Math 2073 -Solution of Nonlinear Equations

Lecture-5

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- Newton's method is one of the most widely used of all iterative techniques for solving equations.
- To use the method we begin with an initial guess x_0, sufficiently close to the root x_*.
- The next approximation x_1 is given by the point at which the tangent line to f at $f(x_0, f(x_0))$ crosses the x-axis.
- It is clear that the value x_1 is much closer to x_* than the original guess x_0.
- If x_{n+1} denotes the value obtained by the succeeding iterations, that is the x-intercept of the tangent line to f at $(x_n, f(x_n))$, then a formula relating x_n and x_{n+1}, known as Newton's method, is given by
- $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0 \tag{1}$$

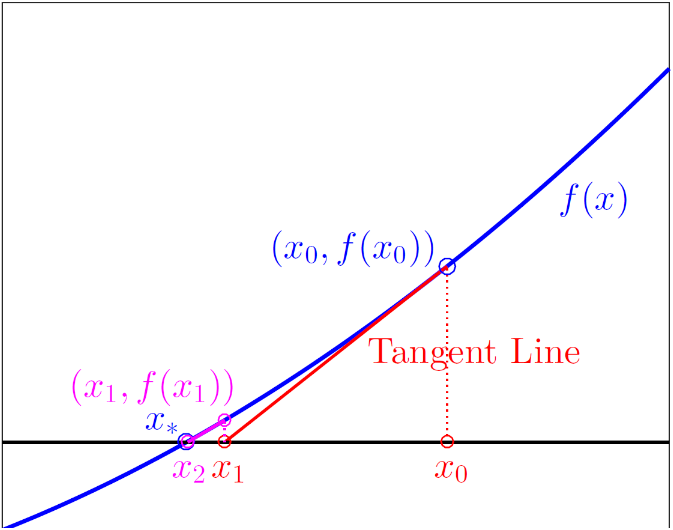

FIGURE: Newton's method.

- If $f \in C^2[a, b]$, and we know $x_1 \in [a, b]$ be an approximation to x_* such that $f'(x_1) \neq 0$ and $|x_1 - x_*|$ is "small."
- Consider the first Taylor polynomial for $f(x)$ expanded about p_1 and evaluated at $x = x_*$:

$$0 = f(x_*) = f(x_1) + (x_1 - x_*)f'(x) + \frac{(x_1 - x_*)^2}{2!}f''(\xi(x_1)) \qquad (2)$$

- where $\xi(x_1) \in [x_1, x_*]$
- Newton's method is derived by assuming that $|x_1 - x_*|$ is small, which means that $|x_1 - x_*|^2 \ll |x_1 - x_*|$, hence we make the approximation:

$$0 = f(x_1) + (x_1 - x_*)f'(x)$$

- Now solve for x_*

$$x_* = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Newton Method

Given a scalar differentiable function in one variable, $f(x)$:

1. Start from an initial guess x_0.
2. For $n = 0, 1, 2, \cdots$, set

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

until x_{n+1} satisfies termination criteria.

Convergence

Theorem

Let $f(x) \in C^2[a, b]$. If $x_* \in [a, b]$ such that $f(x_*) = 0$ and $f'(x_*) \neq 0$, then there exists a $\sigma > 0$ such that Newton's method generates a sequence $\{x_n\}_{n=1}^{\infty}$ converging to x_* for any initial approximation $x_1 \in [x_* - \sigma, x_* + \sigma]$.

Example

Using Newton's method, solve $f(x) = x^6 - x - 1$.

