Math 2073 -Solution of Nonlinear Equations

Lecture-4

Dejen Ketema

Department of Mathematics
Arba Minch University
https://elearning.amu.edu.et/course/view.php?id=279

Fall 2019

- The false position method retains the main features of the Bisection method, that the root is trapped in a sequence of intervals of decreasing size.
- This method uses the point where the secant lines intersect the x-axis.
- The secant line over the interval $[a, b]$ is the chord between $(a, f(a))$ and $(b, f(b))$.
- The two right angles in the figure are similar, which mean that

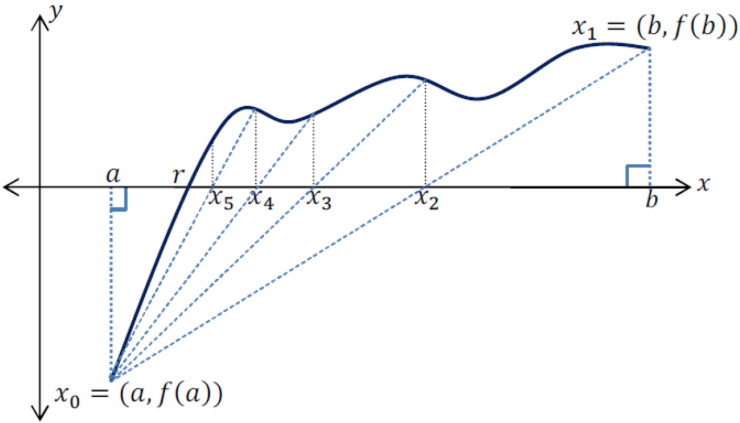

Figure: Root finding using Regula-Falsi method

$$\frac{b-c}{f(b)} = \frac{c-a}{f(a)}$$

This implies that

Formula

$$c = \frac{af(b)-bf(a)}{f(b)-f(a)} = b - f(b)\frac{(b-a)}{f(b)-f(a)} \tag{1}$$

then we can compute $f(c)$ and repeat the process with the interval $[a, c]$, if $f(a) \times f(c) < 0$ or to the interval $[c, b]$, if and only if $f(c) \times f(b) < 0$.

Regula-Falsi Method: Given a continuous function $f(x)$

1. Choose the first interval by finding points a and b such that a solution exists between them and $(a < b)$. This means that $f(a)$ and $f(b)$ have different signs such that $f(a)f(b) < 0$. The points can be determined by looking at a plot of $f(x)$ versus x.
2. Calculate the first estimate of the numerical solution c by using Eq. (1).
3. Determine whether the actual solution is between a and c or between c and b. This is done by checking the sign of the product $f(a) \times f(c)$:
   1. If $f(a) \times f(c) < 0$, the solution is between a and c?.
   2. If $f(a) \times f(c) > 0$, the solution is between c and b.
4. Select the subinterval that contains the solution (a to c, or c to b) as the new interval $[a, b]$, and go back to step 2.

Steps 2 through 4 are repeated until a specified tolerance or error bound is attained.

EXAMPLE

Using the False Position method, find a root of the function $f(x) = e^x - 3x^2$ to an accuracy of 5 digits. The root is known to lie between 0.5 and 1.0.

SOLUTION

We apply the method of False Position with $a = 0.5$ and $b = 1.0$

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0).$$

The calculations based on the method of False Position are shown in the following Table

Solution

Example

Iteration	a	b	$f(a)$	$f(b)$	x	$f(x)$
1	0.5	1	0.89872	-0.28172	0.88067	0.08577
2	0.88067	1	0.08577	-0.28172	0.90852	0.00441
3	0.90852	1	0.00441	-0.28172	0.90993	0.00022
4	0.90993	1	0.00022	-0.28172	0.91000	0.00001
5	0.91000	1	0.00001	-0.28172	0.91001	0

