Quotient set
The set of all possible equivalence classes of X *by* $\sim$*, denoted*

$$X/\!\sim\, := \{[x] \mid x \in X\}$$

, is the quotient set of X *by* $\sim$*.*

The strictly-less-than and proper-subset relations are not partial order because they are not reflexive. They are examples of some relation called quasi order.

Definition 1.5.2. *(quasi order): A binary relation* R *on a set* A *is a quasi order if and only if it is*
(1) irreflexive, and
(2) transitive.

A ***quasi order*** *is necessarily antisymmetric as one can easily verify.*

Example 1.5.4. *: The less-than relation on the set of integers* I *is a quasi order.*

Example 1.5.5. *: The proper subset relation on the power set of a set, say* $\{1, 2\}$ *, is also a quasi order. A digraph of a binary relation on a set can be simplified if the relation is a partial order. Hasse diagrams defined as follows are such graphs.*

Definition 1.5.3. *(Hasse diagram): A Hasse diagram is a graph for a poset which does not have loops and arcs implied by the transitivity. Further, it is drawn so that all arcs point upward eliminating arrowheads.*

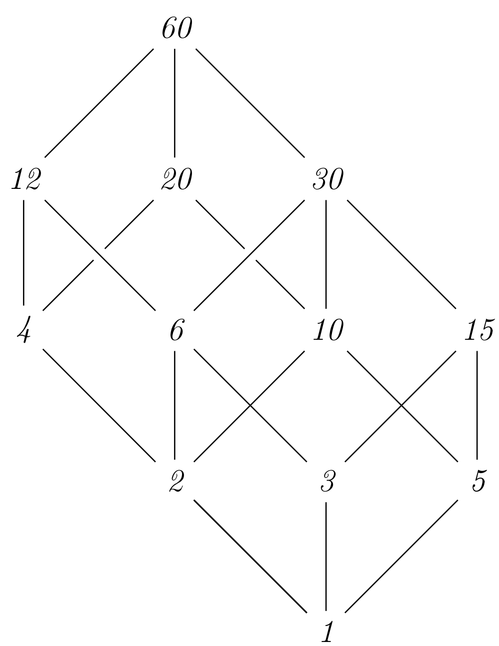
To obtain the Hassse diagram of a poset, first remove the loops, then remove arcs $< a, b >$ *if and only if there is an element* c *that* $< a, c >$ *and* $< c, b >$ *exist in the given relation.*

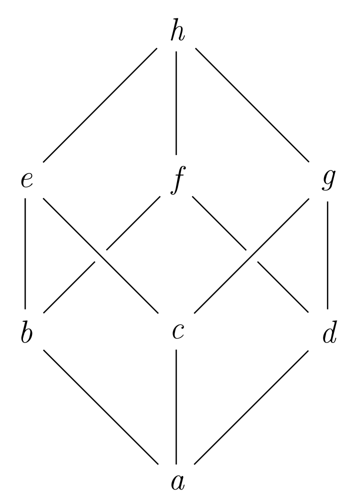
Exercise 1.5.2. *Identify extreme elements in the following posets:*

a) The divisors of 60, ordered by divisibility.

b) The set $\{a, b, c, d, e, f, g, h\}$*, ordered like the subsets of* $\{0, 1, 2\}$*.*

Solution
The Hasse diagrams of these posets are given below. Note how the "dimensionality" of the first diagram corresponds to the number of prime factors in the factorization of 60.





