Chapter I: characteristic equation and matrix diagonalization

Eigen values and Eigen vector:

Definition: The real number λ is said to be an **eigenvalue** of the *n* x *n* matrix *A* provided that there exists a **nonzero** vector **v** such that, $\boldsymbol{Av = \lambda v}$**.**

The vector **v** is called the **eigenvector** of the matrix *A* associated with the Eigen valueλ.

Eigenvalues and eigenvectors are also called **characteristic values** and **characteristic vectors** respectively

From definition: $Av = \lambda v$ gives

$$Av - \lambda v = 0$$

$$= (A - \lambda I)v = 0$$

From the property of H SLE (homogenous system of linear equation) the system has nontrivial solution and infinite solution if the coefficient matrix is singular or non-invertible matrix

That means $det(A - \lambda I) = 0$ this is characteristic equation of matrix A

finding characteristic equation, Eigen value, and Eigen vector

Example: Finding Eigen value and Eigen vector

Let A=$\begin{pmatrix} 11 & 9 & -2 \\ -8 & -6 & 2 \\ 4 & 4 & 1 \end{pmatrix}$ then $(A-xI)=\begin{pmatrix} 11-x & 9 & -2 \\ -8 & -6-x & 2 \\ 4 & 4 & 1-x \end{pmatrix}$ and

$$det(A-xI)=\det\left(\begin{pmatrix} 11-x & 9 & -2 \\ -8 & -6-x & 2 \\ 4 & 4 & 1-x \end{pmatrix}\right)=$$

Characteristic polynomial o A

$$p(x)=6-11\,x+6\,x^2-x^3$$

Step3: Factor characteristic equation$-(x-3)\,(x-2)\,(x-1)=0$, So $eigen\ \ value\ are\ \lambda=3,2,1$

Step4: finding eigenvector: to find Eigen vector we have to solve the following S.L.E

$$For\ eigen\ \ value\ \lambda=3\,,(A-3I)V=0\ldots\ldots\ldots\ldots..(1)$$

$$For\ eigen\ \ value\ \lambda=2\,,(A-2I)V=0\ldots\ldots\ldots..(2)$$

$$For\ eigen\ \ value\ \lambda=1,(A-1I)V=0\ldots\ldots\ldots.(3)$$

For Eigen value=3 , $(A-3I)V=0$ AND $v=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so $\begin{pmatrix} 8 & 9 & -2 \\ -8 & -9 & 2 \\ 4 & 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ by ERO we have

reduce$\left(\begin{pmatrix} 11-3 & 9 & -2 \\ -8 & -6-3 & 2 \\ 4 & 4 & 1-3 \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & -\frac{5}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

so $z\ is\ free\ \ variable\ and\ y=2z\,,x=\frac{5}{2}z\,,eigen\ vectorv=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} \frac{5}{2}z \\ 2z \\ z \end{pmatrix}=\begin{pmatrix} \frac{5}{2} \\ -2 \\ 1 \end{pmatrix}z\,,such\ that$

$$z\in IR\ and\ z\neq 0$$

Similar way for λ =2:reduce $\left(\begin{pmatrix} 11-2 & 9 & -2 \\ -8 & -6-2 & 2 \\ 4 & 4 & 1-2 \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so Eigen vector is

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} y, suh\ that\ y \neq 0\ and\ y\ is\ free\ varibles$$

$\lambda = 1,$ reduce $\left(\begin{pmatrix} 11-1 & 9 & -2 \\ -8 & -6-1 & 2 \\ 4 & 4 & 1-1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

so Eigen vector is $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ -2z \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} z\ , z\ is\ free\ variable$

Eigen vector matrix is $P = \begin{pmatrix} 2.5 & -1 & 2 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$ $and\ P^{-1} =$ inverse $\left(\begin{pmatrix} 2.5 & -1 & 2 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}\right) =$ $\begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 2 \\ -2 & -2 & 1 \end{pmatrix}$ using suitable method

Theorems on eigenvalues and eigenvectors

Theorem : If $\mathbf{A}$ is a $n \times n$ triangular matrix– upper triangular, lower triangular or diagonal, the eigenvalues of $\mathbf{A}$ are the diagonal entries of $\mathbf{A}$.

Theorem : $\lambda = 0$ is an eigenvalue of $\mathbf{A}$ if $\mathbf{A}$ is a singular (noninvertible) matrix.

Theorem : $\mathbf{A}$ and $\mathbf{A}^T$ has the same eigenvalues.

Theorem : Eigenvalues of a symmetric matrix are real.

Theorem : Eigenvectors of a symmetric matrix are orthogonal, but only for distinct eigenvalues.

Theorem : $|\det(A)|$ is the product of the absolute values of the eigenvalues of $\mathbf{A}$.

