Problem Two

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Solve the flowing problems manually (if possible) and by using computer Lab.

Bisection exercises

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1. Find the root of $f(x) = e^x - 2 - x$ in the interval $[-2.4, -1.6]$ accurate to 10^{-4} using the bisection method.

   Solution: Using the bisection method gives $a_1 = -2.4$ and $b_1 = -1.6$, so $f(-2.4) = 0.4907 > 0$ and $f(-1.6) = -0.1981 < 0$. Thus $\alpha \in [-2.4, -1.6]$ we have $x_1 = \frac{1}{2}(a_1 + b_1) = -2.0$ and $f(x_1) = 0.1353 > 0$. Since $f(x_1)f(b_1) < 0$ so $\alpha \in [-2.4, -1.6]$ we have $x_2 = \frac{1}{2}(x_1 + b_1) = -1.8$ and $f(x_2) = -0.0347 < 0$. Continuing in this manner, the bisection gives $x_{12} = -1.84121094$, accurate to within 10^{-4}

n	a	b	p	f(p)
1	-2.40000000	-1.60000000	-2.00000000	0.13533528
2	-2.00000000	-1.60000000	-1.80000000	-0.03470111
3	-2.00000000	-1.80000000	-1.90000000	0.04956862
4	-1.90000000	-1.80000000	-1.85000000	0.00723717
5	-1.85000000	-1.80000000	-1.82500000	-0.01378236
6	-1.85000000	-1.82500000	-1.83750000	-0.00328503
7	-1.85000000	-1.83750000	-1.84375000	0.00197298
8	-1.84375000	-1.83750000	-1.84062500	-0.00065680
9	-1.84375000	-1.84062500	-1.84218750	0.00065789
10	-1.84218750	-1.84062500	-1.84140625	0.00000050
11	-1.84140625	-1.84062500	-1.84101563	-0.00032817
12	-1.84140625	-1.84101563	-1.84121094	-0.00016384

2. Use the bisection method to find solutions accurate to within 10^{-4} on the interval $[-5, 5]$ of the following functions:

(a) $f(x) = x^5 - 10x^3 - 4$

(b) $f(x) = 2x^2 + \ln(x+6) - 3$

(c) $f(x) = \ln(x+1) + 30e^{-x} - 3$

Solution:

(a) The bisection method gives $x_{17} = 3.1818$, accurate to within 10^{-4}.

n	a	b	p	f(p)
1	-5.00000000	5.00000000	0.00000000	-4.00000000
2	0.00000000	5.00000000	2.50000000	-62.59375000
3	2.50000000	5.00000000	3.75000000	210.23339844
4	2.50000000	3.75000000	3.12500000	-11.15255737
5	3.12500000	3.75000000	3.43750000	69.78041744
6	3.12500000	3.43750000	3.28125000	23.08243206
7	3.12500000	3.28125000	3.20312500	4.54498532
8	3.12500000	3.20312500	3.16406250	-3.64232635
9	3.16406250	3.20312500	3.18359375	0.36467328
10	3.16406250	3.18359375	3.17382813	-1.66023578
11	3.17382813	3.18359375	3.17871094	-0.65316528
12	3.17871094	3.18359375	3.18115234	-0.14559598
13	3.18115234	3.18359375	3.18237305	0.10920065
14	3.18115234	3.18237305	3.18176270	-0.01828210
15	3.18176270	3.18237305	3.18206787	0.04543816
16	3.18176270	3.18206787	3.18191528	0.01357275

(b) The bisection method fails because $f(-5) = 47$ and $f(5) = 49.3979$, which gives $f(-5)f(5) > 1$.

(c) The bisection method gives $x_{17} = 2.9084$, accurate to within 10^{-4}.

n	a	b	p	f(p)
1	-5.00000000	5.00000000	0.00000000	27.00000000
2	0.00000000	5.00000000	2.50000000	0.71531293
3	2.50000000	5.00000000	3.75000000	-0.73632301
4	2.50000000	3.75000000	3.12500000	-0.26482597
5	2.50000000	3.12500000	2.81250000	0.13992518
6	2.81250000	3.12500000	2.96875000	-0.08052443

7	2.81250000	2.96875000	2.89062500	0.02481446
8	2.89062500	2.96875000	2.92968750	-0.02902827
9	2.89062500	2.92968750	2.91015625	-0.00240610
10	2.89062500	2.91015625	2.90039063	0.01112863
11	2.90039063	2.91015625	2.90527344	0.00434247
12	2.90527344	2.91015625	2.90771484	0.00096350
13	2.90771484	2.91015625	2.90893555	-0.00072247
14	2.90771484	2.90893555	2.90832520	0.00012022
15	2.90832520	2.90893555	2.90863037	-0.00030120
16	2.90832520	2.90863037	2.90847778	-0.00009051

3. The following equations have a root in the interval $[0, 1.6]$. Determine these with an error less than 10^{-4} using the bisection method.