Extreme Elements in Posets *In a totally ordered set, any two elements can be compared, without exception. The concept of least/greatest number in a set of integers can be generalized for a general poset. We start with the concepts of minimal/maximal elements.*

Definition 1.5.4. ***(Extremal Elements)***
Suppose $(A, \leq)$ *is a poset,* M *and* m *are elements of* A*, and* S *is a subset of* A*.*

- ***Upper bound:*** *An element $M \in A$ is called an upper bound of S iff $x \leq S$, $\forall x \in S$.*

- ***Least Upper bound:*** *An element $lub(S) \in A$ is called the least upper bound of S (or* ***supremum*** *of S) if it is an upper bound of S and if $lub(S) \leq M$ for every upper bound M .*

- ***Lower Bound:*** *An element $m \in A$ is called a lower bound of S iff $s \in S$, $m \leq s$.*

- *Greatest Lower Bound An element $glb(S) \leq U$ is called the* ***greatest lower bound*** *of S (or infimum) if it is a lower bound of S and if $l \leq glb(S)$ for every lower bound l of S .*

- *Let P be a partially ordered set. An element x* ***covers*** *the element y in the partially ordered set if $x > y$ and there is no element $z \in P$ such that $x > z > y$.*

- *M is a* ***maximal element*** *of S iff M is in S and there is no x in S such that $M < x$;*
  M is a ***maximum of*** *S iff M is in S and $x \leq M$ for all x in S.*

- *m is a* ***minimal element of*** *S iff m is in S and there is no x in S such that $x < m$;*
  m is a ***minimum of*** *S iff m is in S and $m \leq x$ for all x in S.*

Two elements x and y in P are ***comparable*** *if $x \leq y$ or $y \leq x$; they are* ***incomparable*** *if neither $x \leq y$ nor $y \leq x$. A subset $C \subseteq P$ is a* ***chain*** *if any two elements in C are* ***comparable****. A subset $A \subseteq P$ is an* ***antichain*** *if any two elements in A are* ***incomparable****.*
If C is a finite chain and $|C| = n + 1$, then the elements in C can be linearly ordered, so that

$$x_0 < x_1 < x_2 < \cdots < x_n.$$

The length of the chain C is n, 1 less than the number of elements in C.
A chain

$$x_0 < x_1 < \cdots < x_n$$

in the partial order P is maximal or saturated if x_{i+1} covers x_i for $1 \leq i \leq n$.
A function r defined from P to the nonnegative integers is a rank function if $r(x) = 0$ for every minimal element and $r(y) = r(x) + 1$ whenever y covers x. The partial order P is ranked if there exists a rank function on P. The rank of the entire partially ordered set P is the maximum

$$max\{r(x) : x \in P\}$$

. If $x \leq y$ in P, the interval $[x, y]$ is the set

$$\{z : x \leq z \leq y\}.$$

Note: *All these extremal elements belong to the set of elements they bound: that's part of their definition. Extrema (minimum, maximum) may or may not exist for a given subset, but if they do, they will be unique.*

Example 1.5.6. *(Multiples and Divisors of 24)*
Given the set

$$A = \{1, 2, 3, 4, 6, 8, 12, 24\}$$

of positive divisors of 24, define two partial orders on A by

$$aMb \Leftrightarrow \ a \text{ is a multiple of } b$$

$$aDb \Leftrightarrow a \text{ divides } b$$

Then find the minimal and maximal element of the poset.
Solution

Divide relation and multiple relation respectively.

In the first hasse diagram, 1 *is minimal element,* 24 *is maximal, whereas in the secoond* 1 *is maximal and* 24 *is minimal.*

Example 1.5.7. *: The set of* $\{\{1\}, \{2\}, \{1, 2\}\}$ *with has two minimal elements* $\{1\}$ *and* $\{2\}$*. Note that* $\{1\}$*, and* $\{2\}$ *are not related to each other in . Hence we can not say which is "smaller than" which, that is, they are not comparable.*
Note that the least element of a poset is unique if one exists because of the antisymmetry of .

