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Arba Minch University

Numerical Method(Math 2073/53)
Problem set 1

"Numerical methods" are methods devised to solve mathematical problems on a computer.

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April 6, 2019

1. Write down each of these numbers rounded them to **4 decimal places**:

$$0.12345, -0.44444, 0.5555555, 0.000127351, 0.000005$$

Solution

$$0.1235, -0.4444, 0.5556, 0.0001, 0.0000$$

2. Write down each of these numbers, rounding them to **4 significant figures:**

$$0.12345, -0.44444, 0.5555555, 0.000127351, 25679$$

Solution

$$0.1235, -0.4444, 0.5556, 0.0001274, 25680$$

3. Show that the evaluation of the function

$$f(x) = x^2 - x - 1500$$

near $x = 39$ is an ill-conditioned problem.

Solution: Consider $f(39) = -18$ and $f(39.1) = -10.29$. In changing x from 39 to 39.1 we have changed it by about 0.25%. But the percentage change in f is greater than 40%. The demonstrates the ill-conditioned nature of the problem.
One reason that this matters is because of rounding error. Suppose that, in the example above, we know is that x is equal to 39 to 2 significant figures. Then we have no chance at all of evaluating f with confidence, for consider these values

$$f(38.6) = -48.64$$
$$f(39) = -18$$
$$f(39.4) = 12.96.$$

All of the arguments on the left-hand sides are equal to 39 to 2 significant figures so all the values on the right-hand sides are contenders for $f(x)$. The ill-conditioned nature of the problem leaves us with some serious doubts concerning the value of f. It is enough for the time being to be aware that ill-conditioned problems exist.

4. Consider the function

$$f(x) = x^2 + x - 1975$$

and suppose we want to evaluate it for some x.

(a) Let $x = 20$. Evaluate $f(x)$ and then evaluate f again having altered x by just 1%. What is the percentage change in f? Is the problem of evaluating $f(x)$, for $x = 20$, a well-conditioned one?

(b) Let $x = 44$. Evaluate $f(x)$ and then evaluate f again having altered x by just 1%. What is the percentage change in f? Is the problem of evaluating $f(x)$, for $x = 44$, a well-conditioned one?

(Answer: the problem in part (a) is well-conditioned, the problem in part (b) is illconditioned.)

5. Perform the following calculations

Questions	a	b	c	d
Exact	17/15	4/15	139/660	301/660
3-digit chopping				
Relative error				
3-digit rounding				
Relative error				

Answer: We have

Questions	a	b	c	d
Exact	17/15	4/15	139/660	301/660
3-digit chopping	1.13	0.266	0.211	0.455
Relative error	0.003	0.0025	0.002	0.00233
3-digit rounding	1.13	0.266	0.21	0.456
Relative error	0.003	0.0025	0.0029	0.000133

6. Suppose that two points (x_0, y_0) and (x_1, y_1) are on a straight line with $y_1 \neq y_0$. Two formulas are available to compute the x-intercept of the line:

$$x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0} \quad \text{and} \quad x = x_0 - \frac{(x_1 - x_0) y_0}{y_1 - y_0}.$$

(a) Show that both formulas are algebraically correct.

(b) Suppose that $(x_0, y_0) = (1.31, 3.24)$ and $(x_1, y_1) = (1.93, 4.76)$. Use three-digit rounding arithmetic to compute the x-intercept using both of the formulas. Which method is better and why?

Answer: (*a*) We should be a bit careful here to avoid dividing by 0. It is potentially unsafe to write that the equation of the line is $\frac{y-y_0}{x-x_0}=\frac{y_1-y_0}{x_1-x_0}$, because potentially $x_0=x_1$. However, we are told that $y_0 \neq y_1$, so we can instead write the equation of the line as $\frac{x-x_0}{y-y_0}=\frac{x_1-x_0}{y_1-y_0}$. We can cross-multiply and write this instead as $x-x_0=y-y_0\left(\frac{x_1-x_0}{y_1-y_0}\right)$.

The x-intercept is the point on the line at which $y=0$, so we can substitute $y=0$ into this equation and get $x-x_0=(-y_0)\left(\frac{x_1-x_0}{y_1-y_0}\right)$, or $x=x_0-\frac{(x_1-x_0)y_0}{y_1-y_0}$, which is the given formula.