   (a) $2x - e^{-x} = 0$

   (b) $e^{-3x} + 2x - 2 = 0.$

   Solution:

   (a) The bisection method gives $x_{16} = 0.35173$, accurate to within 10^{-4}.

   (b) The bisection method gives $x_{15} = 0.9730$, accurate to within 10^{-4}..

4. Estimate the number of iterations needed to achieve an approximation with accuracy 10^{-4} to the solution of $f(x) = x^3 + 4x^2 + 4x - 4$ lying in the interval $[0, 1]$ using the bisection method.

5. Use the bisection method for $f(x) = x^3 - 3x + 1$ in $[1, 3]$ to find:

   (a) The first eight approximation to the root of the given equation.

   (b) Find an error estimate $\mid \alpha - x_8 \mid$.

False Position Exercises

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1. Solve the Problem 1 of bisection by the false position method.

n	a	b	f(a)	f(b)	p	f(p)
1	-2.40000000	-1.60000000	0.49071795	-0.19810348	-1.83007818	-0.00952079
2	-2.40000000	-1.83007818	0.49071795	-0.00952079	-1.84092522	-0.00040423
3	-2.40000000	-1.84092522	0.49071795	-0.00040423	-1.84138538	-0.00001707

2. Use the false position method to find the root of $f(x) = x^3 + 4x^2 + 4x - 4$ on the interval $[0, 1]$ accurate to 10^{-4}.

n	a	b	f(a)	f(b)	p	f(p)
1	0.00000000	1.00000000	-4.00000000	5.00000000	0.44444444	-1.34430727
2	0.44444444	1.00000000	-1.34430727	5.00000000	0.56216216	-0.30958814
3	0.56216216	1.00000000	-0.30958814	5.00000000	0.58769134	-0.06473274
4	0.58769134	1.00000000	-0.06473274	5.00000000	0.59296109	-0.01325746
5	0.59296109	1.00000000	-0.01325746	5.00000000	0.59403749	-0.00270360
6	0.59403749	1.00000000	-0.00270360	5.00000000	0.59425688	-0.00055087

3. Use the false position method to find solution accurate to within 10^{-4} on the interval $[1, 1.5]$ of the equation $2x^3 + 4x^2 - 2x - 5 = 0$.

n	a	b	f(a)	f(b)	p	f(p)
1	1.00000000	1.50000000	-1.00000000	7.75000000	1.05714286	-0.28125948
2	1.05714286	1.50000000	-0.28125948	7.75000000	1.07265198	-0.07462622
3	1.07265198	1.50000000	-0.07462622	7.75000000	1.07672775	-0.01949217
4	1.07672775	1.50000000	-0.01949217	7.75000000	1.07778965	-0.00507039
5	1.07778965	1.50000000	-0.00507039	7.75000000	1.07806570	-0.00131752
6	1.07806570	1.50000000	-0.00131752	7.75000000	1.07813742	-0.00034226
7	1.07813742	1.50000000	-0.00034226	7.75000000	1.07815605	-0.00008890

4. Use the false position method to find solution accurate to within 10^{-4} on the interval $[3, 4]$ of the equation $e^x - 3x^2 = 0$.

n	a	b	f(a)	f(b)	p	f(p)
1	3.00000000	4.00000000	-6.91446308	6.59815003	3.51170436	-3.49087822
2	3.51170436	4.00000000	-3.49087822	6.59815003	3.68065826	-0.96923545
3	3.68065826	4.00000000	-0.96923545	6.59815003	3.72155975	-0.22121453
4	3.72155975	4.00000000	-0.22121453	6.59815003	3.73059212	-0.04815826
5	3.73059212	4.00000000	-0.04815826	6.59815003	3.73254421	-0.01037526
6	3.73254421	4.00000000	-0.01037526	6.59815003	3.73296411	-0.00223023
7	3.73296411	4.00000000	-0.00223023	6.59815003	3.73305434	-0.00047917
8	3.73305434	4.00000000	-0.00047917	6.59815003	3.73307372	-0.00010294