Example 1.5.8. *: The poset of the set of natural numbers with the less-than-or-equal-to relation has the least element* 0*.*

Example 1.5.9. *: The poset of the powerset of* $\{1, 2\}$ *with has the least element .*

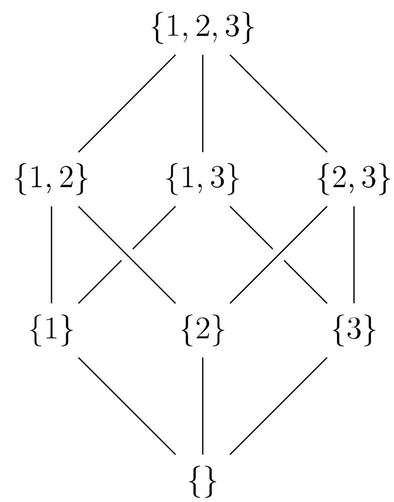
Every non-empty subset S of the given posets will have maximal and minimal elements (nothing above them or below them respectively), though they need not have either a maximum or a minimum. The set $S = \{b, d\}$ *from the second poset is such an example; both elements are maximal as well as minimal for S, but S has no maximum or minimum.*

Example 1.5.10. *: Let* $A = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ *with the partial order . This given A has three minimal elements* $\{1\}, \{2\}$*, and* $\{3\}$*.*
Select $\{2\}$ *and remove it from A. Let A denote the resultant set i.e.* $A := A - \{2\}$*. The new A has two minimal elements* $\{1\}$*, and* $\{3\}$*.*
Select $\{1\}$ *and remove it from A. Denote by A the resultant set, that is*

$$A = \{\{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

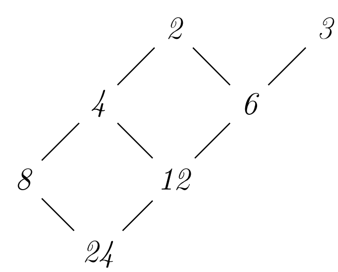
This new A has two minimal elements $\{3\}$ *and* $\{1,2\}$.
Select $\{1,2\}$ *and remove it from A.*
Proceeding in like manner, we can obtain the following linear order:

$$\{\{2\},\{1\},\{1,2\},\{3\},\{1,3\},\{2,3\},\{1,2,3\}\}.$$



A certain poset may or mayn't have maximal and/or minimal element.

Example 1.5.11. *:*



Here, 2 and 3 are upper bounds but they are not comparable as well as there is element of the set exceeds both elements.
Hence the poset doesn't have maximal element, minimal element 24.

Topological Sorting *The elements in a finite poset can be ordered linearly in a number of ways while preserving the partial order. For example* $\{,\{1\},\{2\},\{1,2\}\}$ *with the partial order , can be ordered linearly as ,* $\{1\},\{2\},\{1,2\}$, *or ,* $\{2\},\{1\},\{1,2\}$.
In these orders a set appears before (to the left of) another set if it is a subset of the other.
In real life, tasks for manufacturing goods in general can be partially ordered based on the prerequisite relation, that is certain tasks must be completed before certain other tasks can be started. For example the arms of a chair must be carved before the chair is assembled. Scheduling those

tasks is essentially the same as arranging them with a linear order (ignoring here some possible concurrent processing for simplicity's sake).

The topological sorting is a procedure to find from a partial order on a finite set a linear order that does not violate the partial order.
It is based on the fact that a finite poset has at least one minimal element. The basic idea of the topological sorting is to first remove a minimal element from the given poset, and then repeat that for the resulting set until no more elements are left.
The order of removal of the minimal elements gives a linear order.
The following algorithm formally describes the topological sorting.

Algorithm Topological Sort
Input: A finite poset $< A, R >$.
Output: A sequence of the elements of A *preserving the order* R.
integer i;
$i := 1$;
while (A)

$$\begin{aligned} \{pick\ a\ minimal\ element\ b\ from\ A; \\ A := A - \{b\}; \\ i := i + 1; \\ output\ b \\ \} \end{aligned}$$

Definition 1.5.5. *Let* P *and* Q *be partially ordered sets.*
A function $f : P \to Q$ *is* ***order preserving*** *if for elements* x *and* y *in* P, $x \leq_P y$ *implies* $f(x) \leq_Q f(y)$ *or* $f(x) \leq_Q f(y)$.
A subset $I \subseteq P$ *is an (order) ideal of* P *if it is "down-closed;" that is,* $y \leq x$ *and* $x \in I$ *imply* $y \in I$.