Now we can simplify:

$$x=x_0-\frac{(x_1-x_0)y_0}{y_1-y_0}=\frac{x_0(y_1-y_0)}{y_1-y_0}-\frac{(x_1-x_0)y_0}{y_1-y_0}=\frac{x_0y_1-x_1y_0}{y_1-y_0}.$$

b) The first formula gives the answer -0.00658, while the second formula gives the answer -0.0100. In this case, the second formula is better. The

first one involved subtracting $x_0y_1-x_1y_0$. Because $x_0y_1=6.24$ and $x_1y_0=6.25$, the result of the subtraction has only one significant digit.

We can check this by working to 10 significant digits. In that case, the first formula gives-0.0115789474 and the second gives -0.0115789470. Surely the answer is closer to-0.01 than to -0.00658.

7. Compute $0.1+0.2-0.3$ in MATLAB

8. The distance from the Earth to the Moon varies between $356400km$ and $406700km$. Give a bound on the absolute and relative errors incurred when using one of both values as the "real distance." This requires giving two bounds

9. When computing $1-\sin(\pi/2+x)$ for small x, one commits a specific type of error. Which one? Can this be prevented?

10. Assume the Euro-peseta exchange value is $1Eu=166.386birr$. However, Law requires that these transactions be rounded to the nearest monetary unit (i.e. either 1 birr or 1 cent). Compute

    (a) The absolute and relative errors incurred when exchanging 1 Euro for 166 birr.

    (b) The absolute and relative errors incurred when exchanging 1 birr for its "equivalent" in Euros.

    (c) The absolute and relative errors incurred when exchanging 1 cent for its "equivalent" in pesetas.

11. Consider the following two computations, the first one is assumed to be "correct" while the second one is an approximation:

$$26493-\frac{33}{0.0012456}=-0.256 \text{ (the exact result)}$$

.

$$26493\frac{33}{0.0012456}=8.2488 \text{ (approximation)}$$

12. Consider $f(x)=\frac{1}{1+x}$

    (a) What is the second order Taylor polynomial of $f(x)$ around 0?

(b) Use part (a) to approximate $f(0.1)$. What is the relative and absolute error of the approximation?

13. Convert the binary number to decimal format.

    (a) 1010100

    (b) 1101.001

    (c) 10110001110001.01010111

14. Converting decimal to binary

    (a) 157

    $157 \div 2 = 78$ with a remainder of 1
    $78 \div 2 = 39$ with a remainder of 0
    $39 \div 2 = 19$ with a remainder of 1
    $19 \div 2 = 9$ with a remainder of 1
    $9 \div 2 = 4$ with a remainder of 1
    $4 \div 2 = 2$ with a remainder of 0
    $2 \div 2 = 1$ with a remainder of 0
    $1 \div 2 = 0$ with a remainder of $1 <---$to convert write this remainder first.

    Next, write down the value of the remainders from bottom to top (in other words write down the bottom remainder first and work your way up the list) which gives:

    $$10011101 = 157$$

    (b) 256.1875

15. Consider the function:

    $$f(x) = x(\sqrt{x} - \sqrt{x-1})$$

    (a) Use MATLAB to calculate the value of $f(x)$ for the following three values of x: $x = 10, x = 1000, \quad \text{and } x = 100000$.

    (b) Use the decimal format with six significant digits to calculate $f(x)$ for the values of x in part(a). Compare the results with the values in part (a).

    (c) Change the form of $f(x)$ by multiplying it by $\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}}$.Using the new form with numbers in decimal format with six significant digits, calculate the value of $f(x)$ for the three values of x. Compare the results with the values in part (a).

SOLUTION

(a)

```
>> format long g
>> x = [10  1000  100000];
>> Fx = x.*(sqrt(x) - sqrt(x-1))
Fx =
         1.6227766016838          15.8153431255776           158.114278298171
```

(b) Using decimal format with six significant digits in Eq. (1.12) gives the following values for $f(x)$:

$f(10) = 10(\sqrt{10} - \sqrt{10-1}) = 10(3.16228 - 3) = 1.62280$

This value agrees with the value from part (a), when the latter is rounded to six significant digits.

$f(1000) = 1000(\sqrt{1000} - \sqrt{1000-1}) = 1000(31.6228 - 31.6070) = 15.8$

When rounded to six significant digits, the value in part (a) is 15.8153.

