Math 2073 -Solution of Nonlinear Equations
Lecture-3

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- The root-finding problem consists of the following: given a continuous function f, find the values of x that satisfy the equation

$$f(x) = 0. \tag{1}$$

- The solutions of this equation are called the zeros of f or the roots of the equation.
- In general, Eqn. (1) is impossible to solve exactly.
- Therefore, one must rely on some numerical methods for an approximate solution.
- The methods we will discuss in this chapter are iterative and consist basically of two types:
  - one in which the convergence is guaranteed and
  - the other in which the convergence depends on the initial guess.

The methods used for solving equations numerically can be divided into two groups: bracketing methods and open methods.

Stopping Criteria

- Many ways to decide when to stop:
- Let c^* be the true (exact) solution such that $f(c^*) = 0$, and let m_k be a numerically approximated solution such that $f(m_k) = \epsilon$ (where ϵ is a small number).
- Tolerance in the solution A tolerance is the maximum amount by which the true solution can deviate from an approximate numerical solution.

Stopping Criteria

When do we stop?

Let m_k is approximate solution of $f(x)$ at k^{th} iteration.
We can

1. keep going until successive iterates are close: $|m_k - m_{k-1}| < \epsilon$
2. close in relative terms: $\dfrac{|m_k - m_{k-1}|}{|m_k|} < \epsilon$
3. the function value is small enough: $|f(m_k)| < \epsilon$

No choice is perfect. In general, where no additional information about f is known, the second criterion is the preferred one (since it comes the closest to testing the relative error).

Suppose an algorithm generates a sequence of approximations, c_n, which approaches a limit, c_*, or

$$\lim_{n\to\infty} c_n = c_*$$

How quickly does $c_n \to c_*$?

RATE OF CONVERGENCE

If a sequence $c_1, c_2, \cdots, c_n$ converges to a value c_* and if there exist real numbers $\lambda > 0$ and $\alpha \geq 1$ such that

$$\lim_{n\to\infty} \frac{|c_{n+1} - c_*|}{|c_n - c_*|^\alpha} = \lambda$$

then we say that α is the rate of **convergence of the sequence**.

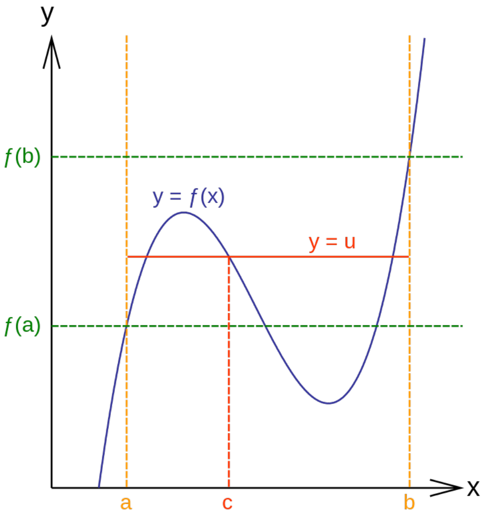
Theorem (The intermediate value theorem)

The intermediate value theorem states that if a continuous function, f, with an interval, $[a, b]$, as its domain, takes values $f(a)$ and $f(b)$ at each end of the interval, then it also takes any value between $f(a)$ and $f(b)$ at some point within the interval.

Theorem (Bolzano's theorem)

If a continuous function has values of opposite sign inside an interval, then it has a root in that interval. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) \cdot f(b) < 0$. Then there exist at least one point x in the open interval (a, b) such that $f(x) = 0$.

Intermediate value theorem



The input for the method is a continuous function f, an interval $[a_0, b_0]$, and the function values $f(a_0)$ and $f(b_0)$. The function values are of opposite sign (there is at least one zero crossing within the interval).