Definition 1.5.6. *(well order): A total/Linear order* R *on a set* A *is a well order if every non-empty subset of* A *has the least element.*

Lemma 1.5.1. *If* $(W, <)$ *is a well-ordered set and* $f : W \to W$ *is an increasing function, then* $f(x) \geq x$ *for each* $x \in W$.
Proof. *Assume that the set* $X = \{x \in W : f(x) < x\}$ *is nonempty and let* z *be the least element of* X.
If $w = f(z)$, *then* $f(w) < w$, *a contradiction.*

1.6 Cardinal and Ordinal Numbers

The idea is to define ordinal numbers so that $\alpha < \beta$ *if and only if* $\alpha \in \beta$, *and* $\alpha = \{\beta : \beta < \alpha\}$.

Definition 1.6.1. *A set* T *is transitive if every element of* T *is a subset of* T.
(Equivalently,

$$\cup T \subset T, \text{ or } T \subset \mathcal{P}(T).$$

Colloquially, an ordinal number is a number that tells the position of something in a list. Such as First, Second, Third, etc. This basic understanding extends to the meaning of ordinal numbers in set theory. In an ordered set, that is a collection of objects placed in some order, ordinal numbers (also called ordinals) are the ***labels for the positions*** *of those ordered objects.*
Ordinal numbers: which are obtainable by counting a given set. It is also an adjective which describes the numerical position of an object. e.g., first,second,..., (rank).

Definition 1.6.2. *A set is an ordinal number (an ordinal) if it is transitive and well-ordered by* $\in$.
We shall denote ordinals by lowercase Greek letters $\alpha, \beta, \gamma, \ldots$ *. The class of all ordinals is denoted by* Ord.
We define $\alpha < \beta$ *if and only if* $\alpha \in \alpha$.

Definition 1.6.3. ***cardinal/counting numbers****, or cardinals for short, are a generalization of the natural numbers used to measure the* ***cardinality (size)*** *of sets.*
The cardinality of a finite set is a natural number: the number of elements in the set. The ***transfinite*** *cardinal numbers describe the sizes of infinite sets.*
A Cardinal number says "how many" of something there are, such as one, two, three,....
Cardinality is defined in terms of bijective functions. Two sets have the same cardinality if, and only if, there is a one-to-one correspondence (bijection) between the elements of the two sets. In the case of finite sets, this agrees with the intuitive notion of size.
cardinal number is what is normally referred to as a counting number, provided that 0 is included: $0, 1, 2, \cdots$.
They may be identified with the natural numbers beginning with 0. *The counting numbers are exactly what can be defined formally as the finite cardinal numbers. Infinite cardinals only occur in higher-level mathematics and logic.*

Exercise 1.6.1. *:*

1. *Let* R *be the relation on the set* $\mathbb{R}$ *real numbers defined by* xRy *iff* $x - y$ *is an integer. Prove that* R *is an equivalence relation on* $\mathbb{R}$.

2. *(Testing for an Order Relation) Tell whether the following relations on* $A = \{1, 2, 3\}$ *are reflexive, antisymmetric, and transitive. Plot the points of the Cartesian product* $A \times A$ *and denote the members of* $R \subset A \times A$. *If the relation is an order relation, draw a Hasse diagram and directed graph.*

   a) $R = \{(1, 1), (2, 2), (3, 3)\}$
   b) $R = \{(1, 1), (1, 2), (2, 1)\}$
   c) $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)\}$
   d) $R = \{(1, 2), (2, 3), (1, 3)\}$

3. *(Graphing Order Relations) Sketch the points in each of the following relations* R *on the given universe* U .

a) $R = \{(x, y) : x\check{c} \leq y\},\ U = \mathbb{R}$

b) $R = \{(x, y) : x \geq y\},\ U = \mathbb{R}$

c) $R = \{(x, y) : x \leq y\},\ U = \{1, 2, 3\}$

d) $R = \{(x, y) : x \geq y\},\ U = \{1, 2, 3\}$

4. *(Finding Relations) Find a relation on the $A = \{1, 2, 3, 4\}$ with the following properties.*

   a) reflexive but not antisymmetric

   b) antisymmetric and reflexive

   c) not reflexive but transitive

   d) not reflexive, not antisymmetric, not transitive.