$f(100000) = 100000(\sqrt{100000} - \sqrt{100000-1}) = 100000(316.228 - 316.226) = 200$

When rounded to six significant digits, the value in part (a) is 158.114.

The results show that the rounding error due to the use of six significant digits increases as x increases and the relative difference between $\sqrt{x}$ and $\sqrt{x-1}$ decreases.

(c) Multiplying the right-hand side of Eq. (1.12) by $\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}}$ gives:

$$f(x) = x(\sqrt{x} - \sqrt{x-1})\frac{\sqrt{x}+\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}} = \frac{x[x-(x-1)]}{\sqrt{x}+\sqrt{x-1}} = \frac{x}{\sqrt{x}+\sqrt{x-1}} \tag{1.13}$$

Calculating $f(x)$ using Eq. (1.13) for $x = 10$, $x = 1000$, and $x = 100000$ gives:

$$f(10) = \frac{10}{\sqrt{10}+\sqrt{10-1}} = \frac{10}{3.16228+3} = 1.62278$$

$$f(1000) = \frac{1000}{\sqrt{1000}+\sqrt{1000\quad 1}} = \frac{1000}{31.6228+31.6070} = 15.8153$$

$$f(100000) = \frac{100000}{\sqrt{100000}+\sqrt{100000-1}} = \frac{1000}{316.228+316.226} = 158.114$$

Now the values of $f(x)$ are the same as in part (a).

16. Find the rates of convergence of the following functions as $n \to \infty$:

(a) $\lim_{n\to\infty} \sin(\frac{1}{n}) = 0$

(b) $\lim_{n\to\infty} \sin(\frac{1}{n^2}) = 0$

(c) $\lim_{n\to\infty} (\sin(\frac{1}{n}))^2 = 0$

(d) $\lim_{n\to\infty} (\log(n+1) - \log(n))$

(e) $\lim_{h\to 0} \frac{\sin h}{h}$

Answer: The first three problems can be answered much more easily if we know that $\sin x \le x$ for $0 \le x < 1$. (Much more than this is true, but this inequality suffices.) As a result, we have $|\sin\left(\frac{1}{n}\right)| < \frac{1}{n}$, and so $\sin\frac{1}{n} = O(\frac{1}{n})$. Similarly, $|\sin\left(\frac{1}{n^2}\right)| < |\frac{1}{n^2}|$, and so $\sin\left(\frac{1}{n^2}\right) = O(\frac{1}{n^2})$. We also can take the inequality $|\sin\left(\frac{1}{n}\right)| < \frac{1}{n}$ and square both sides, giving $|\sin\left(\frac{1}{n}\right)|^2 < \frac{1}{n^2}$, and therefore $\left(\sin\frac{1}{n}\right)^2 = O(\frac{1}{n^2})$.

The last one is a bit more interesting. We rewrite $\log(n+1) - \log(n)$ as $\log\left(1+\frac{1}{n}\right)$, and now use the fact that $|\log(1+x)| < |x|$ for $0 < x < 1$. Therefore, $|\log(n+1) - \log(n)| < \frac{1}{n}$, and so $\log(n+1) - \log(n) = O(\frac{1}{n})$.

Answer: Here, Maclaurin series are the easiest way to get a solution:

$$\frac{\sin h}{h} = \frac{h - \frac{h^3}{6} + \cdots}{h} = 1 - \frac{h^2}{6} + \cdots$$

and so $\frac{\sin h}{h} = 1 + O(h^2)$.

For **b**, we have

$$\frac{1 - \cos h}{h} = \frac{1 - (1 - \frac{h^2}{2} + \cdots)}{h} = \frac{h}{2} + \cdots,$$

so $\frac{1-\cos h}{h} = O(h)$.

For **c**, we have

$$\frac{\sin h - h\cos h}{h} = \frac{(h - \frac{h^3}{6} + \cdots) - h(1 - \frac{h^2}{4} + \cdots)}{h} = \frac{-h^2}{6} + \frac{h^2}{4}$$

so $\frac{\sin h - h\cos h}{h} = O(h^2)$.

Finally, for **d**, we have

$$\frac{1 - e^h}{h} = \frac{1 - (1 + h + \frac{h^2}{2} + \cdots)}{h} = -1 - \frac{h}{2} + \cdots,$$