Theorem : If λ is an eigenvalue of $\mathbf{A}$, then λ^n is an eigen value of $\mathbf{A}^n$.

Theorem : If λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}, provided that $\lambda \neq 0$.

Example: What are the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 9 & 5 & 7.5 & 0 \\ 2 & 6 & 0 & -7.2 \end{bmatrix}.$$

Solution: Since the matrix $\mathbf{A}$ is a lower triangular matrix, the eigenvalues of $\mathbf{A}$ are the diagonal elements of $\mathbf{A}$. Thus, the eigenvalues are, $\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 7.5, \lambda_4 = -7.2$.

Example : One of the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 5 & 6 & 2 \\ 3 & 5 & 9 \\ 2 & 1 & -7 \end{bmatrix}$$ is zero. Is $\mathbf{A}$ invertible?

Example : Given the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 2 & -3.5 & 6 \\ 3.5 & 5 & 2 \\ 8 & 1 & 8.5 \end{bmatrix}$$ are $\lambda_1 = -1.547, \lambda_2 = 12.33, \lambda_3 = 4.711$

What are the eigenvalues of $\mathbf{B}$ if $\mathbf{B} = \begin{bmatrix} 2 & 3.5 & 8 \\ -3.5 & 5 & 1 \\ 6 & 2 & 8.5 \end{bmatrix}$.

Solution: $\lambda = 0$ is an eigenvalue of $\mathbf{A}$, that implies $\mathbf{A}$ is singular and is not invertible.

Solution: Since $\mathbf{B} = \mathbf{A}^T$, the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ are the same. Hence eigenvalues of $\mathbf{B}$ also are

$$\lambda_1 = -1.547, \lambda_2 = 12.33, \lambda_3 = 4.711$$

Example : Given the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 2 & -3.5 & 6 \\ 3.5 & 5 & 2 \\ 8 & 1 & 8.5 \end{bmatrix} \text{ are } \lambda_1 = -1.547, \lambda_2 = 12.33, \lambda_3 = 4.711$$

Calculate the magnitude of the determinant of the matrix.

Solution: Since $|\det A| = |\lambda_1||\lambda_2||\lambda_3|$

The determinant of the matrix becomes $|\det A| = |-1.547||12.33||4.711| = 89.88$.

Cayley- Hamilton theorem and minimal polynomials

EVERY MATRIX Satisfy its characteristic equation: that means $P(A) = 0$ matrix if p is characteristic equation of matrix A

Proof: simple way

let $f(\lambda) = det(A - \lambda I) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 \;\; c_i \epsilon K$(field)

then $p(A) = det(A - AI)I = 0 \; matrix \; of \; the$ same size of A

Let A = an $nx\, n$ matrix over a field K.

Note: $A^{-1} = \frac{adj(A)}{\det(A)}$ implies $A^{-1}\, det(A) = adj(A)$

$$\Longrightarrow \det(A)\, I = A\; adj(A)$$

$$\Longrightarrow (xI - A)adj(xI - A) = \det(xI - A) * I = f(x)I$$

Let $f(x)\ in\ \ K[x]$, suppose $f(x) = a_t x^t + a_{t-1}x^{t-1} + \cdots + a_1 x + a_0$

$By\ f(A)$ we mean $f(A) = a_t A^t + a_{t-1}A^{t-1} + \cdots + a_1 A + a_0 I$ we say f annihilates A if $f(A)I = a_t A^t + a_{t-1}A^{t-1} + \cdots + a_1 A + a_0 I = (A - A)adj(A - A) = 0$

Claim: there exist a non zero polynomial f annihilating A

Proof: all nxn matrices form a vector spaces of n^2 .hence the matrices

$I, A, A^2, \ldots, A^{n^2}$ are linearly independent. Therefore , there exist scalars

$c_{0\,,} c_1, \ldots, c_{n^2}$ not all are zero, such that $c_0\, I + c_1 A + \cdots + c_{n^2} A^{n^2} = 0$

Let $f(x) = c_0\ + c_1 x + \cdots + c_{n^2} x^{n^2}$; $f(x) \neq 0; degf\ \leq n^2\ and\ f(A) = 0$

Among all the non-zero polynomials annihilating A, CONSIDER those of least degree. By multiplying suitable non-zero constants we can ensure that they are monic polynomial.

Claim : there is only one monic polynomial of least degree annihilating A.