Drawback of FPM

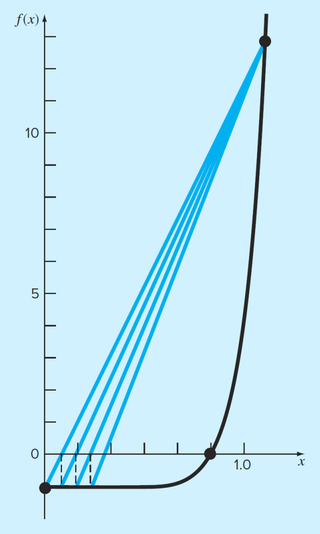

FIGURE: Plot of $f(x) = x^{10} - 1$, illustrating slow convergence of the false-position method.

Example

A Case Where Bisection Is Preferable to False Position Use bisection and false position to locate the root of

$$f(x) = x^{10} - 1$$

between $x = 0$ and 1.3.

Solution

Using bisection, the results can be summarized as

Iteration	a	b	c	ϵ_a	ϵ_t
1	0	1.3	0.65	100.0	35
2	0.65	1.3	0.975	33.3	2.5
3	0.975	1.3	1.1375	14.3	13.8
4	0.975	1.1375	1.05625	7.7	5.6
5	0.975	1.05625	1.015625	4.0	1.6

Thus, after five iterations, the true error is reduced to less than 2%. For false position, a very different outcome is obtained:

Example

Solution

Iteration	a	b	c	ϵ_a	ϵ_t
1	0 1.3	0.09430	90.6		
2	0.09430	1.3	0.18176	48.1	81.8
3	0.18176	1.3	0.26287	30.9	73.7
4	0.26287	1.3	0.33811	22.3	66.2
5	0.33811	1.3	0.40788	17.1	59.2

After five iterations, the true error has only been reduced to about 59%. Insight into these results can be gained by examining a plot of the function. As in Fig.2, the curve violates the premise on which false position was based that is, if $f(a)$ is much closer to zero than $f(b)$, then the root should be much closer to a than to b.

Fixed point Iteration Method

- Fixed-point iteration is a method for solving an equation of the form $f(x) = 0$.
- The method is carried out by rewriting the equation in the form:

$$x = g(x) \tag{2}$$

- Obviously, when x is the solution of $f(x) = 0$, the left side and the right side of Eq. (2) are equal.
- The point of intersection of the two plots, called the fixed point, is the solution.
- It starts by taking a value of x near the **fixed point** as the first guess for the solution and substituting it in $g(x)$.
- The value of $g(x)$ that is obtained is the new (second) estimate for the solution.
- The second value is then substituted x back in $g(x)$, which then gives the third estimate of the solution.
- The iteration formula is thus given by:

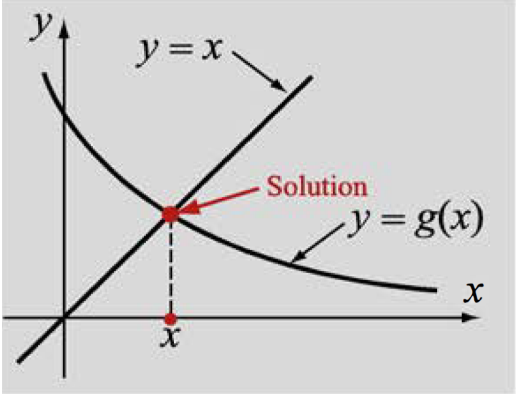

Figure: Fixed-point iteration method.

Definition

If we can write $f(x) = 0$ in the form $x = g(x)$, then the point x would be a fixed point of the function g (that is, the input of g is also the output). Then an obvious sequence to consider is

$$x_{n+1} = g(x_n) \tag{3}$$

The function $g(x)$ is called the **iteration function.**

- When the method works, the values of x that are obtained are successive iterations that progressively converge toward the solution.
- Two such cases are illustrated graphically in Fig.4.