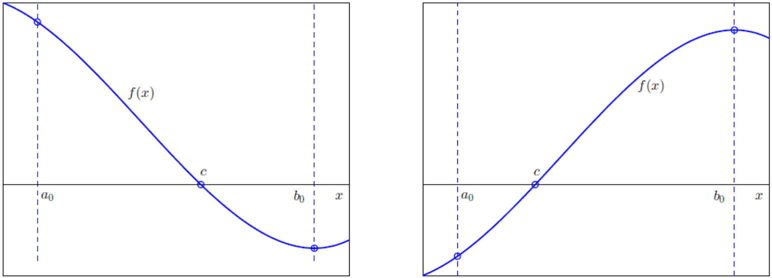


Figure: Bisection Method

THE BISECTION METHOD PROCEDURE IS:

1. Choose a starting interval $[a_0, b_0]$ such that $f(a_0)f(b_0) < 0$.
2. Compute $f(m_0)$ where $m_0 = (a_0 + b_0)/2$ is the midpoint.
3. Determine the next subinterval $[a_1, b_1]$:
    1. If $f(a_0)f(m_0) < 0$, then let $[a_1, b_1]$ be the next interval with $a_1 = a_0$ and $b_1 = m_0$.
    2. If $f(b_0)f(m_0) < 0$, then let $[a_1, b_1]$ be the next interval with $a_1 = m_0$ and $b_1 = b_0$.
4. Repeat (2) and (3) until the interval $[a_N, b_N]$ reaches some predetermined length.
5. Constructs a sequence of intervals containing the root c:

$$(a_0, b_0) \supset (a_1, b_1) \supset \cdots \supset (a_{N-1}, b_{N-1}) \supset (a_N, b_N) \ni c$$

6. After k steps

$$|b_k - a_k| = \frac{1}{2}|b_{k-1} - a_{k-1}| = \left(\frac{1}{2}\right)^k |b_0 - a_0|$$

7. At step k, the midpoint $m_k = \frac{a_k + b_k}{2}$ is an estimate for the root c with

Absolute Error

The bisection method does not (in general) produce an exact solution of an equation $f(x) = 0$. However, we can give an estimate of the absolute error in the approxiation.

Theorem

Let $f(x)$ be a continuous function on $[a, b]$ such that $f(a)f(b) < 0$. After N iterations of the biection method, let x_N be the midpoint in the N^{th} subinterval $[a_N, b_N]$

$$x_N = \frac{a_N + b_N}{2}$$

There exists an exact solution x_{true} of the equation $f(x) = 0$ in the subinterval $[a_N, b_N]$ and the absolute error is

$$|\, x_{\mathrm{true}} - x_N \,| \leq \frac{b-a}{2^{N+1}}$$

Note that we can rearrange the error bound to see the minimum number of iterations required to guarantee absolute error less than a prescribed ϵ :

$$\frac{b-a}{2^{N+1}} < \epsilon$$

$$\frac{b-a}{\epsilon} < 2^{N+1}$$

$$\ln\left(\frac{b-a}{\epsilon}\right) < (N+1)\ln(2)$$

$$\frac{\ln\left(\frac{b-a}{\epsilon}\right)}{\ln(2)} - 1 < N$$

Example

The bisection method applied to $f(x) = \left(\frac{x}{2}\right)^2 - \sin(x) = 0$ with $(a_0, b_0) = (1.5, 2.0)$, and $(f(a_0), f(b_0)) = (-0.4350, 0.0907)$ gives:

k	a_k	b_k	m_k	$f(m_k)$
0	1.5000	2.0000	1.7500	-0.2184
1	1.7500	2.0000	1.8750	-0.0752
2	1.8750	2.0000	1.9375	0.0050
3	1.8750	1.9375	1.9062	-0.0358
4	1.9062	1.9375	1.9219	-0.0156
5	1.9219	1.9375	1.9297	-0.0054
6	1.9297	1.9375	1.9336	-0.0002
7	1.9336	1.9375	1.9355	0.0024
8	1.9336	1.9355	1.9346	0.0011
9	1.9336	1.9346	1.9341	0.0004

ADVANTAGES AND DISADVANTAGES

1. The method is guaranteed to converge
2. The error bound decreases by half with each iteration
3. The bisection method converges very slowly
4. The bisection method cannot detect multiple roots

Example

Let's use our function with input parameters $f(x) = x^2 - x - 1$ and $N = 25$ iterations on $[1, 2]$ to approximate the **golden ratio**

$$\phi = \frac{1+\sqrt{5}}{2}$$

The golden ratio is a root of the quadratic polynomial $x^2 - x - 1 = 0$.

$$1.618033990263939$$

The absolute error is guaranteed to be less than $(2-1)/(2^{26})$ which is

$$2^{(-26)} = 1.4901161193847656e^{-08}$$