PROOF. Suppose $f_1 and\ , f_2$ are two polynomials

Let $g = f_1 - f_2$

g$\neq$ 0 $and\ degg < degf_1$

$g(A) = f_1(A) - f_2(A) = 0$

Thus$f_1\ = f_2$

Theorem: let $f(x)\ , g(x)$ the polynomial in x with matrix coefficient and suppose that they are related by the equation $g(x) = f(x)(xI - A)$, where A is a given matrix. Then $g(A) = 0$

Proof: $g(x) = f(x)(xI - A) = (c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k)(xI - A)$

$$= -c_0 A + xIc_0 - c_1 Ax + c_1 Ix^2 + \cdots + (-c_k A + c_k Ix^{k+1})$$

$$= -c_0 A + (Ic_0 - c_1 A)x + (c_1 I - c_2 A)x^2\ + (c_2 I - c_3 A)x^3 + \quad \ldots + (-c_k A + Ic_{k-1})x^k + c_k Ix^{k+1}$$

Therefore:

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$$g(A) = -c_0A + (Ic_0 - c_1A)A + (c_1I - c_2A)A^2 \;+\; ... + (-c_kA + Ic_{k-1})A^k \;+ c_kIA^{k+1} = 0$$

Since : all terms are cancelled each other

Theorem :(cayley- Hamilton): every square matrix A is a root of its own characteristic polynomial

That means: $\chi_A(A) = 0$

Proof: let $B = adj(xI - A)$

$$= \begin{pmatrix} p_{11}(x) & p_{12}(x) & \dots & p_{1n}(x) \\ p_{21}(x) & p_{22}(x) & \dots & p_{2n}(x) \\ \dots & \dots & \dots & \dots \\ p_{n1}(x) & p_{n2}(x) & \dots & p_{nn}(x) \end{pmatrix}$$

Where the $p_{ij}(x)$ are polynomial in x . B can be writen as polynomial in x having matrix coefficent. Let

$$B = B_0 + B_1x + B_2x^2 \dots + B_kx^k = f(x)$$

Where $B_0, B_1, \dots ,\ B_k$ are matrices whose entries are scalars

let $p_A(x) = \det(xI - A) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

by property of adjiont we have

$B(xI - A) = \chi_A(x)I = \det(xI - A)\,I$;Where B$= B_0 + B_1x + B_2x^2 \dots + B_kx^k = f(x)$

$$(B_0 + B_1x + B_2x^2 \dots + B_kx^k)(xI - A) = P(x)I$$

$$= -B_0A + (IB_0 - B_1A)x + (B_1I - B_2A)x^2 \;+ \cdots + (-B_kA + IB_{k-1})x^k \;+ B_kIx^{k+1} = P(x)I$$

$$= -B_0A + (IB_0 - B_1A)A + (B_1I - B_2A)A^2 \;+ \cdots + (-B_kA + IB_{k-1})A^k \;+ B_kIA^{k+1} = P(A)I$$

Left side form telescoping series sum, completely canceled each other

$$0 = p(A)I = p(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_k A^k$$

Note :Manipulation of matrices is often greatly facilitated by the the cayley hamilton theorem, which provides an easy method for expressing any plynomial n A as polynomial of degree not exceding $n-1$

Minimal polynomial of matrix(square matrix)

Minimal polynomial is a least degree polynomial which is factor of characteristic polynomials. Every matrix satisfies its minimal polynomials. That mean
$M(A) = 0\ matrix\ if\ M\ is\ minimal\ polynomial\ of\ matrix\ A$

Definition. Le A be an nxn matrix over a field K . the unique monic polynomial of least degree, which annihilates A is called the minimal polynomial of A and is denoted by $m_A(x)$

Theorem; let A be an nxn matrix over field K. A is non-singular iff the constant term of

$m_A(x)$is non zero

Proof: let $m_A(x) = x^t + a_{t-1}x^{t-1} + \cdots + a_1 x + a_0$

$\Rightarrow$ A is non singular matrix

$$m_A(A) = A^t + a_{t-1}A^{t-1} + \cdots + a_1 A + a_0 I = 0$$

If $a_0 = 0$ then A $(A^{t-1} + a_{t-1}A^{t-2} + \cdots + a_1 I) = 0$

Hence $(A^{t-1} + a_{t-1}A^{t-2} + \cdots + a_1 I) = 0$, contradict with minimalist of $m_A(x)$

Therefore $a_0 \neq 0$

$$\Leftarrow suppose\ a_0 \neq 0$$

$$m_A(A) = A^t + a_{t-1}A^{t-1} + \cdots + a_1 A + a_0 I = 0$$

$$A^t + a_{t-1}A^{t-1} + \cdots + a_1 A = -a_0 I$$

$$(A^{t-1} + a_{t-1}A^{t-2} + \cdots + a_1 I)A = -a_0 I$$

$$(A^{t-1} + a_{t-1}A^{t-2} + \cdots + a_1 I) = -a_0 A^{-1}$$

$$IA^{-1} = \frac{-(A^{t-1} + a_{t-1}A^{t-2} + \cdots + a_1 I)}{a_0}$$

Hence A is non singular or invertable

Note:

If the Eigen values of A are distinct, then $m_A(x) = X_A(x)$

How does one can find $m_A(x)$

One way is to find all factors of $X_A(x)$ and test whether they have A as

a root.