- The solution process starts by choosing point x_1 on the x-axis and drawing a vertical line that intersects the curve $y = g(x)$ at point $g(x_1)$. Since $x_2 = g(x_1)$, a horizontal line is drawn from point $(x_1, g(x_1))$ toward the line $y = x$.
- The intersection point gives the location of x_2 .
- From x_2 a vertical line is drawn toward the curve $y = g(x)$.
- The intersection point is now $(x_2, g(x_2))$, and $g(x_2)$ is also the value of x_3.
- From point $(x_2, g(x_2))$ a horizontal line is drawn again toward $y = x$, and the intersection point gives the location of x_3?
- As the process continues the intersection points converge toward the fixed point, or the true solution x_{rs}.

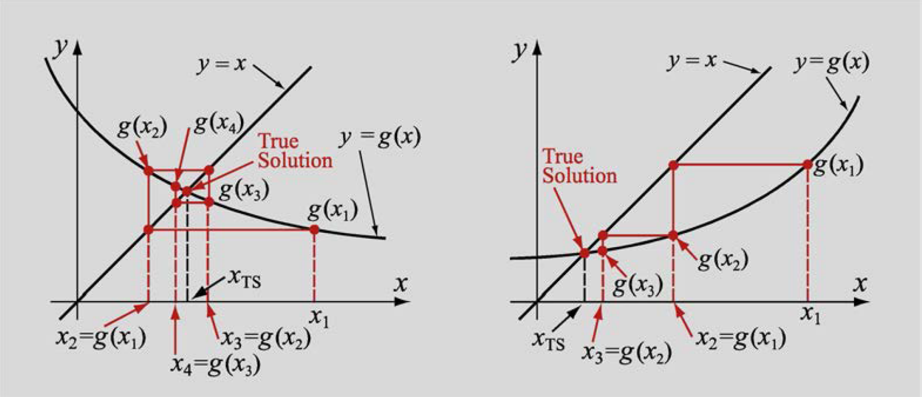

FIGURE: Convergence of the fixed-point iteration method.

- It is possible, however, that the iterations will not converge toward the fixed point, but rather diverge away.
- This is shown in Fig. 5.
- The figure shows that even though the starting point is close to the solution, the subsequent points are moving farther away from the solution.
- Sometimes, the form $f(x) = 0$ does not lend itself to deriving an iteration formula of the form $x = g(x)$.
- In such a case, one can always add and subtract x to $f(x)$ to obtain $x + f(x) - x = 0$.
- The last equation can be rewritten in the form that can be used in the fixed-point iteration method: $x = x + f(x) = g(x)$

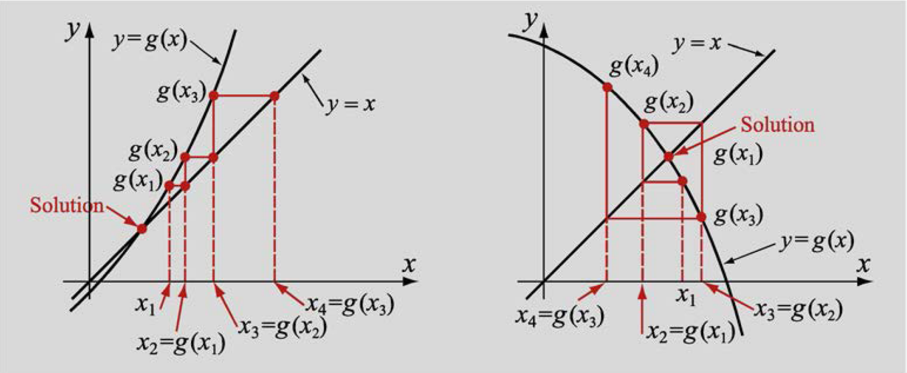

Figure: Divergence of the fixed-point iteration method.