Fixed Point Iteration Exercises

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1. The cubic equation $x^3 - 3x - 20 = 0$ can be written as

   (a) $x = \frac{x^3-20}{3}$

   (b) $x = \frac{20}{x^3-3}$

   (c) $x = (3x+20)^{1/3}$

   Choose the form which satisfies the condition $\mid g'(x) \mid < 1$ on $[1,4]$ and then find third approximation x_3 when $x_0 = 1.5$.

   Solution For (a) and (b), $|g'(x)| > 1$ on $[1,4]$, but for (c), we have $g'(x) = \frac{1}{()3x+20)^{3/2}} < 1$ on $[1,4]$. So by using (c), the third approximation by the fixed-point method is, $x_3 = 3.0789$.

2. Consider the nonlinear equation $g(x) = \dfrac{1}{2}e^{0.5x}$ defined on the interval $[0,1]$. Then

   (a) Show that there exists a unique fixed-point for g in $[0,1]$.

   (b) Use the fixed-point iterative method to compute x^3, set $x_0 = 0$.

   (c) Compute an error bound for your approximation in part (b).

   Solution:

   (a) Since given g is continuous in $[0,1]$ and $g(0) = 0.5 \in [0,1]$ and $g(1) = 0.8243 \in [0,1]$. Also, $g'(x) = 1/4e^{0.5x}$ and $g'(0) = 0.25$, $g'(1) = 0.41218$, so $|g'(x)| < 1$ for $x \in [0,1]$.

   (b) The fixed-point iterative method using $x_0 = 0$ gives $x_1 = g(x_0) = 0.5, x_2 = g(x_1) = 0.64201, x_3 = g(x_2) = 0.58705$

   (c) The error bound for the approximation is

   $$|\alpha - x_3| \leq \frac{k^3}{1-k}|x-1-x_0|$$

   where $k = \max_{0 \leq x \leq 1}|g'(x)| = g'(1) = 0.41218$. Thus

   $$|\alpha - x_3| \leq \frac{(0.41218)^3}{1-0.41218}|0.5-0| = 0.05957$$

3. An equation $x^3 - 2 = 0$ can be written in form $x = g(x)$ in two ways:

   (a) $x = g_1(x) = x^3 + x - 2$

   (b) $x = g_2(x) = \frac{2+5x-x^3}{5}$ Generate first four approximations from $x_{n+1} = g_i(x_n),\ i = 1, 2$ by using $x_0 = 1.2$.
   Show which sequence converges to $2^{1/3}$ and why ?

   Solution:

   (a) The fixed-point iterative method using $x_0 = 0$ gives $x_4 = -16.3514$.

   (b) The fixed-point iterative method using $x_0 = 0$ gives $x_4 = 1.2599$. The second sequence converges to $2^{1/3}$ because $|g_2'(2^{1/3})| = 0.0476 < 1$ whereas the first sequence does not converge to $2^{1/3}$ because $|g_1'(2^{1/3})| = 4.7622 > 1$.

4. Find value of k such that the iterative scheme $x_{n+1} = \frac{x_n^2 - 4kx_n + 7}{4}, n \geq 0$ converges to 1. Also, find the rate of convergence of the iterative scheme

5. Write the equation $x^2 - 6x + 5 = 0$ in the form $x = g(x)$, where $x \in [0, 2]$, so that the iteration $x_{n+1} = g(x_n)$ will converge to the root of the given equation for any initial approximation $x_0 \in [0, 2]$.

6. Which of the following iterations

   (a) $x_{n+1} = \frac{1}{4}(x_n^2 + \frac{6}{x_n})$

   (b) $x_{n+1} = (4 - \frac{6}{x_n})$

   is suitable to find a root of the equation $x^3 = 4x^2 - 6$ in the interval $[3, 4]$? Estimate the number of iterations required to achieve 10^{-3} accuracy, starting from $\mathrm{x}_0 = 3$.