Surprisingly, one does not need to find the eigenvalues of A to find

$X_A(x)$, as shown in the theorem below.

Theorem : let $b(x)$ be the monic gcd of the of entries of the adjiont of xI-A. then b(x) divides X(A) and $m_A(x) = \frac{X_A(x)}{b(x)}$

Proof: let $B = adj(xI - A)$

We write $B = b(x)C$, where gcd of entries of C is 1

$$B(xI - A) = \chi_A(x)I = \det(xI - A)\, I$$

We get $b(x)(xI - A)C = \chi_A(x)I$……………………………….(*) shows b(x) divedes $\chi_A(x)$

Let $t(x) = \frac{X_A(x)}{b(x)}$ we get $(xI - A)C = \frac{\chi_A(x)I}{b(x)} = t(x)I$

Let $t(x) = \dfrac{\chi_A(x)}{b(x)}$

We get (xI – A). C = t(x).I

By reasoning as in the proof of the C - H theorem, we can substitute A for x. Therefore, t(A) = 0. So, $m_A(x)$ must divide t(x).

On the other hand, since $m_A(A) = 0$, we have a factorization

$$m_A(xI) = Q(x)\,(xI - A)$$

[Analogous to p(x) = q(x) (x-a) if p(a) = 0]

with suitable matrix polynomial Q(x).

Multiplying by C and using (xI – A) C = t(x) I we get:

$$m_A(x).\ C = t(x)\ Q(x).$$

Since $m_A(x)$ is the gcd of the entries of the matrix on the left, t(x) must divide $m_A(x)$.

Therefore, $m_A(x) = t(x)$

Examples .let A= $\begin{pmatrix} 11 & 9 & -2 \\ -8 & -6 & 2 \\ 4 & 4 & 1 \end{pmatrix}$ then the Chacteristic polynomial of A is

$$p(x) = -(x-3)\,(x-2)\,(x-1) = x^3 - 6x^2 + 11x - 6$$

$p(A) = -(A-3I)\,(A-2I)\,(A-I) = 0\ matrix$ That means

$$\left(-\left(\begin{pmatrix} 11-3 & 9 & -2 \\ -8 & -6-3 & 2 \\ 4 & 4 & 1-3 \end{pmatrix}\right)\right)\left(\begin{pmatrix} 11-1 & 9 & -2 \\ -8 & -6-1 & 2 \\ 4 & 4 & 1-1 \end{pmatrix}\right)\left(\begin{pmatrix} 11-2 & 9 & -2 \\ -8 & -6-2 & 2 \\ 4 & 4 & 1-2 \end{pmatrix}\right) =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Or $A^3 - 6A^2 + 11A - 6I =$

$$= \begin{pmatrix} 11 & 9 & -2 \\ -8 & -6 & 2 \\ 4 & 4 & 1 \end{pmatrix}^3 - 6\begin{pmatrix} 11 & 9 & -2 \\ -8 & -6 & 2 \\ 4 & 4 & 1 \end{pmatrix}^2 + 11\begin{pmatrix} 11 & 9 & -2 \\ -8 & -6 & 2 \\ 4 & 4 & 1 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And minimal polynomial is $M(x) = -(x-3)\,(x-2)\,(x-1)$ why? Since Eigen values are distinct

Similar matrix and characteristic polynomials

Definition: Similar matrices

Let A and B be $n \times n$. We say that A is similar to B if there exist an invertible nxn matrix P such that $B = P^{-1}\,AP$.denoted by $A \sim B$

Remarks:

1) If A is similar to B we can write equivalently, that $A = PBP^{-1}$ or $AP = PB$
2) . The matrix P depends on A and B. it is not unique for a given pair of similar matrices A&B

Theorem: Let A, B and C is square matrices. Then

i. A$\sim A$
ii. If A$\sim B$ then B$\sim A$
iii. If A$\sim B$ then B$\sim C$ then A$\sim C$

Proof: $A = I^{-1}\,AI$ $implies$ $A \backsim A$

$B = P^{-1}AP$ $then$ $A = (P^{-1})^{-1}BP^{-1}$ Implies B$\sim A$

$$B = P^{-1}AP \quad and\ C = S^{-1}BS\ then$$

$C = s^{-1}p^{-1}APS = (PS)^{-1}APS$

Implies A $\sim C$

Theorem: let A and B be similar an nxn matrices then

i. $det(A) = det(B)$
ii. A is invertible if and only if B is invertible
iii. A and B have the same Eigen value
iv. A and B have the same characteristic polynomials