5. *(Upper and Lower Bounds) For the partially ordered set $\mathcal{P}(A)$ with order relation Í in Example 6, find an upper bound, least upper bound, a lower bound, and the greatest lower bound for the following subsets of $\mathcal{P}(A)$.*

   a) $B = \{\{a\}, \{a, b\}\}$

   b) $B = \{\{a\}, \{b\}\}$

   c) $B = \{\{a\}, \{a, b\}, \{a, b, c\}\}$

   d) $B = \{\{a\}, \{c\}, \{a, c\}\}$

   e) $B = \{\emptyset, \{a, b, c\}\}$

   f) $B = \{\{a\}, \{b\}, \{c\}\}$

6. *(Divide Ordering). Prove that D is reflexive, antisymmetric, and transitive.*

7. *(Hasse Diagram) Draw the Hasse diagram for the power set $\mathcal{P}(A)$ with ordering $\subset$ when $A = \{a, b, c, d\}$.*

8. *(Hasse Diagram for Multiples) Let M be the order relation " a is a multiple of b " defined on the set of positive divisors of 15. Draw a Hasse diagram for M .*

9. *(A Partial Order of Points in the Plane)*
   There are various ways to construct new orders from existing orders. A partial order can be constructed on the Cartesian product of two partially ordered sets by defining

   $$(a, x) \preceq (b, y) \Leftrightarrow (a \preceq b) \wedge (x \preceq y)$$

   Use this order to

   a) Order the set $A = \{(-1, 3), (3, 0), (0, 5), (0, 0), (-2, 9)\}$

   b) Construct a Hasse diagram for the set A in part a).

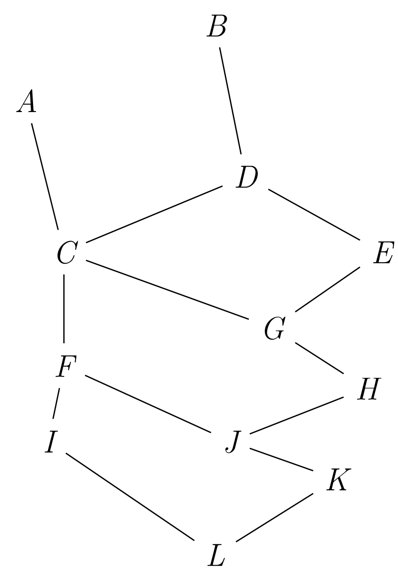
   c) Draw a digraph for the set A .

10. *(Equivalent form of Antisymmetry) State the contrapositive form of the antisymmetry condition*

    $$(x \leq y) \wedge (y \leq x) \Rightarrow x = y.$$

11. *(Ordering the Complex Numbers) Suppose you order the complex numbers* $z = a+bi$ *according to their magnitude* $|z| = \sqrt{(a)^2 + (b)^2}$ *, that is* $z_1 \preceq z_2 \Leftrightarrow |z_1| \leq |z_2|$ *Is this a partial order ?*

12. *(Hasse Diagram) For the "starred" subset* $S = \{C, F, G, I, J, H\}$ *of the partially ordered set* $U = \{A, B, C, D, E, F, G, H, I, J, K, L\}$ *illustrated in Figure below, find (if they exist) the following: find (if they exist) the following:*



a) *upper bound(s)*
b) *lower bound(s)*
c) *the least upper bound*
d) *greatest lower bound*
e) *maximal element(s)*
f) *minimal element(s)*
g) *maximum*
h) *minimum*