Solution: **(a)** Let $g_1(x) = \frac{1}{4}\left(x^2 + \frac{6}{x}\right)$ which is continuous in $[3,4]$, but $g_1'(x) > 1$ for all $x \in (3,4)$. So $g_1(x)$ is not suitable.

(b) $g_2(x) = \left(4 - \frac{6}{x_n^2}\right)$ which is continuous in $[3,4]$ and $g(x) \in [3,4]$ for all $x \in [3,4]$. Also, $|g_2'(x)| = |12/x^3| < 1$ for all $x \in (3,4)$. Then from the Theorem **??** implies that a unique fixed-point exists in $[3,4]$. To find an approximation of that is accurate to within 10^{-3}, we need to determine the number of iterations n so that

$$|\alpha - x_n| \leq \frac{k^n}{1-k}|x_1 - x_0| < 10^{-3}$$

With $k = \max_{3 \leq x \leq 4} |g'(x)| = 4/9$ and using the fixed-point method by taking $x_0 = 3$, we have $x_1 = 10/3$, we have and

$$|\alpha - x_n| \leq \frac{(4/9)^n}{(1-4/9)}|10/3 - 3| < 10^{-3}$$

Thus a bound for the number of iterations is

$$\frac{(4/9)^n}{(1-4/9)}|10/3 - 3| < 10^{-3}$$

and solving for n, we get, $n = 8$.

7. An equation $e^x = 4x^2$ has a root in $[4,5]$. Show that we cannot find that root using $x = g(x) = \frac{1}{2}e^{x/2}$ for the fixed-point iteration method. Can you find another iterative formula which will locate that root ? If yes, then find third iterations with $x_0 = 4.5$. Also find the error bound

Solution: Since $g'(x) = \frac{1}{4}e^{x/2} > 0$ for all $x \in (4,5)$, therefore, the fixed-point iteration fails to converge to the root in $[4,5]$. Consider $x = g(x) = \ln(4x^2)$ and its derivative can be found as, $g'(x) = \frac{2}{x}$. Note that $|g'(x)| < 1$ for all $x \in (4,5)$ and the fixed-point iteration converges to the root in $[4,5]$. Using the fixed-point iteration method gives the third iteration as $x_3 = 4.3253$. With $k = \max_{4 \leq x \leq 5} |g'(x)| = g'(4) = 0.5$ and taking $x_0 = 4.5$, we have $x_1 = 4.3944$. Thus the error bound for the above approximation, gives

$$|\alpha - x_3| \leq \frac{(0.5)^3}{(1-0.5)}|4.3944 - 4.5| = 0.0264$$

Newton Exercises

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1. Solve the Problem 1 of bisection' by the Newton's method by taking initial approximation $x_0 = -2$.

   Solution: Since $f(x) = e^x - 2 - x$ and its derivative is $f'(x) = e^x - 1$. Using the Newton's iterative formula, we get

   $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -2 - \frac{(0.1353)}{(-0.8647)} = -1.8435$$

   and

   $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1.8435 - \frac{(0.0017)}{(-0.8417)} = -1.8414$$

   Thus, the Newton's method gives $x_2 = -1.8414$, accurate to within 10^{-4}.

2. Let $f(x) = e^x + 3x^2$

   (a) Find the Newton's formula $g(x_k)$.

   (b) Start with $x_0 = 4$ and compute x_4.

   (c) Start with $x_0 = -0.5$ and compute x_4.

   Solution: **(a)** Using the Newton's formula gives

   $$x_{k+1} = g(x_k) = x_k - \frac{e^{x_k} + 3x_k^2}{e^{x_k} + 6x_k} = \frac{(x_k - 1)e^{x_k} + 3x_k^2}{e^{x_k} + 6x_k}$$

   (b) For $x_0 = 4, \; x_4 = 0.1215$

   (c) For $x_0 = -0.5, \; x_4 = -32.1077$

3. Use the Newton's formula for the reciprocal of square root of a number 15 and then find the 3^{rd} approximation of number, with $x_0 = 0.05$.