Proof: $Det(B - \lambda I) = Det(P^{-1}\,AP - \lambda I) = Det(P^{-1}\,AP - \lambda P^{-1}PI) =$

$$Det(P^{-1}(A - \lambda I)P) = \det(P^{-1})(\det(A - \lambda I)\det(P) = \det(A - \lambda I)$$

$$and\ \det(B) = det(A)$$

Suppose A and B are similar then

$$B = P^{-1}AP \ \ and \ \ A = (P^{-1})^{-1}BP^{-1} = PBP^{-1}$$

and Suppose A is invertible then $B^{-1} = ([P^{-1}AP)^{-1} = P^{-1}A^{-1}P$

Suppose B is invertible then $A = (P^{-1})^{-1}BP^{-1} = PBP^{-1}$ then

$$A^{-1} = (PBP^{-1})^{-1} = PB^{-1}P^{-1}$$

Example. 'Let. A= $\begin{pmatrix} 11 & 9 & -2 \\ -8 & -6 & 2 \\ 4 & 4 & 1 \end{pmatrix}$ and P= $\begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$ then find similar matrix to A

Solution: $P^{-1} =$ inverse $\left(\begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & -2 & -\frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & 1 \end{pmatrix}$ $so\ p\ is$ invertable

$$B = p^{-1}AP = \left(\left(\begin{matrix} 1 & \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & -2 & -\frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & 1 \end{matrix}\right)\right)\left(\begin{pmatrix} 11 & 9 & -2 \\ -8 & -6 & 2 \\ 4 & 4 & 1 \end{pmatrix}\right)\left(\begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}\right) = \begin{pmatrix} -\frac{11}{2} & -\frac{15}{2} & -\frac{1}{2} \\ \frac{17}{2} & \frac{21}{2} & \frac{1}{2} \\ -14 & -14 & 1 \end{pmatrix}$$

This is similar to A. so B has similar Eigen value and characteristic equation

To Show assume . $(B - xI) = \begin{pmatrix} -\frac{11}{2} - x & -\frac{15}{2} & -\frac{1}{2} \\ \frac{17}{2} & \frac{21}{2} - x & \frac{1}{2} \\ -14 & -14 & 1 - x \end{pmatrix} and$

$\det(B - xI) = 6 - 11x + 6x^2 - x^3 = \det(A - xI)$

$$P^{-1} = \begin{pmatrix} -\frac{7}{3} & -\frac{17}{6} & 1 \\ \frac{8}{3} & \frac{19}{6} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \end{pmatrix} \text{ and } B^{-1} = \begin{pmatrix} \frac{35}{12} & \frac{29}{12} & \frac{1}{4} \\ -\frac{31}{12} & -\frac{25}{12} & -\frac{1}{4} \\ \frac{14}{3} & \frac{14}{3} & 1 \end{pmatrix} \text{ and } A^{-1} = \begin{pmatrix} -\frac{7}{3} & -\frac{17}{6} & 1 \\ \frac{8}{3} & \frac{19}{6} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \end{pmatrix}$$

$$B^{-1} = \left(\begin{pmatrix} 1 & \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & -2 & -\frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & 1 \end{pmatrix}\right)\left(\begin{pmatrix} -\frac{7}{3} & -\frac{17}{6} & 1 \\ \frac{8}{3} & \frac{19}{6} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \end{pmatrix}\right)\left(\begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}\right) = \begin{pmatrix} \frac{35}{12} & \frac{29}{12} & \frac{1}{4} \\ -\frac{31}{12} & -\frac{25}{12} & -\frac{1}{4} \\ \frac{14}{3} & \frac{14}{3} & 1 \end{pmatrix}$$

ACTIVTY: both matrices have the same Eigen vector or not? Show .Answer: not

Proof: Note: $Av = \lambda v = APP^{-1}\, v = \lambda v$

$P^{-1}APP^{-1}\, v = P^{-1}\lambda v = \lambda P^{-1}v$ Multiply by P^{-1} from left to right

$P^{-1}APP^{-1}\, v = \lambda P^{-1}v = BP^{-1}v = \lambda P^{-1}v$ Therefore $z = P^{-1}v$ is Eigen vector of similar matrix B and the same Eigen value and different Eigen vector

Exercise: calculate Eigen vector of similar matrix B from the above example.