Solution: Here

$$f(x) = x^6 - x - 1,\ f'(x) = 6x^5 - 1$$

and the iteration

$$x_{n+1} = x_n - \frac{x_n^6 - x_n - 1}{6x_n^5 - 1}, \neq 0$$

n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
1	1.14357584	0.05642416
2	1.13490946	0.00866638
3	1.13472422	0.00018524
4	1.13472414	0.00000008

Secant Method

- Newton's method is an extremely powerful technique, but it has a major weakness; the need to know the value of the derivative of f at each approximation.
- Frequently, $f'(x)$ is far more difficult and needs more arithmetic operations to calculate than $f(x)$.
- The secant method is a variant of Newton's method, where $f'(x_n)$ is replaced by its finite difference approximation based on the evaluated function values at x_n and at the previous iterate x_{n-1}.
- Assuming convergence, observe that near the root

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

- Substitution of this approximation into the formula for Newton's method yields the **Secant method**,

$$x_{n+1} = \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}, \quad n = 0, 1, 2, 3, \cdots$$

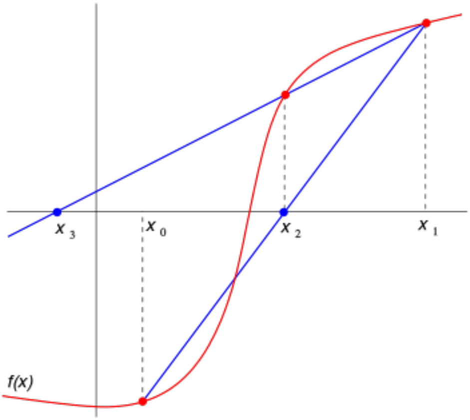

Figure: The first two iterations of the secant method.

Given a scalar differentiable function in one variable, $f(x)$:

1. Start from two initial guesses x_0 and x_1.
2. For $k = 1, 2, \cdots$, set

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

until x_{n+1} satisfies termination criteria.

EXAMPLE

Find a root of the equation $x^6 - x - 1 = 0$
We apply the method of secant method with $x_1 = 1$ and $x_2 = 1.5$.

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

The calculations based on the secant method are shown in the following Table

Solution

Iteration	a	b	$f(a)$	$f(b)$	x
1.	1.000000000	2.000000000	-1.000000000	61.000000000	1.016129032
2.	2.000000000	1.016129032	61.000000000	-0.915367713	1.030674754
3.	1.016129032	1.030674754	-0.915367713	-0.831921414	1.175688944
4.	1.030674754	1.175688944	-0.831921414	0.465227164	1.123679065
5.	1.175688944	1.123679065	0.465227164	-0.110632879	1.133671081
6.	1.123679065	1.133671081	-0.110632879	-0.010805918	1.134752681
7.	1.133671081	1.134752681	-0.010805918	0.000293664	1.134724065
8.	1.134752681	1.134724065	0.000293664	-0.000000748	1.134724138
9.	1.134724065	1.134724138	-0.000000748	-0.000000000	1.134724138
10.	1.134724138	1.134724138	-0.000000000	-0.000000000	1.134724138

(Recall that the true root is $\alpha = 1.134724138$.)

ADVANTAGES AND DISADVANTAGES:

1. The error decreases slowly at first but then rapidly after a few iterations.
2. The secant method is slower than Newton's method but faster than the bisection method.
3. Each iteration of Newton's method requires two function evaluations, while the secant method requires only one
4. The secant method does not require differentiation.

1. The Bisection method
   1. Very stable Algorithm - Good technique to find starting point for Newton's method
   2. Costs only one function evaluation, so rapid iterations
   3. Linear convergence, so slow (3.3 iterations/digit)
2. The Secant method
   1. Hard to find starting points (Unknown basin of attraction)
   2. Costs only two function evaluations, so rapid iterations
   3. Superlinear convergence, $\alpha \approx 1.62$, which is pretty fast
3. The Newton's method
   1. Hard to find starting points (Unknown basin of attraction)
   2. Finding and evaluating derivative requires more machine work at each iteration
   3. Quadratic convergence is very fast- doubling the digits at each iteration.

Example

Summary Find the roots of

$$x^3 + 4x^2 - 10 \quad x \in [1.5, 2]$$

n	Bisection	Secant	Newton
1	1.25	1.33898305084745	1.45454545454545
2	1.375	1.36356284991687	1.36890040106951
3	1.3125	1.36525168742565	1.36523660020211
4	1.34375	1.36522999568865	1.36523001343536
5	1.359375	1.36523001341391	1.36523001341409
6	1.3671875	1.36523001341409	
7	1.36328125		
8	1.365234375		
9	1.3642578125		
10	1.36474609375		
11	1.364990234375		