Solution: Let N be a positive number and $x = 1/\sqrt{N}$. If $f(x) = 0$, then $x = \alpha = 1/\sqrt{N}$ is the exact zero of the function

$$f(x) = 1/x^2 - N, \quad \text{gives } f'(x) = -2/x^3$$

Hence, assuming an initial estimate to the root, say, $x = x_0$ and by using the Newton's iterative formula, we get

$$x_1 = x_0 - \frac{(1/x_0^2 - N)}{(-2/x_0^3)} = x_0(3 - Nx_0^2)/2.$$

In general, we have

$$x_{n+1} = x_n(3 - Nx_n^2)/2, \qquad n = 0, 1, \ldots,$$

Now to find the reciprocal of square root of a number $N = 15$, using an initial gauss of say $x_0 = 0.05$, we have

$$\begin{array}{lcl} n &=& 0, \quad x_1 = x_0(3 - Nx_0^2)/2 = 0.05(3 - (15)(0.05)^2)/2 = 0.0741 \\ n &=& 1, \quad x_2 = x_1(3 - Nx_1^2)/2 = 0.0741(3 - (15)(0.0741)^2)/2 = 0.1081 \\ n &=& 0, \quad x_3 = x_2(3 - Nx_2^2)/2 = 0.1081(3 - (15)(0.1081)^2)/2 = 0.1527 \end{array}$$

4. Use the Newton's method to find solution accurate to within 10^{-4} of the equation $\tan(x) - 7x = 0$, with initial approximation $x_0 = 4$.

   Solution: The Newton's method gives $x_6 = -4.1231e{-}014$, accurate to within 10^{-4}.

5. Find the Newton's formula for $f(x) = x^3 - 3x + 1$ in $[1, 3]$ to calculate x_3, if $x_0 = 1.5$. Also, find the rate of convergence of the method.

   Solution: The Newton's method gives $x_3 = 1.5321$. To find the order of convergence, we do the following:

$$g(x) = x - \frac{x^3 - 3x + 1}{3x^2 - 3} = \frac{2x^3 - 1}{3x^2 - 3}$$

and its derivative is

$$g'(x) = \frac{6x^4 - 18x^2 + 6x}{(3x^2 - 3)^2} = \frac{6x(x^3 - 3x + 1)}{(3x^2 - 3)^2} = 0$$

Thus $g'(\alpha) = 0$, gives at least quadratic convergence.

Secant Exercises

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1. Find the positive root of $f(x) = x^{10} - 1$ by the secant method by using starting values $x_0 = 1.2$ and $x_1 = 1.1$ accurate to within 10^{-4}.

   Solution: Using the secant method gives $x_0 = 1.2$ and $x_1 = 1.1$, so $f(1.2) = 5.1917$ and $f(1.1) = 1.5937$, and the new approximation

   $$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1.2)(1.5937) - (1.1)(5.1917)}{1.5937 - 5.1917} = 1.0557$$

   Similarly, the secant method gives other approximations

   $$x_3 = 1.0192, \quad x_4 = 1.0042, \quad x_5 = 1.0004, \quad x_6 = 1.0000, \quad x_7 = 1.0000$$

   accurate to within 10^{-4}.

2. Find the first three estimates for the equation $x^3 - 2x - 5 = 0$ by the secant method using $x_0 = 2$ and $x_1 = 3$.

   Solution: Using the secant method gives $x_0 = 2$ and $x_1 = 3$, so $f(2) = -1$ and $f(3) = 16$, and the first approximation

   $$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(2)(16) - (3)(-1)}{16 + 1} = 2.0588$$

   second approximation using $x_1 = 3, x_2 = 2.0588, f(3) = 16$, and $f(2.0588) = -0.3908$, gives

   $$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(3)(-0.3908) - (2.0588)(16)}{-0.3908 - 16} = 2.0813$$

   and third approximation using $x_2 = 2.0588, x_3 = 2.0813, f(2.0588) = -0.3908$, and $f(2.0813) = -0.1472$ is

   $$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{(2.0588)(-0.1472) - (2.0813)(-0.3908)}{-0.1472 + 0.3908} = 2.0948$$

3. Solve the equation $e^{-x} - x = 0$ by using the secant method, starting with $x_0 = 0$ and $x_1 = 1$, accurate to 10^4.

   Solution: The secant method gives $x_4 = 0.56714$, accurate to within 10^{-4}.

4. Use the secant method to find a solution accurate to within 10^{-4} for $\ln(x) + x - 5 = 0$ on $[3, 4]$.

   Solution: The secant method gives $x_3 = 3.6934$, accurate to within 10^{-4} .