Solution: $z_1 = P^{-1}v_1 = \begin{pmatrix} -\frac{7}{3} & -\frac{17}{6} & 1 \\ \frac{8}{3} & \frac{19}{6} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \end{pmatrix}\begin{pmatrix} \frac{5}{2} \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ for $\lambda = 3$

For $\lambda = 2; z_2 = P^{-1}v_2 = \begin{pmatrix} -\frac{7}{3} & -\frac{17}{6} & 1 \\ \frac{8}{3} & \frac{19}{6} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$

For $\lambda = 1; z_3 = P^{-1}v_3 = \begin{pmatrix} -\frac{7}{3} & -\frac{17}{6} & 1 \\ \frac{8}{3} & \frac{19}{6} & -1 \\ -\frac{4}{3} & -\frac{4}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

The spectral radius of matrix

Let $\lambda_1, \lambda_2, \dots \lambda_n$ be the real or complex Eigen values of a matrix $A \in C^{nxn}$ then its spectral radius of $\rho(A)$ is defined as

$$\rho(A) = \max\{ abs(\lambda_1), abs(\lambda_2), \dots, abs(\lambda_n)\}.$$

The condition number of matrix A can be expressed using the spectral radius as $\rho(A) * \rho(A^{-1})$

Examples: let matrix A=$\begin{bmatrix} 9 & -1 & 2 \\ -2 & 8 & 4 \\ 1 & 1 & 8 \end{bmatrix}$ then Eigen values of a are $\lambda = 5,10,10\ and \rho(A) = 10$ by definition

DIAGONALIZATION OF MATRIX

Definition: Diagonalization of matrices

An nxn matrix called to be diagonalizable if it is similar to a diagonal matrix. That is A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal matrix.this matrix is said to be diagonalizable A. if P is an orthogonal matrix i.e.$P^T = P^{-1}$ such that $P^{-1}AP = P^TAP$ is diagonal matrix then A is orthogonally diagonalizable and P is said to be orthogonally diagonalizable A

Diagonalization procedure

1. Finding characteristic polynomial P(λ)of A
2. Find Eigen values λ of the matrix A and their algebraic multiplicities from the characteristic polynomial P(λ)of A
3. For each Eigen values λ of the matrix A.. find linear independent eigenvectors
4. Combine all Eigen vectors in matrix P

Examples1 : let A= $\begin{pmatrix} 11 & 9 & -2 \\ -8 & -6 & 2 \\ 4 & 4 & 1 \end{pmatrix}$ then From the above examples Eigen values are $\lambda = 3, 2, 1$

For eigen value=3 , $(A - 3I)V = 0$ AND $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ so $\begin{pmatrix} 8 & 9 & -2 \\ -8 & -9 & 2 \\ 4 & 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ by ERO we have

reduce$\left(\begin{pmatrix} 11-3 & 9 & -2 \\ -8 & -6-3 & 2 \\ 4 & 4 & 1-3 \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{5}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

so $z\ is\ free\ \ variable\ and\ y = 2z\ , x = \frac{5}{2}z\ , eigen\ vectorv = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{2}z \\ 2z \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{2} \\ -2 \\ 1 \end{pmatrix} z\ , such\ that$

$$z \in IR\ and\ z \neq 0$$

Similar way for λ =2:reduce$\left(\begin{pmatrix} 11-2 & 9 & -2 \\ -8 & -6-2 & 2 \\ 4 & 4 & 1-2 \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so Eigen vector is

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} y, suh\ that\ y \neq 0\ and\ y\ is\ free\ varibles$$

$\lambda = 1$, reduce$\left(\begin{pmatrix} 11-1 & 9 & -2 \\ -8 & -6-1 & 2 \\ 4 & 4 & 1-1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

so Eigen vector is $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ -2z \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} z\ , z\ is\ free\ variable$

Eigen vector matrix is $P = \begin{pmatrix} 2.5 & -1 & 2 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$ $and\ P^{-1}$ = inverse$\left(\begin{pmatrix} 2.5 & -1 & 2 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}\right) =$ $\begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 2 \\ -2 & -2 & 1 \end{pmatrix}$

Hence $D = P^{-1}AP = \left(\begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 2 \\ -2 & -2 & 1 \end{pmatrix}\right)\left(\begin{pmatrix} 11 & 9 & -2 \\ -8 & -6 & 2 \\ 4 & 4 & 1 \end{pmatrix}\right)\left(\begin{pmatrix} 2.5 & -1 & 2 \\ -2 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Example 2: Diagonalize $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, if possible.