Theorem

The fixed-point iteration method converges if, in the neighborhood of the fixed point, the derivative of $g(x)$ has an absolute value that is smaller than 1 (also called Lipschitz continuous):

$$\left|g'(x)\right| < 1 \tag{4}$$

1. Take an initial approximation x_0
2. Find the next (first) approximation x_1 by using $x_1 = g(x_0)$
3. Follow the above procedure to find the successive approximation

$$x_{n+1} = g(x_n), \; n = 1, 2, 3, \cdot$$

4. Stop evaluation where relative error less than the prescribed accuracy ϵ.

Example

Consider the equation $f(x) = x^2 - 3x + 1 = 0$ (whose true roots are $a_1 = 0.381966$ and $a_2 = 2.618034$. This can be rearranged as a fixed point problem in many different ways. Compare the following two algorithms.

- $x_{n+1} = \frac{1}{3}(x_n^2 + 1) \equiv g_1(x_n)$
- $x_{n+1} = 3 - \frac{1}{x_n} \equiv g_2(x_n)$

Condition for the fixed point iteration scheme

Consider the equation $f(x) = 0$, which has the root α and can be written as the fixed point problem $g(x) = x$. If the following conditions hold

1. $g(x)$ and $g'(x)$ are continuous functions;
2. $|g'(\alpha)| < 1$ then the fixed point iteration scheme based on the function g will converge to α.
3. Alternatively, if $|g'(\alpha)| > 1$ then the iteration will not converge to α.
4. Note that when $|g'(\alpha)| = 1$ no conclusion can be reached.

Example

- For the previous example, we have
- $g_1(x) = \frac{1}{3}(x_2 + 1) \Rightarrow g_1'(x) = \frac{2x}{3}$
- Evaluating the derivative at the two roots (or fixed points):
- $|g_1'(\alpha_1)| = 0.254 \cdots < 1$ and $|g_1'(\alpha_2)| = 1.745 \cdots > 1$
- so the first algorithm converges to $\alpha_1 = 0.3819 \cdots$ but not to $\alpha_2 = 2.618 \cdots$.
- The second algorithm is given by
- $g_2(x) = 3 - \dfrac{1}{x} \Rightarrow g_2'(x) = \dfrac{1}{x^2}$ which gives
- $|g_2'(\alpha_1)| = 6.92 \cdots > 1$ and $|g_2'(\alpha_2)| = 0.13 \cdots < 1$
- so the second algorithm converges to $\alpha_2 = 2.61 \cdots$ but not to $\alpha_1 = 0.38 \cdots$.

n	x_n	x_{n+1}	$f(x_n)$	$x_n - x_{n-1}$
1.	0.500000000	0.416666666	-0.076388888	-0.083333333
2.	0.416666666	0.391203703	-0.020570773	-0.025462963
3.	0.391203703	0.384346779	-0.005317891	-0.006856924
4.	0.384346779	0.382574148	-0.001359467	-0.001772630
5.	0.382574148	0.382120993	-0.000346526	-0.000453155
6.	0.382120993	0.382005484	-0.000088263	-0.000115508
7.	0.382005484	0.381976063	-0.000022477	-0.000029421
8.	0.381976063	0.381968571	-0.000005723	-0.000007492
9.	0.381968571	0.381966663	-0.000001457	-0.000001907
10.	0.381966663	0.381966177	-0.000000371	-0.000000485

For the second root we use $g_2(x)$ and the result is

n	x_n	x_{n+1}	$f(x_n)$	$x_n - x_{n-1}$
1.	2.750000000	2.636363636	0.041322314	-0.113636363
2.	2.636363636	2.620689655	0.005945303	-0.015673981
3.	2.620689655	2.618421052	0.000865651	-0.002268602
4.	2.618421052	2.618090452	0.000126259	-0.000330600
5.	2.618090452	2.618042226	0.000018420	-0.000048225
6.	2.618042226	2.618035190	0.000002687	-0.000007035
7.	2.618035190	2.618034164	0.000000392	-0.000001026
8.	2.618034164	2.618034014	0.000000057	-0.000000149
9.	2.618034014	2.618033992	0.000000008	-0.000000021
10.	2.618033992	2.618033989	0.000000001	-0.000000003