Solution: $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2.$

When $\lambda = 1$

$$\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Leftrightarrow \left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ -3 & -6 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x_2 = -x_3, x_1 = -x_2 = x_3 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, k \neq 0$$

When $\lambda = -2$

$$\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Leftrightarrow \left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_3 = t, x_2 = s, x_1 = -s - t$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, t^2 + s^2 \neq 0$$

$A=\begin{bmatrix}1 & 3 & 3\\ -3 & -5 & -3\\ 3 & 3 & 1\end{bmatrix}$ has three linearly independent eigenvectors, therefore $A=\begin{bmatrix}1 & 3 & 3\\ -3 & -5 & -3\\ 3 & 3 & 1\end{bmatrix}$ is diagonalizable and $\begin{pmatrix}-1 & -1 & 1\\ 0 & 1 & -1\\ 1 & 0 & 1\end{pmatrix}^{-1}\begin{bmatrix}1 & 3 & 3\\ -3 & -5 & -3\\ 3 & 3 & 1\end{bmatrix}\begin{pmatrix}-1 & -1 & 1\\ 0 & 1 & -1\\ 1 & 0 & 1\end{pmatrix}=\begin{pmatrix}-2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{pmatrix}$

theorem. $if\ A = PDP^{-1}\ then\ A^n = PD^n\ P^{-1}$

Proof: $A^2 = A * A = [PDP^{-1}][PDP^{-1}] = PD^2\ \ P^{-1}]$ and $A^3 = A * A * A = [PDP^{-1}][PDP^{-1}]PDP^{-1} = PD^3\ \ P^{-1}$ so that by induction: $if A = PDP^{-1}\ then\ A^n = PD^n\ P^{-1}$

Exercise. Calculate $D^n\ and\ A^n\ for\ n = 8\ ,9\ ,12\ from\ the\ above$

More examples and exercise: do more by your self

<u>ORTHOGONAL MATRIX</u>

Definition: orthogonal matrices: A square matrix A aver Real for which $A^T = A^{-1}$ is called orthogonal matrix . orthogonal matrix is non-singular

Examples :show that the given matrix is orthogonal matrices

$$\begin{bmatrix}cos\theta & -sin\theta & 0\\ -sin\theta & cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

Solution show that $AA^T\ = I = \begin{bmatrix}cos\theta & -sin\theta & 0\\ -sin\theta & cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}cos\theta & -sin\theta & 0\\ -sin\theta & cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

Example 2: Let

$$A=\begin{bmatrix}0 & 2 & 2\\ 2 & 0 & 2\\ 2 & 2 & 0\end{bmatrix}.$$

Find an orthogonal matrix ***P*** and a diagonal matrix ***D*** such that $D = P^T AP$.

Solution**:** We need to find the orthonormal eigenvectors of ***A*** and the associated eigenvalues first. The characteristic equation is

$$f(\lambda)=\det(\lambda I-A)=\begin{vmatrix}\lambda & -2 & -2\\ -2 & \lambda & -2\\ -2 & -2 & \lambda\end{vmatrix}=(\lambda+2)^2(\lambda-4)=0$$. Thus, $\lambda=-2,-2,4.$

1. As $\lambda=-2,$ solve for the homogeneous system
   $(-2I-A)x=0$.

The eigenvectors are

$$t\begin{bmatrix}-1\\1\\0\end{bmatrix}+s\begin{bmatrix}-1\\0\\1\end{bmatrix}, t,s\in R,\ t\neq 0 \text{ or } s\neq 0. \Rightarrow v_1=\begin{bmatrix}-1\\1\\0\end{bmatrix} \text{ and } v_2=\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

are two eigenvectors of **A**. However, the two eigenvectors are not orthogonal. We can obtain two orthonormal eigenvectors via ***Gram-***
Schmidt process. The orthonormal eigenvectors are

$$v_1^*=v_1=\begin{bmatrix}-1\\1\\0\end{bmatrix},\quad v_2^*=v_2-\frac{v_2\cdot v_1}{v_1\cdot v_1}v_1=\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}.$$

Standardizing these two eigenvectors results in

$$w_1=\frac{v_1^*}{\|v_1^*\|}=\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix},\quad w_2=\frac{v_2^*}{\|v_2^*\|}=\begin{bmatrix}-1/\sqrt{6}\\-1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix}.$$

2. As $\lambda=4,$ solve for the homogeneous system, $(4I-A)x=0$

The eigenvectors are $r\begin{bmatrix}1\\1\\1\end{bmatrix}, r\in R,\ r\neq 0$.

$\Rightarrow v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of **A**. Standardizing the eigenvector results in

$$w_3 = \frac{v_3}{\|v_3\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}.$$ Thus,

$$P = [w_1 \quad w_2 \quad w_3] = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$, is orthogonal matrix

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$, and

$$D = P^T AP.$$

Hence A is orthogonal

WORK SHEET#1: ASSIGNMENT:

1) Find Eigen values and Eigen vectors of the following matrices, diagonalze,check cayley hamilton and find minimal polynomial of the following matrices . Calculate for $A^n = PD^n P^{-1}$ $for\ n = 6\ and\ 12$

{Assignment problem . do by group 1-5}

a) $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

c) $C = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

d) $,D = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

e) $D' = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

f) $E = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}$,

g) $F = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$;

h) $G = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

i) $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{pmatrix}$ k) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

j) $A = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$ l) $A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$

2) $Let\ A$ be a square matrix. Show that the Eigen value of A^T are same as those of A. Are the Eigen vectors o A^T the same as those of A?

3) Suppose that A is similar to

$$D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

a) Find characteristic polynomial and Eigen value of A

b) Let $D = p^{-1}AP$ where

$$P = \begin{pmatrix} 2 & 1 & 0 & 5 \\ 0 & 3 & 0 & 2 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 4 \end{pmatrix}$$

Find a basis for each Eigen spaces of A

4) Diagonalizable the following matrix

a) $A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}$

b) $B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

Decomposable matrices

Block matrix: let A be an $n\ xn$ over field K. partition A into blocks by means of horizontal and vertical lines.

The parts are smaller matrices and are called **block** of matrix A. blocks are maybe considered as element of A

Diagonal block

Given a square matrix A , it is often necessary to partition in to blocks so that the diagonal blocks are also square matrices.

Block matrices of the form $A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_r \end{pmatrix}$

Where $A_1, A_2 \dots . A_r$ square matrices and each zero are is zero matrix of proper dimension, is called a diagonal block matrix.

Note:

1. We say that
   i. A is decomposed into blocks $A_1, A_2 \dots . A_r$
   ii. Or A is direct sum of matrices $A_1, A_2 \dots . A_r$
2. By analogy with diagonal matrices, one writes symbolically

$$A = diag(A_1, A_2 \dots . A_r)$$

Properties of Decomposable matrices

Let A be a square matrix and suppose $A = diag(A_1, A_2, \dots, A_r)$. Then

i) $|A| = |A_1|. |A_2| \dots |A_r|$

ii) If A is non-singular, $A^{-1} = diag(A_1^{-1}, A_2^{-1}, \dots, A_r^{-1})$.

iii) If $B = diag(B_1, B_2, \dots, B_r)$ also with A_i the same size as B_i for $i = 1, 2, \dots, r$, then $AB = diag(A_1B_1, A_2B_2, \dots, A_rB_r)$.

Example: $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 7 & 0 \\ 8 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Theorem: the characteristics polynomial of a decomposable matrix is the product of the the characteristic polynomial of its diagonal blocks**.**

Proof: let
$A = diag(A_1, A_2 \dots . A_r)$ where $A_1, A_2 \dots . A_r$ square matrices of order are n_j

For $\boldsymbol{i = 1, 2, \dots, r}$ **then** $\boldsymbol{xI - A = diag(xI_{n_1}} - A_1, xI_{n_2} - A_2 \dots . xI_{n_r} - A_r)$, where I_{n_i} is an indent matrix of order n_i

$$\boldsymbol{det}(\boldsymbol{xI - A}) = |\boldsymbol{xI_{n_1}} - A_1||xI_{n_2} - A_2| \dots |xI_{n_r} - A_r|$$

$$\boldsymbol{i.e\ X_A(x)} = \boldsymbol{det}(\boldsymbol{xI_{n_1}} - A_1) \det(xI_{n_2} - A_2) \dots . \det(xI_{n_r} - A_r)$$

Lemma; suppose M = $\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$ where $A_1 and$, A_2 are square matrices then

$$X_M(x) = X_{A_1}(x) * X_{A_2}(x)$$

Proof:$xI - M = \begin{pmatrix} xI - A_1 & -B \\ 0 & xI - A_2 \end{pmatrix}$

$$\mathrm{X(x)} = \det(xI - M) = \det(xI - A_1) \det(xI - A_2)$$

Generalizing the lemma by induction we may obtain

Theorem: suppose $\boldsymbol{M} = \begin{pmatrix} A_1 & B & \dots & C \\ 0 & A_2 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_r \end{pmatrix}$ where $A_1, A_2 \dots . A_r$ square matrices are then $X_M(x) = X_{A_1}(x) * X_{A_2}(x) * \dots * X_{A_r}(x)$

Work sheet #2

3. Show that if A is diagonalizable, so is A^2.
4. Suppose that $n \times n$ matrices A and B have exact the same linearly independent eigenvectors that span R^n. Show that (a) both A and B can be simultaneously diagonalized (i.e., there are the same P such that $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices), and (b) AB is also diagonalizable.
5. for each statement, determine and explain true or false.
   (a) A is diagonalizable if $A = P^{-1}DP$ for some diagonal matrix D.
   (b) If A is diagonalizable, then A is invertible.
   (c) If $AP = PD$ with D diagonal matrix, then nonzero columns of P are eigenvectors of A.
   (d) Let A be a symmetric matrix and λ be an eigenvalue of A with multiplicity 5. Then the eigenspace $Null\,(A - \lambda I)$ has dimension 5.